*Discussiones Mathematicae*
*Graph Theory 19 (1999) 159–166*

# REMARKS ON THE EXISTENCE OF UNIQUELY PARTITIONABLE PLANAR GRAPHS

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**Abstract**

We consider the problem of the existence of uniquely partitionable planar graphs. We survey some recent results and we prove the nonexistence of uniquely $(\mathcal{D}_1,\mathcal{D}_1)$-partitionable planar graphs with respect to the property $\mathcal{D}_1$ "to be a forest".

**Keywords:** property of graphs, additive, hereditary, vertex partition, uniquely partitionable graphs.

**1991 Mathematics Subject Classification:** 05C15, 05C35, O5C75.

## 1. Introduction and Notation

Let us denote by $\mathcal{I}$ the class of all simple finite graphs. A *graph property* is any isomorphism closed nonempty proper subclass $\mathcal{P}$ of $\mathcal{I}$. A graph property is said to be *hereditary* if whenever $G \in \mathcal{P}$ and $H \subseteq G$ ($H$ is a subgraph of $G$), then also $H \in \mathcal{P}$ and $\mathcal{P}$ is *additive* if $\mathcal{P}$ is closed under disjoint union of graphs.

Every additive and hereditary property $\mathcal{P}$ is uniquely determined by the set $\boldsymbol{F}(\mathcal{P})$ of its minimal forbidden subgraphs. A property $\mathcal{P}$ is said to be *degenerate* if there exists a bipartite graph in $\boldsymbol{F}(\mathcal{P})$ , *very degenerate* if there is a forbidden tree for $\mathcal{P}$ and *defective* if the forbidden tree is a star $K_{1,n}$.

The lattice $(\mathbb{L}^a, \subseteq)$ of all additive and hereditary graph properties partially ordered by set-inclusion is investigated in the survey [5].

For an arbitrary hereditary property there exists a number $c(\mathcal{P})$, called *the completeness of* $\mathcal{P}$ defined in the following way: $c(\mathcal{P}) = \max\{k : K_{k+1} \in \mathcal{P}\}$. It is easy to see that the unique hereditary additive property whose completeness is zero, is the property of all edgeless graphs, denoted by $\mathcal{O}$, while the class $\mathcal{D}_1$ of all acyclic graphs has completeness 1.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of graphs. A vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of a graph $G$ is a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that each partition class $V_i$ induces a subgraph $G[V_i]$ of property $\mathcal{P}_i, i = 1, 2, ..., n$. A graph $G$ is said to be uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable if $G$ has exactly one vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. Let us denote by $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ the class of all graphs which have a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition and by $\boldsymbol{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n)$ the set of uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs.

Some basic results on uniquely partitionable graphs may be found in [11, 10, 14, 4, 5, 7, 12].

It is easy to prove that if $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n \in \mathbb{L}^a$, then $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n \in \mathbb{L}^a$, too.

A graph property $\mathcal{R} \in \mathbb{L}^a$ is said to be *reducible* in $\mathbb{L}^a$ if there exist $\mathcal{P}, \mathcal{Q} \in \mathbb{L}^a$ such that $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$ and *irreducible* otherwise.

We say that a property $\mathcal{P} \in \mathbb{L}^a$ is generated by a set $\mathcal{G}$ of graphs if $G \in \mathcal{P}$ if and only if there is a graph $F \in \mathcal{G}$ such that $G \subseteq F$.

In [12, 13] the following general results have been presented:

**Theorem 1.** *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be irreducible properties of graphs. Then the property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ is generated by the class $\boldsymbol{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n)$ of uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs. Moreover the factorization of $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ into irreducible factors is unique apart from the order of the factors.*

A necessary and sufficient condition of the existence of uniquely $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$-partitionable graphs in general is presented in [6].

In this paper we consider the problem of the existence of uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable planar graphs for $\mathcal{P}, \mathcal{Q} \neq \mathcal{O}$. We shall show that there are no uniquely $(\mathcal{D}_1, \mathcal{D}_1)$-partitionable planar graphs proving that every $(\mathcal{D}_1, \mathcal{D}_1)$-partitionable planar graph $G$ has at least three different $(\mathcal{D}_1, \mathcal{D}_1)$-partitions. We shall show that the above mentioned result is sharp i.e., for any $\mathcal{P} \circ \mathcal{Q} \subset \mathcal{D}_1 \circ \mathcal{D}_1$ the class $\boldsymbol{U}(\mathcal{P} \circ \mathcal{Q}) \cap Planar \neq \emptyset$.

Concerning $(\mathcal{D}_1, \mathcal{D}_1)$-partitionable planar graphs from the algorithmic point of view, it is an interesting open problem whether a second $(\mathcal{D}_1, \mathcal{D}_1)$-partition can be found in polynomial time if one such partition is given. (As we prove here, at least two others exist.) The background of this problem and approximation results can be found in [1].

## 2. Existence of Uniquely Partitionable Planar Graphs

Since the maximal uniquely $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$-partitionable graphs generating the property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n$ are joins of graphs of order at least 3, they contain subgraphs isomorphic to $K_{3,3}$ and therefore they are not planar. The problem of the existence of uniquely $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$-partitionable planar graphs has been investigated in [9] and [8]. In this section we summarize the known results:

**Proposition 1** [9]**.** *Let $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ be any additive and hereditary properties of graphs. Then*

1. *if $\boldsymbol{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n) \cap Planar \neq \emptyset$, then $n \leq 4$*
2. *$\boldsymbol{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \mathcal{P}_3 \circ \mathcal{P}_4) \cap Planar \neq \emptyset$ if and only if $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}_3 = \mathcal{P}_4 = \mathcal{O}$.*

**Theorem 2** [9]**.** *If $\boldsymbol{F}(P)$ contains a star $K_{1,k+1}$, $k \geq 1$ (i.e., $\mathcal{P}$ is defective), then there exists a uniquely $(\mathcal{O}, \mathcal{P})$-partitionable outerplanar graph $G$.*

**Theorem 3** [9]**.** *If $c(\mathcal{P}) = 1$ and $\boldsymbol{F}(\mathcal{P})$ contains a tree $T$ ($\mathcal{P}$ is very degenerate), then there exists a uniquely $(\mathcal{O}, \mathcal{P})$-partitionable outerplanar graph $G$.*

**Theorem 4** [9]**.** *Let $\mathcal{P}$ be an additive hereditary property of completeness 1. Then there exists a uniquely $(\mathcal{O}, \mathcal{P})$-partitionable planar graph $G$ if and only if some odd cycle $C_{2q+1}$ has property $\mathcal{P}$ or there is a bipartite planar graph $H$ in $\boldsymbol{F}(P)$.*

Theorem 2 was generalized by Bucko and Ivančo:

**Theorem 5** [8]. *Let $\mathcal{P}$ be an additive hereditary property. If there is a tree $T \in \boldsymbol{F}(\mathcal{P})$ (i.e., $\mathcal{P}$ is very degenerate), then there exists a uniquely $(\mathcal{O}, \mathcal{P})$-partitionable planar graph.*

They also present a construction of uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable planar graphs for properties of completeness 1 provided that one of them is very degenerate.

**Theorem 6** [8]. *Let $\mathcal{P}$, $\mathcal{Q}$ be the additive hereditary properties of graphs with completeness 1. If there is a tree $T \in \boldsymbol{F}(\mathcal{P})$ , then there exists a uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable planar graph.*

The next result presents the existence of uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable graphs with respect to the defective properties $\mathcal{P}$ and $\mathcal{Q}$ of arbitrary large completeness.

**Theorem 7.** *Let $\mathcal{P}, \mathcal{Q} \in \mathbb{L}^a$ be defective properties of graphs. Then there exists a uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable planar graph.*

***Proof.*** Let $K_{1,r} \in \boldsymbol{F}(\mathcal{P})$ and $K_{1,s} \in \boldsymbol{F}(\mathcal{Q})$ and without loss of generality let us assume $r \leq s$. Let us denote by $H$ the graph consisting of $s-1$ disjoint copies of $K_{1,2s-1}$. Then (see Figure 1 for $r = s = 4$) the planar graph $K_1 + H$ is uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable. Indeed, the vertex $v$ of $K_1$ and the $s-1$ centers of the stars $K_{1,2s-1}$ must belong into the same partition class, forcing the remaining vertices of $H$ to be in the other. ■

## 3. Non-Existence of Uniquely $(\mathcal{P}, \mathcal{Q})$-Partitionable Planar Graphs

The Four Colour Theorem immediately implies that every planar graph has at least three $(\mathcal{O}^2, \mathcal{O}^2)$-partitions. We are going to prove the following

**Theorem 8.** *If a planar graph of order $n \geq 3$ is $(\mathcal{D}_1, \mathcal{D}_1)$-partitionable, then it has at least three $(\mathcal{D}_1, \mathcal{D}_1)$-partitions.*

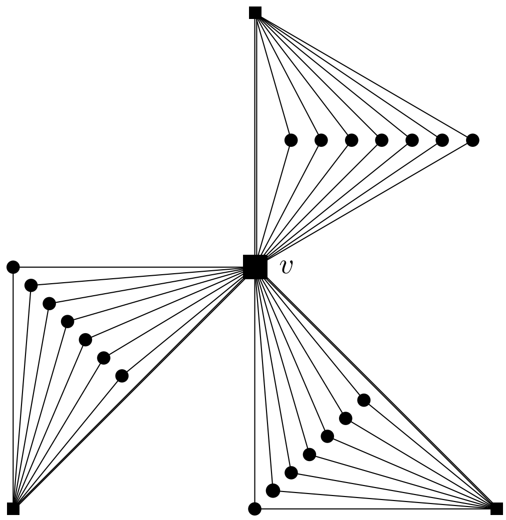


Figure 1

***Proof.*** The assertion is trivial for $n = 3$. Below we assume that $G = (V, E)$ is a $(\mathcal{D}_1, \mathcal{D}_1)$-partitionable plane graph of order $n \geq 4$ with a fixed planar embedding, and denote by $V_1$ and $V_2$ the partition classes in one of its $(\mathcal{D}_1, \mathcal{D}_1)$-partitions. For $i = 1, 2$ let $G_i = G[V_i] = (V_i, E_i)$ be the subgraphs of $G$ induced by $V_i$.

**Lemma 1.** *If Theorem* 8 *holds true for all triangulations, then it is valid for all planar graphs.*

***Proof.*** We apply induction on $3|V| - |E| - 6$. One can clearly assume that $G$ is connected (and even 2-connected). If $G$ is not a triangulation, then it has a face $F$ with four consecutive vertices $v_1, v_2, v_3, v_4$ such that $v_2$ and $v_3$ belong to distinct partition classes. Say, $v_2 \in V_2$ and $v_3 \in V_1$. If the new edge $e_1 := v_1v_3$ cannot be inserted without destroying the $(\mathcal{D}_1, \mathcal{D}_1)$-partition $V_1 \cup V_2$, then $v_1 \in V_1$, and $v_1$ and $v_3$ belong to the same connected component of $G_1$. Similarly, if the edge $e_2 := v_2v_4$ cannot be inserted, then $v_2$ and $v_4$ belong to the same component of $G_2$. But then the $v_1$–$v_3$ path of $G_1$ and the $v_2$–$v_4$ path of $G_2$ share a vertex by planarity, a contradiction.

Consequently, there is an edge $e$ (either $e_1$ or $e_2$) such that $G + e$ is $(\mathcal{D}_1, \mathcal{D}_1)$-partitionable. By the induction hypothesis, $G + e$ has at least three $(\mathcal{D}_1, \mathcal{D}_1)$-partitions. Observe that removing $e$, each of them remains a $(\mathcal{D}_1, \mathcal{D}_1)$-partition of $G$.

From now on we assume that $G$ is a triangulation, and together with its planar embedding we also draw the planar dual $G^*$ in the natural way (one dual vertex inside each face, one dual edge crossing each edge of $G$, no two dual edges cross, each dual edge lies in the interior of the quadrangle formed by the union of the corresponding two triangles). Note that $|V| = n$, $|E| = |E^*| = 3n - 6$, $|V^*| = 2n - 4$, and $G^*$ is 3-regular. Denote $E_3 := E \setminus (E_1 \cup E_2)$. ■

**Lemma 2.** *We have $|E_3| = 2n - 4$, and $G_i$ is connected for $i = 1, 2$.*

***Proof.*** The inequality $|E_3| \leq 2n-4$ follows from the fact that the edges of $E_3$ form a planar bipartite graph on (at most) $n$ vertices. This also implies $|E_1| + |E_2| \geq n - 2$. On the other hand, $G_1$ and $G_2$ are acyclic, therefore $|E_1| + |E_2| = n - c$, where $c \geq 2$ is the number of connected components in $G_1 \cup G_2$. As $G_1$ and $G_2$ are vertex disjoint, this implies $c = 2$ and connectivity for both $G_i$. ■

Denote by $E_3^*$ the set of dual edges corresponding to $E_3$.

**Lemma 3.** *The edges of $E_3^*$ form a Hamiltonian cycle in $G^*$.*

***Proof.*** As the vertex set of each triangle of $G$ is 2-colored, each triangle contains precisely two edges of $E_3$. Therefore, $E_3^*$ forms a 2-regular subgraph in $G^*$. We have to show that it consists of precisely one cycle. Observe that the cycles of $E_3^*$ are drawn as mutually disjoint Jordan curves, therefore $k$ such curves divide the plane into $k+1$ regions. By the conventions as $G^*$ is drawn, each region contains at least one vertex of $G$, therefore $k = 1$ follows from the fact that each $G_i$ $(i = 1, 2)$ is connected. ■

Next, we apply a well-known theorem of Smith 1946 (see in: Berge [2] p. 185, or [3] p. 190) to find further Hamiltonian cycles in $G^*$.

**Lemma 4.** *The graph $G^*$ contains at least three distinct Hamiltonian cycles.*

***Proof.*** By Smith's theorem, every edge of a cubic graph is contained in an even number of Hamiltonian cycles. We have already found one, namely $E_3^*$, in the graph $G^*$. Choosing one of its edges arbitrarily, we find a second Hamiltonian cycle, say ${E'_3}^*$. Finally, taking any edge in the symmetric difference of $E_3^*$ and ${E'_3}^*$, we get a third cycle. ■

The proof of Theorem 8 will be completed by the following observation.

**Lemma 5.** *If $E'^*$ is a Hamiltonian cycle of $G^*$ for some $E' \subseteq E$, then the subgraph formed by the edges of $E \setminus E'$ is acyclic and has exactly two connected components; i.e., it defines a $(\mathcal{D}_1, \mathcal{D}_1)$-partition of $G$.*

***Proof.*** We have seen that the curve $E'^*$ splits the plane into two regions, each containing at least one vertex of $G$. Moreover, the two subgraphs induced by the vertices belonging to the corresponding regions have precisely $n-2$ edges altogether. Thus, in order to prove that they are acyclic, it suffices to show that they are connected.

Let $G'$ be any one of the two induced subgraphs whose vertex set $V'$ contains two or more vertices, and let $v_1, v_2 \in V'$ be arbitrary. Choose any two triangles $T_i$ incident to $v_i$ $(i = 1, 2)$. By assumption, $E'$ contains precisely two edges from each $T_i$. Let $e_i \in E' \cap E(T_i)$ be an edge containing $v_i$. Since $E'^*$ is a Hamiltonian cycle in $G^*$, the dual edges $e_1^*$ and $e_2^*$ are joined by a path along $E'^*$. Each dual edge $e^*$ of this path corresponds to an edge $e \in E$ having precisely one vertex $v_e$ in $G'$. For any two consecutive edges $e^*, e'^*$, the edges $e$ and $e'$ are contained in a triangle of $G$, therefore in such a situation $v_e$ and $v_{e'}$ are either identical or adjacent; and in the latter case, the edge joining them belongs to $G'$. Consequently, the path from $e_1^*$ to $e_2^*$ defines a walk (possibly with repeated vertices) from $v_1$ to $v_2$. Thus, $G'$ is connected. ■

As distinct edge-cuts $E_3$ cannot belong to the same $(\mathcal{D}_1, \mathcal{D}_1)$-partition, the three Hamiltonian cycles found above yield three distinct $(\mathcal{D}_1, \mathcal{D}_1)$-partitions of $G$. The proof of Theorem 8 is complete. ■

The sharpness of Theorem 8 follows by Theorem 6 and next Theorem presented in [5].

**Theorem 9** [5]**.** *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be additive degenerate hereditary properties and suppose that $\mathcal{P}_1 \circ \mathcal{P}_2 \subseteq \mathcal{R}_1 \circ \mathcal{R}_2$ for $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{L}^a$. Then $\mathcal{P}_1 \subseteq \mathcal{R}_1$ and $\mathcal{P}_2 \subseteq \mathcal{R}_2$ or $\mathcal{P}_1 \subseteq \mathcal{R}_2$ and $\mathcal{P}_2 \subseteq \mathcal{R}_1$.*

## References


[1] C. Bazgan, M. Santha and Zs. Tuza, *On the approximation of finding a(nother) Hamiltonian cycle in cubic Hamiltonian graphs*, in: Proc. STACS'98, Lecture Notes in Computer Science **1373** (Springer-Verlag, 1998) 276–286. Extended version in the J. Algorithms **31** (1999) 249–268.

[2] C. Berge, Theorie des Graphes, Paris, 1958.

[3] C. Berge, Graphs and Hypergraphs, North-Holland, 1973.

[4] B. Bollobás and A.G. Thomason, *Uniquely partitionable graphs*, J. London Math. Soc. **16** (1977) 403–410.

[5] M. Borowiecki, I. Broere, M. Frick, P. Mihók and G. Semanišin, *Survey of hereditary properties of graphs*, Discuss. Math. Graph Theory **17** (1997) 5–50.

[6] I. Broere and J. Bucko, *Divisibility in additive hereditary graph properties and uniquely partitionable graphs*, Tatra Mountains **18** (1999) 79–87.

[7] J. Bucko, M. Frick, P. Mihók and R. Vasky, *Uniquely partitionable graphs*, Discuss. Math. Graph Theory **17** (1997) 103–114.

[8] J. Bucko, J. Ivančo, *Uniquely partitionable planar graphs with respect to properties having a forbidden tree*, Discuss. Math. Graph Theory **19** (1999) 71–78.

[9] J. Bucko, P. Mihók, M. Voigt *Uniquely partitionable planar graphs*, Discrete Math. **191** (1998) 149–158.

[10] G. Chartrand, J.P. Geller, *Uniquely colourable planar graphs*, J. Combin. Theory **6** (1969) 271–278.

[11] F. Harary, S.T. Hedetniemi and R.W. Robinson, *Uniquely colourable graphs*, J. Combin. Theory **6** (1969) 264–270; MR**39**#99.

[12] P. Mihók, *Reducible properties and uniquely partitionable graphs*, in: Contemporary Trends in Discrete Mathematics, From DIMACS and Dimatia to the Future, Proceedings of the DIMATIA-DIMACS Conference, Stiřin May 1997, Ed. R.L. Graham, J. Kratochvil, J. Nešetřil, F.S. Roberts, DIMACS Series in Discrete Mathematics, Volume 49, AMS, 213–218.

[13] P. Mihók, G. Semanišin and R. Vasky, *Hereditary additive properties are uniquely factorizable into irreducible factors* (to appear in J. Graph Theory).

[14] J. Mitchem, *Uniquely k-arborable graphs*, Israel J. Math. **10** (1971) 17–25; MR**46**#81.

Received 2 February 1999
Revised 21 September 1999