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A NOTE ON THE RAMSEY NUMBER AND THE PLANAR RAMSEY NUMBER FOR C_4 AND COMPLETE GRAPHS

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Abstract

We give a lower bound for the Ramsey number and the planar Ramsey number for C_4 and complete graphs. We prove that the Ramsey number for C_4 and K_7 is 21 or 22. Moreover we prove that the planar Ramsey number for C_4 and K_6 is equal to 17.

Keywords: planar graph, Ramsey number.

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1 Introduction

Let F, G, H be simple graphs with at least two vertices. The Ramsey number $R(G, H)$ is the smallest integer n such that in arbitrary two-colouring (say red and blue) of K_n a red copy of G or a blue copy of H is contained (as subgraphs).

Let *the planar Ramsey number* $PR(G, H)$ be the smallest integer n such that any planar graph on n vertices contains a copy of G or its complement contains a copy of H.

So we have an immediate inequality between planar and ordinary Ramsey number, i.e., $PR(G, H) \leq R(G, H)$.

Walker in [9] and Steinberg and Tovey in [8] studied the planar Ramsey number but only in the case when both graphs are complete.

In this paper we will only consider the case when G is a cycle C_4 of order 4 and H is a complete graph K_t of order t. In that case one can say that the Ramsey number is the smallest integer n such that any graph on n vertices contains a copy of C_4 or an independent set of cardinality t. The problem for the case when G, i.e., the first graph of the pair, is a cycle has been studied by J.A. Bondy, P. Erdös in [3] and by P. Erdös, R.J. Faudree, C.C. Rousseau, R.H. Schelp in [6]. We give a lower bound for the Ramsey number and the planar Ramsey number for C_4 and complete graphs. We prove that the Ramsey number for C_4 and K_7 is 21 or 22.

Moreover in Theorem 6 we prove that $PR(C_4, K_6) = 17$.

A graph F is said to be a (G, K_t)-Ramsey-free graph if it does not contain any copy of G and any independent set of cardinality t. For graphs G, H the symbol $G \cup H$ denotes a disjoint union of graphs, tG a disjoint union of t copies of the graph G, $\overline{G}$ a complement of G, $G - S$ a subgraph of G induced by a subset $V(G) - S$ of the vertices of G where $S \subset V(G)$, and $G \supset H$ express the fact that a graph H is a subgraph of G. Then $deg_G(x)$ denotes the degree of the vertex x in the graph G, and $\delta(G)$ is the minimum vertex degree over all vertices of G. Moreover $N(x)$ is the set of vertices adjacent to x, and $N[x]$ is the closed neighbourhood, i.e., $N[x] = N(x) \cup \{x\}$.

The following theorems summarises the results for ordinary and planar Ramsey numbers known so far referring to the cases when the first graph is a cycle of order 4 and the second one is a complete graph.

Theorem 1 [4], [5], [7]**.** (i) $R(C_4, K_3) = 7$;
(ii) $R(C_4, K_4) = 10$;
(iii) $R(C_4, K_5) = 14$;
(iv) $R(C_4, K_6) = 18$.

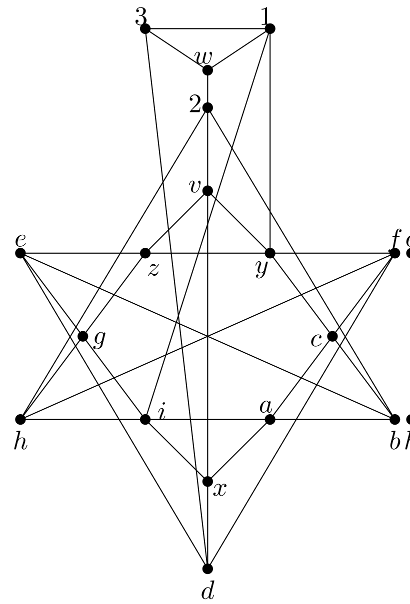
Theorem 2 [1]**.** (i) $PR(C_4, K_3) = 7$;
(ii) $PR(C_4, K_4) = 10$;
(iii) $PR(C_4, K_5) = 13$.

2 Main Results

We use the following lemma to prove some further results for the Ramsey and the planar Ramsey number of pair of graphs.

Lemma 3 [2]**.** *Let G be a graph of order 17 with independence number less than 6 and without C_4. Then G is isomorphic to one of the graphs presented in Figure 1.*

Therefore we have the following simple general observation.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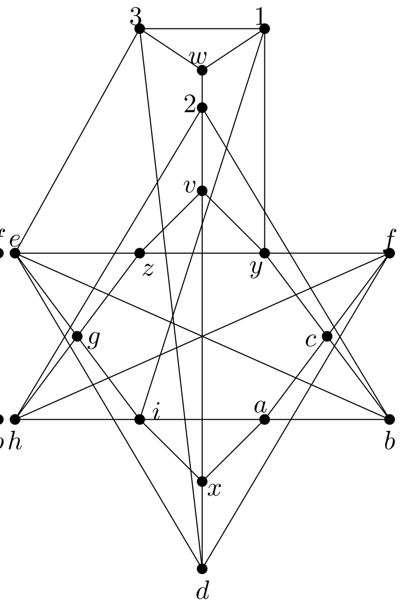
H_1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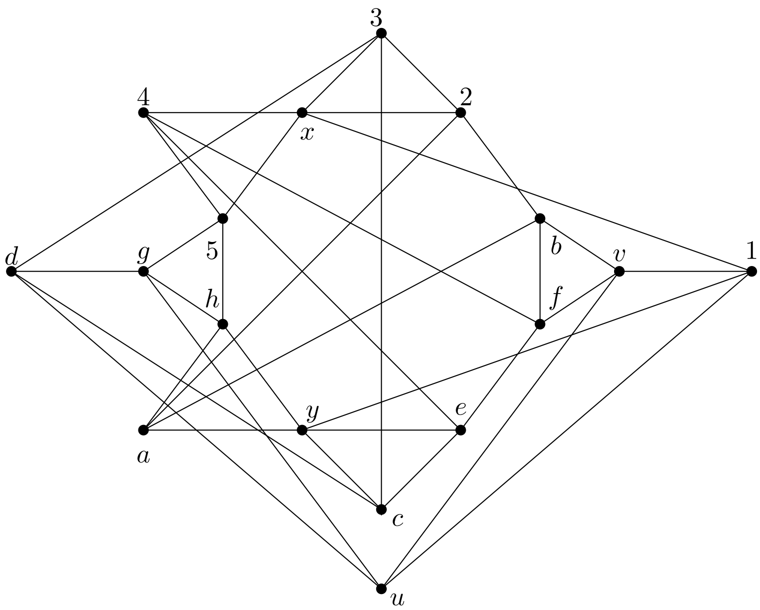
H_2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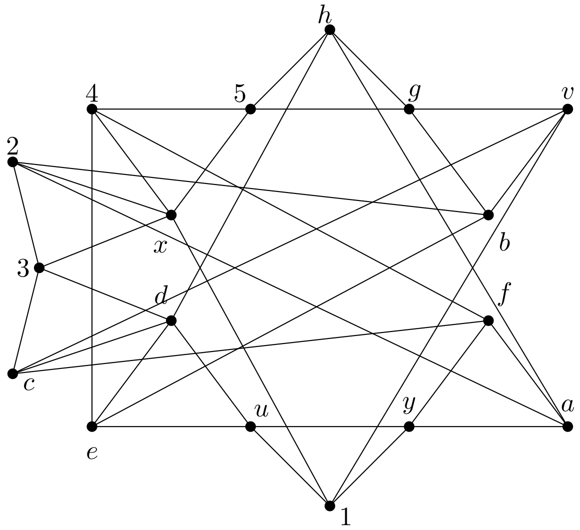
B_1



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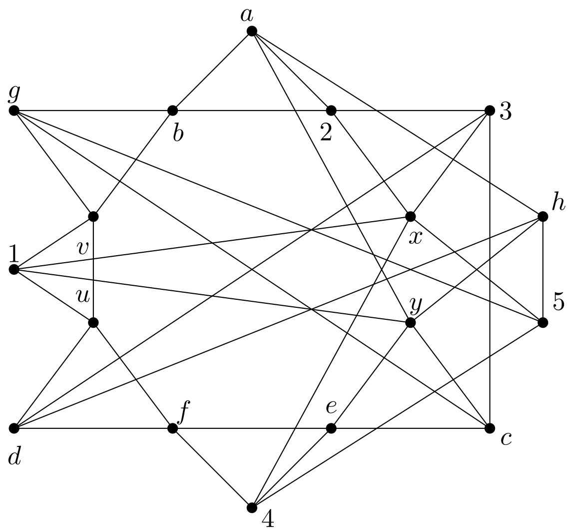

B_3

Figure 1. Graphs of order 17 without C_4 and with $\alpha(G) < 6$.

Proposition 4. *For each integer* $t \geq 6$, $R(C_4, K_{t+1}) \geq 3t + 2[\frac{t}{5}] + 1$.

Proof. Let H be a graph of order 17 presented in Figure 1. Note that H does not contain any subgraph C_4 and $\alpha(H) = 5$. Therefore $[\frac{t}{5}]H \cup (t - [\frac{t}{5}]5)K_3$, $t \geq 5$, shows that $R(C_4, K_{t+1}) \geq 3t + 2[\frac{t}{5}] + 1$. ■

Theorem 5. $21 \leq R(C_4, K_7) \leq 22$.

Proof. Immediately by Proposition 4 we get $21 \leq R(C_4, K_7)$. Suppose for the contrary that $R(C_4, K_7) > 22$. Let G be a (C_4, K_7)-Ramsey-free graph of order 22. Note that $\delta(G) < 5$, else a C_4 should be a subgraph of G.

Let m be an arbitrary vertex of G of the minimum degree $\delta(G)$.

Suppose that $\delta(G) \leq 3$. Then deleting a 3-degree vertex m and all its neighbours we get a graph F of the order at least 18. By Theorem 1(iv) the graph F contains an independent set S of cardinality 6. Thus $S \cup \{m\}$ is an independent set of cardinality 7, a contradiction.

Therefore $\delta(G) = 4$. Let m_i, $i = 1, 2, 3, 4$ be the neighbours of m in G. Let us consider the graph F obtained from G by deleting the vertex m and all its neighbours. Since G does not contain any C_4 then by degree condition each m_i, $i = 1, 2, 3, 4$ has at least two neighbours in F. Evidently the order of F equals 17 and F must be isomorphic to one of the (C_4, K_6)-Ramsey-free graphs presented in Figure 1 (else we get a contradiction as before). Suppose that F is isomorphic to H_1 or H_2. Since the vertex w has degree 3 in F then it must be adjacent to one of the neighbours of m, say m_1. Let us consider the graph $Y = G - N[w]$. Note that the vertex m has degree 3 in Y. Hence Y must be one of the (C_4, K_6)-Ramsey-free graphs H_1 or H_2 presented in Figure 1. Evidently m is not adjacent to any vertex of the set $\{d, v, b, h\}$. Therefore each of the four vertices must be adjacent to a vertex of the set $\{m_2, m_3, m_4\}$. It is impossible without creating C_4 because each two vertices of the set $\{d, v, b, h\}$ are at distance 2. A contradiction. Therefore we can assume that F is not isomorphic to H_i, $i = 1, 2$.

Suppose that F is isomorphic to B_1. Let the vertex x be adjacent to m_1. Then $1m_1 \in E(G)$, else $deg(m_1) < 4$. Moreover without loss of generality $m_1m_2 \in E(G)$. Note that $deg(m_1) = 4$. So we consider the graph $Y = G - N[m_1]$. Since Y cannot be isomorphic to H_i, $i = 1, 2$ then each of the vertices of the set $\{2, 3, 4, 5\}$ must be adjacent to m_3 or m_4 and we get C_4, a contradiction. Therefore $xm_i \notin E(G)$, for $i = 1, 2, 3, 4$. By symmetry, $ym_i \notin E(G)$, for $i = 1, 2, 3, 4$.

Suppose that f is adjacent to m_1. Since C_4 cannot be a subgraph then b, v, e or 4 is not adjacent to m_i, $i = 2, 3, 4$. Therefore $deg(m_1) > 4$, else the graph $G - N[m_1]$ has a 3-degree vertex, so it should be isomorphic to H_i, $i = 1, 2$ and we get a case above. Then m_1 should be adjacent to 3 and h, and without loss of generality $m_1m_2 \in E(G)$. Note that m_2 can be adjacent to one of the vertices $d, 1$ or u. So $deg(m_2) < 4$ or a C_4 exists, a contradiction.

Hence $fm_i \notin E(G)$, for $i = 1, 2, 3, 4$. By symmetry $bm_i \notin E(G)$, for $i = 1, 2, 3, 4$.

If the vertex 2 is adjacent to m_1 then $deg(m_1) > 4$, else $G - N[m_1]$ has a 3-degree vertex, and we get a case above. Then m_1 must be adjacent to e and to one of g, u. Moreover without loss of generality $m_1m_2 \in E(G)$. Note that $deg(m_2) < 4$ or a C_4 exists, a contradiction.

Hence $2m_i \notin E(G)$, for $i = 1, 2, 3, 4$. By symmetry $em_i \notin E(G)$, for $i = 1, 2, 3, 4$.

Similar arguments give that 5 and h cannot be adjacent to m_i, $i = 1, 2, 3, 4$.

Now without loss of generality we can assume that m_1, m_2 and u create an independent set. Therefore $\{m_1, m_2, 2, 5, y, f, u\}$ is an independent set.

Suppose that F is isomorphic to B_2. Let g be adjacent to m_1. Then m_1 must be adjacent to 3 and y, and without loss of generality $m_1m_2 \in E(G)$, else the graph $G - N[m_1]$ has a 3-degree vertex, so it should be isomorphic to H_i, $i = 1, 2$ and we get a case above. So m_2 must be adjacent to 4 and e, and it has degree four. Therefore the vertices $5, b, u, f$ must be adjacent to m_3 or m_4, else we get a 3-degree vertex in $G - N[m_2]$. Without loss of generality we can assume that the vertex m_3 is adjacent to b, f, and the vertex m_4 is adjacent to $5, u$. Note that m_4 has only these two neighbours in B_2. Hence m_4 must be adjacent to m_3 and $deg(m_4) = 4$. Since h cannot be adjacent to m_i, $i = 1, 2, 3, 4$ the graph $G - N[m_4]$ has a 3-degree vertex and we get a case above.

Hence $gm_i \notin E(G)$, for $i = 1, 2, 3, 4$. By symmetry $ym_i \notin E(G)$, for $i = 1, 2, 3, 4$.

Let 2 be adjacent to m_1. Then m_1 should be adjacent to one of the vertices u, a, b. So the graph $G - N[m_1]$ contains a 3-degree vertex g or y, and we get a case above. Hence $2m_i \notin E(G)$, for $i = 1, 2, 3, 4$. By symmetry $cm_i \notin E(G)$, for $i = 1, 2, 3, 4$.

Now without loss of generality we can assume that m_1, m_2 and 4 create an independent set. Therefore $\{m_1, m_2, 4, c, 2, g, y\}$ is an independent set.

Suppose that F is isomorphic to B_3. Let d be adjacent to m_1. Then m_1 must be adjacent to one of the vertices 3,b, g, h, and without loss of generality

$m_1m_2 \in E(G)$. Since $deg(m_1) = 4$ and m_2 cannot be adjacent to $3, h, f, u$, then the graph $G - N[m_1]$ has a 3-degree vertex, and we get a case above.

Hence $dm_i \notin E(G)$, for $i = 1, 2, 3, 4$. By symmetry $gm_i \notin E(G)$, for $i = 1, 2, 3, 4$.

Let a be adjacent to m_1. Then m_1 must be adjacent to one of the vertices $u, f, 4$. As before $deg(m_1) = 4$. Note that one of the vertices $2, b, h, y$ has 3-degree in $G - N[m_1]$, and we get a case above.

Hence a and 4 (by symmetry) cannot be adjacent to m_i, $i = 1, 2, 3, 4$. Now without loss of generality we can assume that m_1, m_2 and 1 create an independent set. Therefore $\{m_1, m_2, 1, 4, a, d, g\}$ is an independent set.

All cases lead to a contradiction ■

For the planar case we get the following theorem.

Theorem 6. $PR(C_4, K_6) = 17$.

Proof. Since by Lemma 3 each (C_4, K_6)-Ramsey-free graph of order 17 is not planar and $R(C_4, K_6) = 18$ we get $PR(C_4, K_6) \leq 17$. The graph presented in Figure 2 is (C_4, K_6)-Ramsey-free planar graph. So PR $(C_4, K_6) > 16$. ■

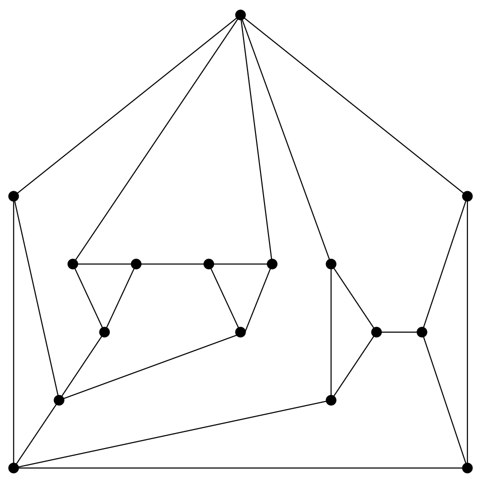

Figure 2. A planar graph of order 16 with independence number less than 6 and without C_4.

Proposition 7. *For each integer* $t \geq 5$, $PR(C_4, K_{t+1}) \geq 3t + [\frac{t}{5}] + 1$.

Proof. Let H be a graph of order 16 presented in Figure 2. Note that H does not contain any subgraph C_4 and $\alpha(H) = 5$. Therefore $[\frac{t}{5}]H \cup (t - [\frac{t}{5}]5)K_3,\ t \geq 6$, shows that $PR(C_4, K_{t+1}) \geq 3t + [\frac{t}{5}] + 1$. ■

Added in Proof. The result cited in Lemma 3 can be also find in: C.J. Jayawardene, C.C. Rousseau, An upper bound for Ramsey number of a quadrilateral versus a complete graph on seven vertices, Congressus Numerantium **130** (1998) 175–188.

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