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FACTORIZATIONS OF PROPERTIES OF GRAPHS

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Abstract

A property of graphs is any isomorphism closed class of simple graphs. For given properties of graphs $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of a graph G is a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that for each $i = 1, 2, \ldots, n$ the induced subgraph $G[V_i]$ has property $\mathcal{P}_i$. The class of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition is denoted by $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$. A property $\mathcal{R}$ is said to be reducible with respect to a lattice of properties of graphs $\mathbb{L}$ if there are $n \geq 2$ properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n \in \mathbb{L}$ such that $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$; otherwise $\mathcal{R}$ is irreducible in $\mathbb{L}$. We study the structure of different lattices of properties of graphs and we prove that in these lattices every reducible property of graphs has a finite factorization into irreducible properties.

Keywords: factorization, property of graphs, irreducible property, reducible property, lattice of properties of graphs.

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1 Introduction and Motivation

For terminology and notation not presented here, we follow [4] and [5]. The *join* $\sum_{i=1}^{n} G_i = G_1 + G_2 + \cdots + G_n$ of n graphs $G_1, G_2, \ldots, G_n$ is the graph consisting of the disjoint union of G_is and all the edges between $V(G_i)$ and $V(G_j)$ for any $1 \leq i < j \leq n$.

In [3] the concept of varieties of graphs with respect to different closure operators on the class of graphs is investigated. In this paper we consider the class $\mathcal{I}$ of all finite simple graphs (without loops and multiple edges) and a finite set of closure operators $\sigma_0, \sigma_1, \ldots, \sigma_n$ defined on $\mathcal{I}$. A *variety* $\mathcal{P}$ of graphs is a subclass of $\mathcal{I}$ *closed under all operators* $\sigma_0, \sigma_1, \ldots, \sigma_n$, i.e., $\sigma_0(\mathcal{P}) = \sigma_1(\mathcal{P}) = \cdots = \sigma_n(\mathcal{P}) = \mathcal{P}$. In order to obtain some interesting results, it is important to choose suitable closure operators in the definition of varieties of graphs.

Throughout the paper we fix six closure operators $\sigma_0, \sigma_1, \ldots, \sigma_5$. These are now defined. For any class $\mathcal{P}$ of graphs we define $\sigma_i(\mathcal{P})$ by:

$\sigma_0(\mathcal{P}) = \{G \in \mathcal{I} : G \cong H$ for some $H \in \mathcal{P}\}$, i.e., σ_0 is derived from the relation "to be isomorphic". A variety of graphs closed under operator σ_0 is called a *graph property*. We also say that a graph G *has property* $\mathcal{P}$ if $G \in \mathcal{P}$. The graph properties $\mathcal{I}$ and $\Theta = \{K_0\}$ are called trivial, all others are nontrivial.

$\sigma_1(\mathcal{P}) = \{G \in \mathcal{I} :$ each component of G is in $\mathcal{P}\}$, i.e., σ_1 is derived from the operation "to take the disjoint union of graphs". A variety of graphs closed under this operator is called *additive*.

$\sigma_2(\mathcal{P}) = \{G \in \mathcal{I} : G \subseteq H$ for some $H \in \mathcal{P}\}$, i.e., σ_2 is derived from the relation "to be a subgraph". A variety of graphs closed under this operator is said to be *hereditary*.

$\sigma_3(\mathcal{P}) = \{G \in \mathcal{I} : G \leq H$ for some $H \in \mathcal{P}\}$, i.e., σ_3 is derived from the relation "to be an induced subgraph". A variety of graphs closed under this operator is said to be *induced hereditary*.

$\sigma_4(\mathcal{P}) = \{G \in \mathcal{I} : G \supseteq H$ for some $H \in \mathcal{P}\}$, i.e., σ_4 is derived from the relation "to be a supergraph". A variety of graphs closed under this operator is said to be *co-hereditary*.

$\sigma_5(\mathcal{P}) = \{G \in \mathcal{I} : G \geq H$ for some $H \in \mathcal{P}\}$, i.e., σ_5 is derived from the relation "to be an induced supergraph". A variety of graphs closed under this operator is said to be *induced co-hereditary*.

Since we are considering properties of graphs, we will always assume that the variety we consider is closed under isomorphism. We will use the notation $\mathbb{L}_{\sigma_{i_1},\ldots,\sigma_{i_k}}$ to denote the set of all properties of graphs closed under each of the operators $\sigma_{i_1},\ldots,\sigma_{i_k}$. If the operator σ_1 is used, we shall omit it in the lower index and we shall add an "a" as an upper index. Thus the set of all additive properties of graphs will be denoted by $\mathbb{L}^a$. In what follows we will consider properties of graphs closed under one of the operators σ_i, $i \in \{2,\ldots,5\}$. In all these cases, the set of all properties closed under the given operator ordered by set inclusion forms a complete lattice, denoted by $(\mathbb{L}_{\sigma_i}, \subseteq)$ (or by $(\mathbb{L}^a_{\sigma_i}, \subseteq)$ if it is also closed under the operator σ_1).

For a property $\mathcal{P}$ the property $\overline{\mathcal{P}} = \mathcal{I} - \mathcal{P}$ is said to be a *co-property.* Note that "to be co-hereditary" is a co-property of "to be hereditary". The set of all "co-properties" of given set of properties $\mathbb{L}$ will be denoted by $\overline{\mathbb{L}}$, i.e., $\overline{\mathbb{L}} = \{\overline{\mathcal{P}} : \mathcal{P} \in \mathbb{L}\}$.

Let $\mathcal{P}$ be a hereditary (an induced hereditary) property of graphs. A graph F is a *forbidden subgraph* for $\mathcal{P}$ provided F is not in $\mathcal{P}$, but all of its proper subgraphs (induced subgraphs) are in $\mathcal{P}$. The set of all forbidden subgraphs for $\mathcal{P}$, denoted by $\boldsymbol{F}(\mathcal{P})$, uniquely determines $\mathcal{P}$.

We omit the proof of the following simple lemma (see [2]).

Lemma 11. *Let $\mathcal{P}$ be a hereditary* (*an induced hereditary*) *property of graphs. Then $\mathcal{P}$ is additive if and only if all the graphs in $\boldsymbol{F}(\mathcal{P})$ are connected.*

In order to determine a hereditary property it is also useful to define the set of so called $\mathcal{P}$-maximal graphs. A graph G is *$\mathcal{P}$-maximal* if $G \in \mathcal{P}$ and for every edge $e \in \overline{G}$ we have that $G + e \notin \mathcal{P}$. Then a graph G has hereditary property $\mathcal{P}$ if there exists $\mathcal{P}$-maximal graph H such that $G \subseteq H$. It is easy to see that the concept of $\mathcal{P}$-maximal graphs cannot be used in the class of induced hereditary properties. Fortunately, a more general concept of generating set is useful also in the class of induced hereditary properties of graphs. We define the set $\mathcal{G} \subseteq \mathcal{I}$ to be a *generating set of* $\mathcal{P}$ if every graph from $\mathcal{P}$ is a subgraph (an induced subgraph) of some graph from $\mathcal{G}$. The fact that $\mathcal{G}$ is a generating set of $\mathcal{P}$ will be written in the following way:

$$[\mathcal{G}]_{\subseteq} = \mathcal{P} \ ([\mathcal{G}]_{\leq} = \mathcal{P} \text{ respectively}).$$

The members of $\mathcal{G}$ are called *generators* of $\mathcal{P}$. The concept of a generating set with respect to an arbitrary partial order was discussed in [1].

Let G be an arbitrary graph. A fundamental colouring problem asks for the minimum positive number k such that there is a partitioning of the vertex set $V(G)$ of G into k independent sets. A natural extension of this type of partitions leads to the generalized graph colouring problems and to the concept of a reducible hereditary property. Let us consider a positive integer $n \geq 2$ and hereditary (or induced hereditary) properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$. A $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$*-partition of a graph* G is a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that for each $i = 1, 2, \ldots, n$ the induced subgraph $G[V_i]$ of G has property $\mathcal{P}_i$. The class $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ is defined as the set of all graphs having a $(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n)$-partition. A nontrivial property $\mathcal{R}$ is called a *reducible property* in some lattice $\mathbf{L}$ of properties if there is an integer $n \geq 2$ and nontrivial properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ in $\mathbb{L}$ such that $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$. The nontrivial property $\mathcal{R}$ is *irreducible* otherwise. If $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_n = \mathcal{P}$ we write $\mathcal{R} = \mathcal{P}^n$. If $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$, we call $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ *a factorization of* $\mathcal{R}$ and we also say that $\mathcal{R}$ is the *product* of $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ and that $\mathcal{R}$ is *divisible* by each $\mathcal{P}_i$, $i = 1, 2, \ldots, n$. The definition of reducible properties appeared in [6]. For more details and some applications we refer the reader to [1] and [2].

The existence of a factorization of a hereditary property of graphs into a finite number of factors follows from the definition of *completeness* of a hereditary property. This is defined, for a nontrivial hereditary property $\mathcal{P}$, as the least integer $c(\mathcal{P})$ such that $K_{c(\mathcal{P})+2} \notin \mathcal{P}$.

The question whether each reducible property has a unique factorization into irreducible factors has been formulated in [5]. This question naturally arises in the study of reducible properties and is motivated by the Fundamental Theorem of Arithmetic. The complete answer for hereditary properties of graphs is given in [8]; we now formulate it as

Theorem 12. *Let $\mathcal{R}$ be an additive hereditary property of graphs. Then $\mathcal{R}$ has a factorization into irreducible additive hereditary factors and this factorization is unique up to the order of the factors.*

In the same paper the authors showed that in the lattice $(\mathbf{L}_{\subseteq}, \subseteq)$ of all hereditary properties of graphs a factorization of a property into irreducible factors need not be unique. The proof of this assertion is given by constructing a property which is not uniquely factorizable into hereditary properties. Since each hereditary property of graphs is also an induced hereditary property, this example shows that the factorization of induced hereditary properties of graphs into irreducible induced hereditary properties is also not unique in general. An unsolved problem in this line of investigation is the question whether the factorization of reducible additive induced hereditary properties of graphs into irreducible factors is unique.

2 Factorization of Reducible Properties of Graphs

We now turn our attention to the factorization of reducible properties of graphs into (finitely many) irreducible factors in the lattices defined above. We will prove that each property in each of the lattices $(\mathbb{L}_{\sigma_2}, \subseteq), (\mathbb{L}^a_{\sigma_2}, \subseteq)$, $(\mathbb{L}_{\sigma_3}, \subseteq)$, $(\mathbb{L}^a_{\sigma_3}, \subseteq)$, $(\mathbb{L}_{\sigma_4}, \subseteq)$ and $(\mathbb{L}_{\sigma_5}, \subseteq)$ has a factorization into a finite number of factors which are irreducible in the corresponding lattice.

Theorem 21. *Let* $(\mathbb{L}, \subseteq)$ *be one of the lattices* $(\mathbb{L}_{\sigma_2}, \subseteq), (\mathbb{L}^a_{\sigma_2}, \subseteq)$, $(\mathbb{L}_{\sigma_3}, \subseteq)$, $(\mathbb{L}^a_{\sigma_3}, \subseteq)$, $(\mathbb{L}_{\sigma_4}, \subseteq)$ *and* $(\mathbb{L}_{\sigma_5}, \subseteq)$ *and let* $\mathcal{P}$ *be any nontrivial property of graphs belonging to* $\mathbb{L}$. *Then* $\mathcal{P}$ *is factorizable into a finite number of factors irreducible over* $(\mathbb{L}, \subseteq)$.

The proof of this theorem can be given using suitable invariants of properties of graphs which we now describe. The first is easy to describe and, as we shall see, related to the completeness of a property. We define the *co-completeness of a co-property* $\overline{c}(\overline{\mathcal{P}})$ as the least value of the integer k such that $K_{k+2} \in \overline{\mathcal{P}}$. If $\mathcal{P}$ is hereditary and $\overline{c}(\overline{\mathcal{P}}) = k$, then $K_{k+1} \in \mathcal{P}$ and k is the least value with this property so that $\overline{c}(\overline{\mathcal{P}}) = c(\mathcal{P})$.

In the next lemma we show how to compute the co-completeness of products of co-hereditary properties of graphs.

Lemma 22. *Let* $\overline{\mathcal{P}}_1$, $\overline{\mathcal{P}}_2$ *be co-hereditary properties of graphs. Then* $\overline{c}(\overline{\mathcal{P}}_1 \circ \overline{\mathcal{P}}_2) = \overline{c}(\overline{\mathcal{P}}_1) + \overline{c}(\overline{\mathcal{P}}_2) + 2$.

Proof. Since $K_{\overline{c}(\overline{\mathcal{P}}_i)+2}$ is the smallest complete graph in $\overline{\mathcal{P}}_i$ for $i = 1, 2$, one can see immediately that $K_{\overline{c}(\overline{\mathcal{P}}_1)+\overline{c}(\overline{\mathcal{P}}_2)+4}$ is the smallest complete graph in $\overline{\mathcal{P}}_1 \circ \overline{\mathcal{P}}_2$. Therefore $\overline{c}(\overline{\mathcal{P}}_1 \circ \overline{\mathcal{P}}_2) = (\overline{c}(\overline{\mathcal{P}}_1) + \overline{c}(\overline{\mathcal{P}}_2) + 4) - 2 = \overline{c}(\overline{\mathcal{P}}_1) + \overline{c}(\overline{\mathcal{P}}_2) + 2$. ∎

The definition of completeness certainly makes sense for induced hereditary properties of graphs, but the concept cannot be used to prove that the number of factors in this lattice is finite. In order to solve this problem we introduce another operation and invariants for graphs and properties.

Definition 1. *We define the operation $*$ as follows: For given graphs G_1, $G_2, \ldots, G_n$, $n \geq 2$,*

$$G_1 * G_2 * \cdots * G_n = \{G : \bigcup_{i=1}^{n} G_i \subseteq G \subseteq \sum_{i=1}^{n} G_i\}.$$

Next we define, for a given graph G in an induced hereditary property of graphs $\mathcal{P}$, the invariant $r_{\mathcal{P}}(G)$ as follows:

$r_{\mathcal{P}}(G) = \max\{n :$ there exist nonempty graphs $G_1, G_2, \ldots, G_n$ in $\mathcal{P}$ and a graph H such that $G \leq H \in G_1 * G_2 * \ldots * G_n \subseteq \mathcal{P}\}$. If $G \notin \mathcal{P}$ we set $r_{\mathcal{P}}(G)$ to be zero.

Now suppose we are given an induced hereditary property $\mathcal{P} \neq \mathcal{I}$. Obviously there exists a finite simple graph $F \notin \mathcal{P}$. For the property $\mathcal{P}$ we can therefore define $f(\mathcal{P})$ to be the least number of vertices of a forbidden subgraph of $\mathcal{P}$, i.e., $f(\mathcal{P}) = \min\{|V(F)| : F \notin \mathcal{P}\}$.

Lemma 23. *Let $\mathcal{P} \neq \mathcal{I}$ be an induced hereditary property of graphs and let $G \in \mathcal{P}$. Then $r_{\mathcal{P}}(G) < f(\mathcal{P})$.*

Proof. Suppose $F \notin \mathcal{P}$ is a graph realizing $f(\mathcal{P})$ and suppose that $n = r_{\mathcal{P}}(G) \geq |V(F)| = f(\mathcal{P})$. Then there exist nonempty graphs $G_1, G_2, \ldots, G_n$ in $\mathcal{P}$ and a graph H such that $G \leq H \in G_1 * G_2 * \ldots * G_n \subseteq \mathcal{P}$. But then, K_1 is a graph in $\mathcal{P}$ and there is a graph H' such that $F \leq H' \in K_1 * K_1 * \ldots * K_1 \subseteq \mathcal{P}$, i.e., $F \in \mathcal{P}$, a contradiction. ∎

This lemma allows us to define the number $r(\mathcal{P})$ for an induced hereditary property $\mathcal{P} \neq \mathcal{I}$ by $r(\mathcal{P}) = \min\{r_{\mathcal{P}}(G) : G \in \mathcal{P}\}$.

We are now ready to prove the main theorem of this section. This is accomplished by choosing a suitable invariant for each of the six lattices listed in Theorem 21 and showing that the number of factors into which reducible graph properties belonging to the chosen lattice can be factorized is bounded by the value of this invariant.

Sketch of the proof of Theorem 21.

1. Consider the lattice $(\mathbb{L}_{\sigma 2}, \subseteq)$. To prove the theorem for this lattice we use the concept of completeness. In [8] it is shown that the completeness of a reducible hereditary property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ satisfies $c(\mathcal{R}) = c(\mathcal{P}_1) + c(\mathcal{P}_2) + 1$. From this fact the proof of the statement in the theorem for this lattice follows immediately.

2. To prove the theorem for the lattice $(\mathbb{L}^a_{\sigma_2}, \subseteq)$ we can use the same argument as in the previous case.

3. The proof for the lattice $(\mathbb{L}_{\sigma_3}, \subseteq)$ uses the invariant $r(\mathcal{R})$. If $\mathcal{R}$ is an induced hereditary property satisfying $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ and $G \in \mathcal{R}$, then it is easy to see that $n \leq r_{\mathcal{R}}(G)$ so that $n \leq r(\mathcal{R})$.

4. In the lattice $(\mathbb{L}^a_{\sigma_3}, \subseteq)$, the same argument as in the previous case can be used.

5. In order to prove the statement of the theorem for the lattice $(\mathbb{L}_{\sigma_4}, \subseteq)$ we can use the concept of $\overline{c}(\overline{\mathcal{P}})$. By Lemma 22 it follows that the number of irreducible factors into which a co-hereditary property can be factorized is finite.

6. Finally we consider the lattice $(\mathbb{L}_{\sigma_5}, \subseteq)\}$. Consider an arbitrary property $\overline{\mathcal{R}} \in \mathbb{L}_{\sigma_5}$ different from $\mathcal{I}$ and suppose that $\overline{\mathcal{R}} = \overline{\mathcal{P}}_1 \circ \overline{\mathcal{P}}_2 \circ \cdots \circ \overline{\mathcal{P}}_n$ with each $\overline{\mathcal{P}}_i \in \mathbb{L}_{\sigma_5}$. Then there exists a graph $G \in \overline{\mathcal{R}}$ with a finite number p of vertices. Then we have for each i that $\overline{\mathcal{P}_i} \neq \mathcal{I}$, since otherwise $\overline{\mathcal{R}} = \mathcal{I}$. Therefore it follows for every factor $\overline{\mathcal{P}_i}$ that $K_0 \notin \overline{\mathcal{P}_i}$ so that every graph in $\overline{\mathcal{R}}$ has at least n vertices. But then $n \leq p$, i.e., the number of factors into which $\mathcal{R}$ can be factorized is finite. ■

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