# A NOTE ON THE RAMSEY NUMBER AND THE PLANAR RAMSEY NUMBER FOR $C_{4}$ AND COMPLETE GRAPHS 

Halina Bielak<br>Institute of Mathematics UMCS<br>M. Curie-Sktodowska University<br>Lublin, Poland<br>e-mail: hbiel@golem.umcs.lublin.pl


#### Abstract

We give a lower bound for the Ramsey number and the planar Ramsey number for $C_{4}$ and complete graphs. We prove that the Ramsey number for $C_{4}$ and $K_{7}$ is 21 or 22. Moreover we prove that the planar Ramsey number for $C_{4}$ and $K_{6}$ is equal to 17 .


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## 1 Introduction

Let $F, G, H$ be simple graphs with at least two vertices. The Ramsey number $R(G, H)$ is the smallest integer $n$ such that in arbitrary two-colouring (say red and blue) of $K_{n}$ a red copy of $G$ or a blue copy of $H$ is contained (as subgraphs).

Let the planar Ramsey number $P R(G, H)$ be the smallest integer $n$ such that any planar graph on $n$ vertices contains a copy of $G$ or its complement contains a copy of $H$.

So we have an immediate inequality between planar and ordinary Ramsey number, i.e., $P R(G, H) \leq R(G, H)$.

Walker in [9] and Steinberg and Tovey in [8] studied the planar Ramsey number but only in the case when both graphs are complete.

In this paper we will only consider the case when $G$ is a cycle $C_{4}$ of order 4 and $H$ is a complete graph $K_{t}$ of order $t$. In that case one can say that the Ramsey number is the smallest integer $n$ such that any graph on $n$ vertices contains a copy of $C_{4}$ or an independent set of cardinality $t$. The
problem for the case when $G$, i.e., the first graph of the pair, is a cycle has been studied by J.A. Bondy, P. Erdös in [3] and by P. Erdös, R.J. Faudree, C.C. Rousseau, R.H. Schelp in [6]. We give a lower bound for the Ramsey number and the planar Ramsey number for $C_{4}$ and complete graphs. We prove that the Ramsey number for $C_{4}$ and $K_{7}$ is 21 or 22.

Moreover in Theorem 6 we prove that $\operatorname{PR}\left(C_{4}, K_{6}\right)=17$.
A graph $F$ is said to be a $\left(G, K_{t}\right)$-Ramsey-free graph if it does not contain any copy of $G$ and any independent set of cardinality $t$. For graphs $G, H$ the symbol $G \cup H$ denotes a disjoint union of graphs, $t G$ a disjoint union of $t$ copies of the graph $G, \bar{G}$ a complement of $G, G-S$ a subgraph of $G$ induced by a subset $V(G)-S$ of the vertices of $G$ where $S \subset V(G)$, and $G \supset H$ express the fact that a graph $H$ is a subgraph of $G$. Then $\operatorname{deg}_{G}(x)$ denotes the degree of the vertex $x$ in the graph $G$, and $\delta(G)$ is the minimum vertex degree over all vertices of $G$. Moreover $N(x)$ is the set of vertices adjacent to $x$, and $N[x]$ is the closed neighbourhood, i.e., $N[x]=N(x) \cup\{x\}$.

The following theorems summarises the results for ordinary and planar Ramsey numbers known so far referring to the cases when the first graph is a cycle of order 4 and the second one is a complete graph.

Theorem 1 [4], [5], [7]. (i) $R\left(C_{4}, K_{3}\right)=7$;
(ii) $R\left(C_{4}, K_{4}\right)=10$;
(iii) $R\left(C_{4}, K_{5}\right)=14$;
(iv) $R\left(C_{4}, K_{6}\right)=18$.

Theorem 2 [1]. (i) $P R\left(C_{4}, K_{3}\right)=7$;
(ii) $P R\left(C_{4}, K_{4}\right)=10$;
(iii) $P R\left(C_{4}, K_{5}\right)=13$.

## 2 Main Results

We use the following lemma to prove some further results for the Ramsey and the planar Ramsey number of pair of graphs.

Lemma 3 [2]. Let $G$ be a graph of order 17 with independence number less than 6 and without $C_{4}$. Then $G$ is isomorphic to one of the graphs presented in Figure 1.

Therefore we have the following simple general observation.

$B_{1}$

$B_{3}$

Figure 1. Graphs of order 17 without $C_{4}$ and with $\alpha(G)<6$.

Proposition 4. For each integer $t \geq 6, R\left(C_{4}, K_{t+1}\right) \geq 3 t+2\left[\frac{t}{5}\right]+1$.
Proof. Let $H$ be a graph of order 17 presented in Figure 1. Note that $H$ does not contain any subgraph $C_{4}$ and $\alpha(H)=5$. Therefore [ $\left[\frac{t}{5}\right] H \cup$ $\left(t-\left[\frac{t}{5}\right] 5\right) K_{3}, t \geq 5$, shows that $R\left(C_{4}, K_{t+1}\right) \geq 3 t+2\left[\frac{t}{5}\right]+1$.

Theorem 5. $21 \leq R\left(C_{4}, K_{7}\right) \leq 22$.
Proof. Immediately by Proposition 4 we get $21 \leq R\left(C_{4}, K_{7}\right)$. Suppose for the contrary that $R\left(C_{4}, K_{7}\right)>22$. Let $G$ be a $\left(C_{4}, K_{7}\right)$-Ramsey-free graph of order 22 . Note that $\delta(G)<5$, else a $C_{4}$ should be a subgraph of $G$.

Let $m$ be an arbitrary vertex of $G$ of the minimum degree $\delta(G)$.
Suppose that $\delta(G) \leq 3$. Then deleting a 3 -degree vertex $m$ and all its neighbours we get a graph $F$ of the order at least 18. By Theorem $1(i v)$ the graph $F$ contains an independent set $S$ of cardinality 6 . Thus $S \cup\{m\}$ is an independent set of cardinality 7 , a contradiction.

Therefore $\delta(G)=4$. Let $m_{i}, i=1,2,3,4$ be the neighbours of $m$ in $G$. Let us consider the graph $F$ obtained from $G$ by deleting the vertex $m$ and all its neighbours. Since $G$ does not contain any $C_{4}$ then by degree condition each $m_{i}, i=1,2,3,4$ has at least two neighbours in $F$. Evidently the order of $F$ equals 17 and $F$ must be isomorphic to one of the ( $C_{4}, K_{6}$ )-Ramseyfree graphs presented in Figure 1 (else we get a contradiction as before). Suppose that $F$ is isomorphic to $H_{1}$ or $H_{2}$. Since the vertex $w$ has degree 3 in $F$ then it must be adjacent to one of the neighbours of $m$, say $m_{1}$. Let us consider the graph $Y=G-N[w]$. Note that the vertex $m$ has degree 3 in $Y$. Hence $Y$ must be one of the ( $C_{4}, K_{6}$ )-Ramsey-free graphs $H_{1}$ or $H_{2}$ presented in Figure 1. Evidently $m$ is not adjacent to any vertex of the set $\{d, v, b, h\}$. Therefore each of the four vertices must be adjacent to a vertex of the set $\left\{m_{2}, m_{3}, m_{4}\right\}$. It is impossible without creating $C_{4}$ because each two vertices of the set $\{d, v, b, h\}$ are at distance 2. A contradiction. Therefore we can assume that $F$ is not isomorphic to $H_{i}, i=1,2$.

Suppose that $F$ is isomorphic to $B_{1}$. Let the vertex $x$ be adjacent to $m_{1}$. Then $1 m_{1} \in E(G)$, else $\operatorname{deg}\left(m_{1}\right)<4$. Moreover without loss of generality $m_{1} m_{2} \in E(G)$. Note that $\operatorname{deg}\left(m_{1}\right)=4$. So we consider the graph $Y=$ $G-N\left[m_{1}\right]$. Since $Y$ cannot be isomorphic to $H_{i}, i=1,2$ then each of the vertices of the set $\{2,3,4,5\}$ must be adjacent to $m_{3}$ or $m_{4}$ and we get $C_{4}$, a contradiction. Therefore $x m_{i} \notin E(G)$, for $i=1,2,3,4$. By symmetry, $y m_{i} \notin E(G)$, for $i=1,2,3,4$.

Suppose that $f$ is adjacent to $m_{1}$. Since $C_{4}$ cannot be a subgraph then $b, v, e$ or 4 is not adjacent to $m_{i}, i=2,3,4$. Therefore $\operatorname{deg}\left(m_{1}\right)>4$, else
the graph $G-N\left[m_{1}\right]$ has a 3-degree vertex, so it should be isomorphic to $H_{i}, i=1,2$ and we get a case above. Then $m_{1}$ should be adjacent to 3 and $h$, and without loss of generality $m_{1} m_{2} \in E(G)$. Note that $m_{2}$ can be adjacent to one of the vertices $d, 1$ or $u$. So $\operatorname{deg}\left(m_{2}\right)<4$ or a $C_{4}$ exists, a contradiction.

Hence $f m_{i} \notin E(G)$, for $i=1,2,3,4$. By symmetry $b m_{i} \notin E(G)$, for $i=1,2,3,4$.

If the vertex 2 is adjacent to $m_{1}$ then $\operatorname{deg}\left(m_{1}\right)>4$, else $G-N\left[m_{1}\right]$ has a 3 -degree vertex, and we get a case above. Then $m_{1}$ must be adjacent to $e$ and to one of $g, u$. Moreover without loss of generality $m_{1} m_{2} \in E(G)$. Note that $\operatorname{deg}\left(m_{2}\right)<4$ or a $C_{4}$ exists, a contradiction.

Hence $2 m_{i} \notin E(G)$, for $i=1,2,3,4$. By symmetry $e m_{i} \notin E(G)$, for $i=1,2,3,4$.

Similar arguments give that 5 and $h$ cannot be adjacent to $m_{i}, i=$ $1,2,3,4$.

Now without loss of generality we can assume that $m_{1}, m_{2}$ and $u$ create an independent set. Therefore $\left\{m_{1}, m_{2}, 2,5, y, f, u\right\}$ is an independent set.

Suppose that $F$ is isomorphic to $B_{2}$. Let $g$ be adjacent to $m_{1}$. Then $m_{1}$ must be adjacent to 3 and $y$, and without loss of generality $m_{1} m_{2} \in E(G)$, else the graph $G-N\left[m_{1}\right]$ has a 3 -degree vertex, so it should be isomorphic to $H_{i}, i=1,2$ and we get a case above. So $m_{2}$ must be adjacent to 4 and $e$, and it has degree four. Therefore the vertices $5, b, u, f$ must be adjacent to $m_{3}$ or $m_{4}$, else we get a 3 -degree vertex in $G-N\left[m_{2}\right]$. Without loss of generality we can assume that the vertex $m_{3}$ is adjacent to $b, f$, and the vertex $m_{4}$ is adjacent to $5, u$. Note that $m_{4}$ has only these two neighbours in $B_{2}$. Hence $m_{4}$ must be adjacent to $m_{3}$ and $\operatorname{deg}\left(m_{4}\right)=4$. Since $h$ cannot be adjacent to $m_{i}, i=1,2,3,4$ the graph $G-N\left[m_{4}\right]$ has a 3-degree vertex and we get a case above.

Hence $g m_{i} \notin E(G)$, for $i=1,2,3,4$. By symmetry $y m_{i} \notin E(G)$, for $i=1,2,3,4$.

Let 2 be adjacent to $m_{1}$. Then $m_{1}$ should be adjacent to one of the vertices $u, a, b$. So the graph $G-N\left[m_{1}\right]$ contains a 3 -degree vertex $g$ or $y$, and we get a case above. Hence $2 m_{i} \notin E(G)$, for $i=1,2,3,4$. By symmetry $c m_{i} \notin E(G)$, for $i=1,2,3,4$.

Now without loss of generality we can assume that $m_{1}, m_{2}$ and 4 create an independent set. Therefore $\left\{m_{1}, m_{2}, 4, c, 2, g, y\right\}$ is an independent set.

Suppose that $F$ is isomorphic to $B_{3}$. Let $d$ be adjacent to $m_{1}$. Then $m_{1}$ must be adjacent to one of the vertices $3, b, g, h$, and without loss of generality
$m_{1} m_{2} \in E(G)$. Since $\operatorname{deg}\left(m_{1}\right)=4$ and $m_{2}$ cannot be adjacent to $3, h, f, u$, then the graph $G-N\left[m_{1}\right]$ has a 3 -degree vertex, and we get a case above.

Hence $d m_{i} \notin E(G)$, for $i=1,2,3,4$. By symmetry $g m_{i} \notin E(G)$, for $i=1,2,3,4$.

Let $a$ be adjacent to $m_{1}$. Then $m_{1}$ must be adjacent to one of the vertices $u, f, 4$. As before $\operatorname{deg}\left(m_{1}\right)=4$. Note that one of the vertices $2, b, h, y$ has 3 -degree in $G-N\left[m_{1}\right]$, and we get a case above.

Hence $a$ and 4 (by symmetry ) cannot be adjacent to $m_{i}, i=1,2,3,4$. Now without loss of generality we can assume that $m_{1}, m_{2}$ and 1 create an independent set. Therefore $\left\{m_{1}, m_{2}, 1,4, a, d, g\right\}$ is an independent set.

All cases lead to a contradiction
For the planar case we get the following theorem.
Theorem 6. $P R\left(C_{4}, K_{6}\right)=17$.
Proof. Since by Lemma 3 each $\left(C_{4}, K_{6}\right)$-Ramsey-free graph of order 17 is not planar and $R\left(C_{4}, K_{6}\right)=18$ we get $P R\left(C_{4}, K_{6}\right) \leq 17$. The graph presented in Figure 2 is $\left(C_{4}, K_{6}\right)$-Ramsey-free planar graph. So $P R$ $\left(C_{4}, K_{6}\right)>16$.


Figure 2. A planar graph of order 16 with independence number less than 6 and without $C_{4}$.

Proposition 7. For each integer $t \geq 5, P R\left(C_{4}, K_{t+1}\right) \geq 3 t+\left[\frac{t}{5}\right]+1$.

Proof. Let $H$ be a graph of order 16 presented in Figure 2. Note that $H$ does not contain any subgraph $C_{4}$ and $\alpha(H)=5$. Therefore $\left[\frac{t}{5}\right] H \cup$ $\left(t-\left[\frac{t}{5}\right] 5\right) K_{3}, t \geq 6$, shows that $P R\left(C_{4}, K_{t+1}\right) \geq 3 t+\left[\frac{t}{5}\right]+1$.

Added in Proof. The result cited in Lemma 3 can be also find in: C.J. Jayawardene, C.C. Rousseau, An upper bound for Ramsey number of a quadrilateral versus a complete graph on seven vertices, Congressus Numerantium 130 (1998) 175-188.

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