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ON 1-DEPENDENT RAMSEY NUMBERS FOR GRAPHS

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Abstract

A set X of vertices of a graph G is said to be 1-dependent if the subgraph of G induced by X has maximum degree one. The 1-dependent Ramsey number $t_1(l, m)$ is the smallest integer n such that for any 2-edge colouring (R, B) of K_n, the spanning subgraph B of K_n has a 1-dependent set of size l or the subgraph R has a 1-dependent set of size m. The 2-edge colouring (R, B) is a $t_1(l, m)$ Ramsey colouring of K_n if B (R, respectively) does not contain a 1-dependent set of size l (m, respectively); in this case R is also called a (l, m, n) Ramsey graph. We show that $t_1(4, 5) = 9$, $t_1(4, 6) = 11$, $t_1(4, 7) = 16$ and $t_1(4, 8) = 17$. We also determine all (4,4,5), (4,5,8), (4,6,10) and (4,7,15) Ramsey graphs.

Keywords: 1-dependence, irredundance, CO-irredundance, Ramsey numbers.

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1 Introduction

Undefined notation and terminology can be found in [6]. The open and closed neighbourhoods of the vertex subset X of a simple graph $G = (V, E)$ are denoted by $N(X)$ and $N[X]$, respectively, and $N(\{x\})$ and $N[\{x\}]$ are abbreviated to $N(x)$ and $N[x]$. The set X is 1-*dependent* if $\Delta(\langle X \rangle) \leq 1$, that is, if $\langle X \rangle \cong \lambda K_1 \cup \mu K_2$. Further, X is *irredundant* if for all $x \in X$, the *private neighbourhood* $pn(x, X)$ *of* x *relative to* X, defined by $pn(x, X) =$

$N[x] - N[X - \{x\}]$, is nonempty. Further, X is called *CO-(Closed-Open) irredundant* if the *CO-private neighbourhood* $PN(x, X)$ *of* x *relative to* X, defined by $PN(x, X) = N[x] - N(X - \{x\})$, is nonempty for each $x \in X$. Note that for $v \in V$ we have $v \in pn(x, X)$ if and only if

(i) $v = x$ and x is isolated in $G[X]$, or

(ii) $v \in V - X$ and $N(v) \cap X = \{x\}$.

Further, $v \in PN(x, X)$ if and only if (i), (ii) or (iii) holds, with

(iii) $v \in X$ and $N(v) \cap X = \{x\}$.

A vertex of $PN(x, X)$ of types (i) or (iii) is called an *internal* XPN *of* x while a vertex of type (ii) is called an *external* XPN *of* x. The definitions imply that $pn(x, X) \subseteq PN(x, X)$ and since $x \in pn(x, X)$ for any vertex x of an independent set X, we have

$$X \text{ independent} \Rightarrow X \text{ irredundant} \Rightarrow X \text{ CO-irredundant.} \tag{1}$$

Clearly, X is 1-dependent if and only if (i) or (iii) holds for each $x \in X$ and so we also have

$$X \text{ independent} \Rightarrow X \text{ 1-dependent} \Rightarrow X \text{ CO-irredundant.} \tag{2}$$

Let $\beta(G)$, $IR(G)$, $COIR(G)$ and $D(G)$ be respectively the largest cardinality among the independent, irredundant, CO-irredundant and 1-dependent sets of G. Then for any graph G, (1) implies that

$$COIR(G) \geq \begin{Bmatrix} IR(G) \\ D(G) \end{Bmatrix} \geq \beta(G). \tag{3}$$

Various generalisations of irredundance, based on the private neighbour properties (i), (ii) and (iii) and the associated generalised Ramsey theory were discussed in [3]. Suppose that each edge of the complete graph K_n is assigned a colour from $\{1, ..., k\}$. For $i = 1, ..., k$, let G_i be the spanning subgraph of K_n induced by the edges with colour i. Then $(G_1, ..., G_k)$ is called a *k-edge colouring* of K_n.

The classical Ramsey numbers are usually defined in terms of complete graphs in G_i. However, since complete graphs in G_i correspond to independent sets in $\overline{G_i}$, they may also be defined in terms of independent sets. Using this approach we now define four types of Ramsey numbers.

Let $k \geq 2$ and $n_i \geq 3$ for $i = 1, ..., k$. The (*classical*) *Ramsey number* $r(n_1, ..., n_k)$ (the *irredundant Ramsey number* $s(n_1, ..., n_k)$, the *CO-irredundant Ramsey number* $t(n_1, ..., n_k)$ and the 1-*dependent Ramsey* *number* $t_1(n_1, ..., n_k)$ respectively) is the smallest integer n such that for any k-edge colouring $(G_1, ..., G_k)$ of K_n, there exists $i \in \{1, ..., k\}$ such that $\beta\left(\overline{G_i}\right)$ $\left(IR\left(\overline{G_i}\right), COIR\left(\overline{G_i}\right), D(\overline{G_i})\right.$ respectively) $\geq n_i$. A special case of Ramsey's theorem [12] guarantees the existence of the classical Ramsey numbers for graphs. Their existence together with (1) and (2) imply the existence of the other types of Ramsey numbers. Moreover, (1) and (2) give

$$t(n_1, ..., n_k) \leq \begin{Bmatrix} s(n_1, ..., n_k) \\ t_1(n_1, ..., n_k) \end{Bmatrix} \leq r(n_1, ..., n_k).$$

Irredundant Ramsey numbers were first defined by Brewster, Cockayne and Mynhardt [1] and as in the case of the classical Ramsey numbers, the determination of exact values proved to be very difficult. A survey of results on these numbers is given in [9].

CO-irredundant Ramsey numbers were first studied by Simmons in the unpublished master's dissertation [13]. The first research paper written on the topic was [4], where it was observed that if $n_i \in \{3, 4\}$ for each $i \in \{1, ..., k\}$, these numbers coincide with other generalised graph Ramsey numbers: Let $F_1, ..., F_k$ be graphs. The generalised Ramsey number $r(F_1, ..., F_k)$ is the smallest n such that in any k-edge-colouring $(G_1, ..., G_k)$ of K_n, for some $i \in \{1, ..., n\}$ the graph G_i has F_i as subgraph. Now let $n_i \in \{3, 4\}$ and $F_i \cong P_3$ (C_4) if $n_i = 3$ (4). Then $t(n_1, ..., n_k) = r(F_1, ..., F_k)$ and the same result is true for 1-dependent Ramsey numbers. (Also see Proposition 2.) For a survey on generalised Ramsey numbers, see [11]. In the case $k = 2$, the following numbers were determined in [4]: $t(4, 4) = 6$, $t(4, 5) = 8$, $t(4, 6) = 11$, while it was shown in [5] that $t\,(4, 7) = 14$. Bounds for $t\,(5, 5)$ were also determined in [13].

In this paper we consider the 1-dependent Ramsey numbers for the case $k = 2$. For ease of presentation (R, B) denotes a 2-edge colouring of K_n and the edges of R and B will be coloured red and blue respectively. The term 1-dependent set of size m is simply denoted by dm. Thus the 1-dependent Ramsey number $t_1(l, m)$ is the smallest integer n such that for any 2-edge colouring of K_n, the subgraph B of K_n has a dl or the subgraph R has a dm. The 2-edge colouring (R, B) is a $t_1\,(l, m)$ *Ramsey colouring* of K_n if B (R, respectively) does not contain a dl (dm, respectively); in this case R is also called an (l, m, n) *1-dependent Ramsey graph* or an (l, m, n) *Ramsey graph* for short.

We determine $t_1\,(4, m)$ for $m = 5, 6, 7$ and 8, as well as all (4,4,5), (4,5,7), (4,6,10) and (4,7,15) Ramsey graphs. Each of these classes of graphs is used to find the next 1-dependent Ramsey number. Note that B has a $d5$ if and only if R has a wheel W_5 (also see Proposition 2) and thus the 1-dependent Ramsey number $t_1(5,5)$ is the same as the generalised Ramsey number $r(W_5, W_5)$, which was determined in [7].

2 Preliminary Results

The following recurrence inequality is well-known for the classical Ramsey numbers and analogous proofs establish it for the other three types.

Proposition 1. *If $\alpha(l,m)$ is any one of the four types of Ramsey numbers defined above, then*

$$\alpha\,(l,m) \leq \alpha(l,m-1) + \alpha\,(l-1,m)\,.$$

Moreover, if $\alpha\,(l,m-1)$ and $\alpha(l-1,m)$ are both even, then this inequality is strict.

To enable us to consider only the red subgraph R of K_n to determine whether B has a $d3$ or a $d4$, we need the following result of [4] for CO-irredundant sets of sizes 3 and 4 and its corollary; the proof for 1-dependent sets of sizes 3 and 4 are similar.

Proposition 2 [4, 13]**.** *Consider a 2-edge colouring (R,B) of K_n. Then*

(i) *B has a $d3$ if and only if R has P_3 as subgraph,*
(ii) *B has a $d4$ if and only if R has C_4 as subgraph,*
(iii) *B has a $d5$ if and only if R has the wheel W_5 as a subgraph.*

Corollary 3 [4]**.** *For any $m \geq 3$,*

(i) $t\,(3,m) = t_1(3,m) = m$,
(ii) $t(4,4) = t_1(4,4) = 6$,
(iii) [7] $t_1(5,5) = 15$.

Given a 2-edge colouring (R,B) of K_n, each vertex v and its neighbours in R and B, respectively, induce a partition (v, R_v, B_v) of $V\,(K_n)$, where

$$R_v = \{u \in V\,(K_n) : uv \in E\,(R)\}$$

and

$$B_v = \{u \in V\,(K_n) : uv \in E\,(R)\}\,.$$

For any $x \in R_v$, define

$$S_x = \{u \in B_v : ux \in E(R)\},$$

and define $T_v \subseteq B_v$ by

$$T_v = B_v - \bigcup_{x \in R_v} S_x.$$

The proof of the following simple results about $t_1(4,m)$ Ramsey colourings of K_n is exactly the same as the proof of Proposition 5 of [5]. We use the terms "(R,B) is a $t_1(4,m)$ Ramsey colouring of K_n" and "R is a $(4,m,n)$ Ramsey graph" interchangeably. In particular, when we say that two vertices u and v are adjacent, we mean that they are adjacent in R and thus that the edge uv is red. Similarly, for $X \subseteq V(K_n)$ the notation $\langle X \rangle$ refers to the subgraph of R induced by X.

Proposition 4. *Consider a $t_1(4,m)$ Ramsey colouring (R,B) of K_n and let $v \in V(K_n)$ be arbitrary. Then*

(i) *Each vertex in B_v is adjacent (in R) to at most one vertex in R_v and hence $S_x \cap S_y = \emptyset$ for distinct vertices $x, y \in R_v$.*
(ii) $\Delta(\langle R_v \rangle) \leq 1$.
(iii) $|R_v| \leq m-1$.
(iv) *For each $x \in R_v$, $|S_x| \leq m - |R_v|$.*
(v) *For each x, $y \in R_v$ with $xy \in E(R)$, $|S_x| + |S_y| \leq m - |R_v| + 1$.*
(vi) *For each x, $y \in R_v$ with $xy \in E(B)$, $\langle S_x \cup S_y \rangle$ consists of paths (possibly including P_1) and cycles C_k, where $k \equiv 0 \,(\mathrm{mod}\, 4)$, $k \geq 8$. There exists a subgraph H of $\langle S_x \cup S_y \rangle$ which contains $\lceil 2q/3 \rceil$ vertices of each path P_q and $\lfloor 2k/3 \rfloor$ vertices of each cycle C_k, such that $\Delta(H) \leq 1$.*

As in the case of CO-irredundant Ramsey numbers Proposition (iii) can be extended to general $t_1(l,m)$ Ramsey colourings. A simple lower bound for $|R_v|$ also exists. Again the proof is the same as for $t(l,m)$.

Proposition 5. *Let (R,B) be a $t_1(l,m)$ Ramsey colouring of K_n and consider an arbitrary vertex v. Then*

$$n - t_1(l, m-1) \leq |R_v| \leq t_1(l-1, m) - 1.$$

In the case $l = 4$ the upper bound in Proposition 5 can sometimes be improved:

Proposition 6. *Let (R, B) be a $t_1\,(4, m)$ Ramsey colouring of K_n and consider an arbitrary vertex v. Then*

$$|R_v| \le 2 + m - n + t_1(4, m-1).$$

Proof. For any $x \in R_v$,

$$\begin{aligned} \deg x &\le 2 + |S_x| \\ &\le 2 + m - |R_v| \end{aligned}$$

by Proposition 4(iv). By Proposition 5,

$$n - t_1(4, m-1) \le \deg x$$

and so

$$n - t_1(4, m-1) \le 2 + m - |R_v|\,,$$

from which the result follows. ■

The following result is frequently used to find (l, m, n) Ramsey graphs or to prove that they do not exist.

Proposition 7. *Consider a $t_1\,(l, m)$ Ramsey colouring of K_n and vertices u and v such that the edge uv is red. Let $X = V\,(K_n) - N\,[\{u, v\}]$. Then $\langle X \rangle$ is a $(l, m-2, |X|)$ Ramsey graph and thus $|X| \le t_1\,(l, m-2) - 1$.*

Proof. Since any *dl* in $\overline{\langle X \rangle}$ is a *dl* in B, it follows that $\overline{\langle X \rangle}$ does not contain a *dl*. Suppose $\langle X \rangle$ contains a $d\,(m-2)$, say Y'. By definition $N\,[\{u, v\}] \cap X = \emptyset$. But then $Y = Y' \cup \{u, v\}$ satisfies $\Delta\,(\langle Y \rangle) \le 1$ and therefore is a *dm* in R, a contradiction. The result follows. ■

3 The values of $t_1(4,5)$ and $t_1(4,6)$, and the (4,4,5), (4,5,8) and (4,6,10) Ramsey Graphs

Recall that $t_1\,(4, 4) = 6$ and let D be the graph obtained by joining the two nonadjacent degree two vertices of P_5.

Proposition 8. *The only $(4, 4, 5)$ graphs are D and C_5.*

Proof. Let (R, B) be any $t_1\,(4, 4)$ colouring of K_5 and consider any vertex v. By Corollary 3 and Proposition 5,

$$1 \le |R_v| \le 3.$$

If R is 2-regular, then $R \cong C_5$ and since C_5 is selfcomplementary and does not contain a C_4, it follows from Proposition 2(ii) that C_5 is a $(4,4,5)$ graph. If $1 = \delta(R) \leq \Delta(R) \leq 2$, then R is the union of paths and contains a 1-dependent set of size $\lceil 10/3 \rceil = 4$, a contradiction. Hence we may assume that there is a vertex v with $|R_v| = 3$, say $R_v = \{1,2,3\}$. Then $|T_v| \leq 1$ and if $|T_v| = 1$, then $S_i = \emptyset$ for each $i = 1,2,3$, implying the vertex T_v is isolated in R, a contradiction. Hence by Proposition 4(i) we may assume that $|S_1| = 1$, say $S_1 = \{4\}$, and $S_2 = S_3 = \emptyset$. To avoid the $d4$ $\{1,4,2,3\}$, $\langle\{1,2,3\}\rangle$ contains at least one and thus by Proposition 4 (ii) exactly one edge. If 2-3 is red, then $\{1,4,2,3\}$ is a $d4$ in any case, hence we may assume that 1-2 is red; clearly all other edges are blue. Hence $R \cong D$ as required. Since D is selfcomplementary and does not contain a C_4, D is a $(4,4,5)$ graph. ∎

Let a and b be the vertices of D of degree three, c (d) the endvertex adjacent to a (b) and e the vertex of degree two. We now determine all $(4,5,8)$ graphs.

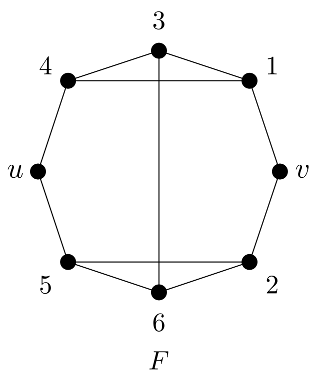

Figure 1. The only $(4,5,8)$ graph

Theorem 9. *The only* $(4,5,8)$ *graph is the graph* F *in Figure* 1.

Proof. Proposition 5 implies that for any $t_1(4,5)$ colouring (R,B) of K_8 and any vertex v,

$$2 \leq |R_v| \leq 4.$$

If $|R_v| = 4$, then by Proposition 4(iv), $|S_x| \leq 1$ for each $x \in R_v$. If $T_v \neq \emptyset$ then for any $y \in T_v$, $|R_v \cup \{y\}| = 5$ and $\Delta(\langle R_v \cup \{y\}\rangle) \leq 1$ which implies that $R_v \cup \{y\}$ is 1-dependent, a contradiction. Thus $T_v = \emptyset$ and a counting argument shows that $S_x = \emptyset$ for exactly one $x \in R_v$ and $|S_y| = 1$ for each $y \in R_v - \{x\}$. Say $R_v = \{1,2,3,4\}$, $S_1 = \emptyset$, $S_2 = \{5\}$, $S_3 = \{6\}$ and $S_4 = \{7\}$. Since $\deg 1 \geq 2$, vertex 1 is adjacent to some vertex in R_v; say without loss of generality that 1-2 is red. By Proposition 4 (ii), 1 and 2 are not adjacent to either 3 or 4. To ensure that $\{1,2,3,6,7\}$ is not 1-dependent, 6-7 is red. To ensure that $\{1,2,3,4,7\}$ is not 1-dependent, 3-4 is red. But then 3-4-7-6 is a 4-cycle, contradicting Proposition 2. Consequently

$$2 \leq |R_v| \leq 3$$

for each vertex v. If R is 2-regular, then by Proposition 2, $R \cong C_8$ or $C_3 \cup C_5$ and it is easy to see that $D(R) = 5$ in each case. Thus

$$|R_v| = 3$$

for some vertex v.

Suppose R is 3-regular. Say $R_v = \{1,2,3\}$. Then the only possibility for the S_i and for red edges in $\langle R_v \rangle$ is, without loss of generality, 1-2 red, $S_1 = \{4\}$, $S_2 = \{5\}$ and $S_3 = \{6,7\}$. To avoid a 4-cycle, 4-5 is blue and since $\deg 4 = 3$, 4 is adjacent to 6 and 7, forming a 4-cycle with 3, a contradiction. This shows that

$$\delta(G) = 2 \text{ and } \Delta(G) = 3.$$

Let v be a vertex with $R_v = \{1,2\}$ and suppose firstly that 1-2 is red. Then $|S_1| = |S_2| = 1$; say $S_1 = \{3\}$, $S_2 = \{4\}$ and $T_v = \{5,6,7\}$. To avoid a 4-cycle, 3-4 is blue. If P_3 is not a subgraph of $\langle\{5,6,7\}\rangle$, then $D(\langle\{5,6,7\}\rangle) \geq 2$ and thus $\{1,2,5,6,7\}$ is 1-dependent, a contradiction. Say 5-6-7 is the vertex sequence of a P_3. Considering $\{v,2,3,5,7\}$, we see that to avoid a $d5$, 3-5 (without loss of generality) and 5-7 are both red. Similarly, $\{v,2,3,6,7\}$ implies that (without loss of generality) 3-6 is red, thus forming a 4-cycle 3-5-7-6, a contradiction.

Hence we may assume that 1-2 is blue and $|S_1| = |S_2| = 2$. Say $S_1 = \{3,4\}$, $S_2 = \{5,6\}$ and $T_v = \{u\}$. Since $\deg(u) \geq 2$ and u is adjacent to at most one vertex in each S_i (to avoid 4-cycles), u is adjacent to exactly one vertex in each S_i. Suppose without losing generality that u-4 and u-5 are red. Considering $\{v,2,3,4,u\}$, we see that 3-4 is red and similarly, 5-6 is red. Moreover, $\{1,3,2,6,u\}$ implies that 3-6 is red. The only vertices incident with only two red edges are u and v and since R is not 3-regular, uv is blue and $R \cong F$ (see Figure 1).

Note that F has two disjoint triangles. If F has a $d5$, say X, then X contains two vertices of the same triangle and at least one of the vertices u

and v. It is easy to check that this is impossible and thus F is the unique (4,5,8) graph. ■

Corollary 10. $t_1\,(4,5) \geq 9$.

We now show that there are no (4,5,9) graphs and thus $t_1(4,5) = 9$. This number was also determined as the generalised Ramsey number $r(C_4, W_5)$ by [14], as cited in [11].

Theorem 11. $t_1(4,5) = 9$.

Proof. Suppose to the contrary that (R,B) is a $t_1\,(4,5)$ colouring of K_9. By Proposition 5,

$$3 \leq |R_v| \leq 4$$

for each vertex v and since R is not 3-regular, there exists a vertex v with $\deg v = 4$. The degree conditions and a counting argument show that $\langle R_v \rangle \cong 2K_2$, $|S_x| = 1$ for each $x \in R_v$ and $T_v = \emptyset$. Let $R_v = \{1,2,3,4\}$ with 1-2 and 3-4 red, and $S_i = \{i+4\}$ for each $i = 1,...,4$. To avoid 4-cycles, 5-6 and 7-8 are blue. Since $\deg 5 \geq 3$, 5-7 and 5-8 are red; similarly, 6-7 and 6-8 are both red. But then 5-7-6-8 is a 4-cycle, a contradiction. ■

It was suggested in [4], and proved in [5], that for CO-irredundant Ramsey numbers, the only (4,6,10) graphs are G_1, G_2 and G_3 in Figure 2. This result also holds for 1-dependent Ramsey numbers and the proof is exactly the same as the proof of Theorem 10 of [5].

Theorem 12. *The only* $(4,6,10)$ *graphs are the graphs* G_1, G_2 *and* G_3 *in Figure* 2.

Corollary 13. $t_1(4,6) \geq 11$.

In order to use Theorem 12 to determine $t_1(4,7)$ and $t_1(4,8)$, we determine the maximum number of vertices which can be chosen from F, G_1, G_2 and G_3 so that no two chosen vertices are joined by a path of length two.

Proposition 14. *For a set X of vertices of a graph, let $\mathcal{P}(X)$ be the property "no two vertices of X are joined by a path of length two". Suppose that $\mathcal{P}(X)$ holds.*

(i) *If* $X \subseteq V(F)$, $X \subseteq V(G_1)$ *or* $X \subseteq V(G_2)$, *then* $|X| \leq 2$.
(ii) *If* $X \subseteq V(G_3)$ *and* $t \notin X$, *then* $|X| \leq 2$; *otherwise* $|X| \leq 3$.

Proof. (i) Suppose $X \subseteq V(F)$. The result obviously holds if X consists of vertices of degree 2. If each vertex of X is of degree 4, then X contains at most one vertex from each triangle and the result holds. If X contains vertices of degree 2 and 4, then X consists of one vertex of degree 4 and the unique vertex of degree 2 adjacent to it.

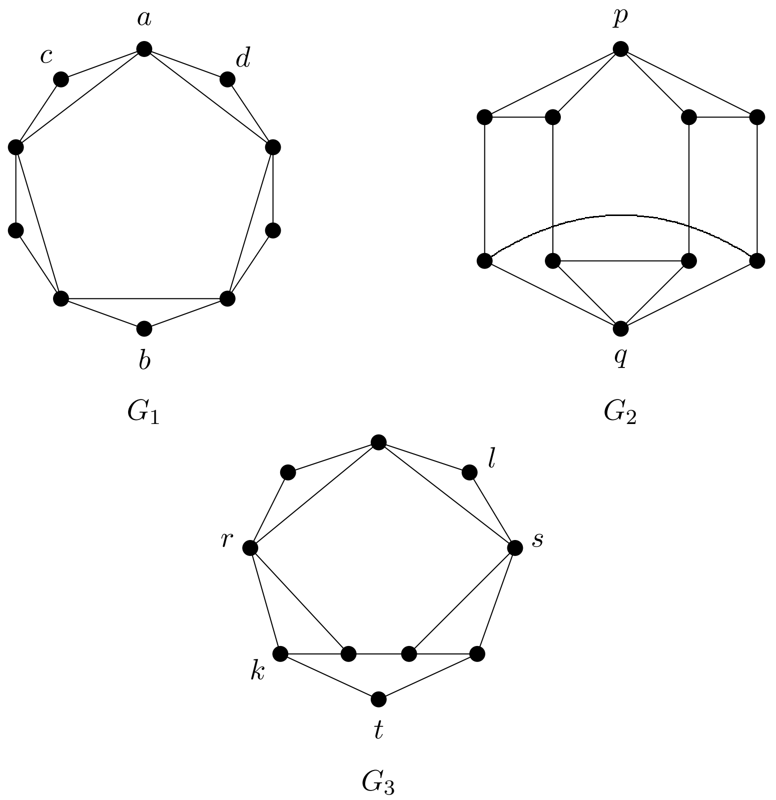

Figure 2. The only $(4, 6, 10)$ graphs (see [4])

Suppose $X \subseteq V(G_1)$. If $\max\{\deg v : v \in X\} = 4$, say $a \in X$, then

$$|X| = \begin{cases} 1, & \text{if } b \notin X, \\ 2, & \text{otherwise.} \end{cases}$$

If $\max\{\deg v : v \in X\} = 2$, say $b \in X$, then

$$|X| = \begin{cases} 1, & \text{if } c, d \notin X, \\ 2, & \text{otherwise.} \end{cases}$$

Suppose $X \subseteq V(G_2)$ and note that $d\,(p,q) = 3$ and each vertex in $N\,(p)$ ($N\,(q)$, respectively) lie on a common triangle with p (q). Thus if $p \in X$, then

$$|X| = \begin{cases} 1, & \text{if } q \notin X, \\ 2, & \text{otherwise.} \end{cases}$$

If $X \cap \{p,q\} = \emptyset$, then $X \subseteq N\,(p) \cup N\,(q)$ and so $|X| \leq 2$ to avoid a P_3.

(ii) If $X \subseteq V(G_3)$ and $t \notin X$, then $X \subseteq N\,(r) \cup N\,(s)$ and each vertex in $N\,(r)$ ($N\,(s)$, respectively) lie on a common triangle with r (s). Hence $|X| \leq 2$. It follows immediately that $|X| \leq 3$ if $t \in X$ and it is easy to see that equality can be obtained. ■

We conclude this section by showing that there are no (4,6,11) graphs. This result was also obtained as the generalised Ramsey number $r(C_4, K_6 - 3K_2)$ in [8], as cited in [11].

Theorem 15. $t_1\,(4,6) = 11$.

Proof. Suppose to the contrary that (R,B) is a $t_1\,(4,6)$ colouring of K_{11}. By Proposition 5,

$$2 \leq |R_v| \leq 5$$

for each vertex v. Consider v with $|R_v| = 5$. By Proposition 4(iv) $|S_x| \leq 1$ for each $x \in R_v$. Since $|R_v| = 5$ and $\Delta\,(\langle R_v \rangle) \leq 1$, there exists $x' \in R_v$ which is isolated in $\langle R_v \rangle$. If $S_{x'} = \emptyset$ then $T_v \neq \emptyset$ and so $R_v \cup T_v$ contains a $d6$. If $S_{x'} \neq \emptyset$ then $R_v \cup S_{x'}$ is a $d6$, a contradiction in each case. This shows that for each vertex v,

$$2 \leq |R_v| \leq 4.$$

Now consider v with $|R_v| = 2$; say $R_v = \{1, 2\}$. By Proposition 7, $|S_i| \geq 3$ for each $i = 1, 2$ and it follows from the degree conditions that $|S_i| = 3$ and hence $|S_i \cup T_v| = 5$ for each i. Thus $\langle S_i \cup T_v \rangle \cong C_5$ or D. Let $T_v = \{u, w\}$. Now, $\langle S_i \rangle$ contains at most one edge, $\langle T_v \rangle$ contains at most one edge and each of u and w is adjacent to at most one vertex of S_i to avoid a 4-cycle. Thus $\langle S_i \cup T_v \rangle$ has at most four edges, a contradiction. This proves that for each v,

$$3 \leq |R_v| \leq 4.$$

Since R is not 3-regular it follows that R has a vertex v of degree four. By Propositions 4(iv) and 7,

$$1 \leq |S_x| \leq 2$$

for each $x \in R_v$. If $|S_x| = 1$ for each x, then $|T_v| = 2$ and $R_v \cup T_v$ is a $d6$, a contradiction. Say $R_v = \{1,2,3,4\}$, $S_1 = \{5,6\}$, $S_2 = \{7\}$, $8 \in S_3$, $S_4 = \{9\}$ and $10 \in S_3 \cup T_v$. Let $X_1 = \{5,6,7,8,10\}$ and $X_2 = \{5,6,8,9,10\}$. By Proposition 7, $\langle X_i \rangle \cong C_5$ or D. If 5-6 is blue, then to avoid 4-cycles the only possibility is that $\langle X_i \rangle \cong D$ with $\{5,6\} = \{c,d\}$. But since neither 7 nor 9 is adjacent to both of 5 and 6, it then follows that $\min\{\deg 5,\ \deg 6\} = 2$, a contradiction. Thus we may assume that 5-6 is red. If $10 \in S_3$, then by Proposition 4(v), 1-3 is blue. If $10 \notin S_3$, then since $\Delta(\langle R_v \rangle) \le 1$, we may assume without loss of generality that 1-3 is blue anyway. Since $|R_2|$, $|R_4| \ge 3$, we may also assume without loss of generality that 1-2 and 3-4 are red.

If $\langle X_1 \rangle \cong C_5$ then, since 5-7 and 6-7 are blue, we may assume that 10-5-6-8-7 is the vertex sequence of the 5-cycle. Note that 5-8 and 8-9 are blue. Therefore $\langle X_2 \rangle \cong C_5$ is impossible and thus $\langle X_2 \rangle \cong D$. But 6-10 and at least one of 9-5 and 9-6 is blue, so this is impossible too.

Therefore $\langle X_1 \rangle \cong D$ with, without loss of generality, $a = 6$ and $c = 5$. Then $d = 7$ and it follows that 6-8, 6-10 and 8-10 are red. Since $|R_5| \ge 3$ and 5-7, 5-8 and 5-10 are blue, 5-9 is red. But then $\langle X_2 \rangle \ncong C_5$ or D, a contradiction. ■

4 The (4,7,15) Graphs and the Value of $t_1(4,7)$

The calculation of $t_1(4,7)$ proves to be surprisingly simple, perhaps because of the existence of a $t_1(4,7)$ Ramsey colouring of K_{15} and the bounds given in Propositions 5 and 6.

Proposition 16. *$t_1(4,7) \le 16$.*

Proof. Suppose to the contrary that (R,B) is a $t_1(4,7)$ Ramsey colouring of K_{16}. By Propositions 5 and 6, any vertex v satisfies

$$5 = 16 - t_1(4,6) \le |R_v| \le 2 + 7 - 16 + t_1(4,6) = 4,$$

which is impossible. ■

We now illustrate the use of the (4,5,8) and (4,6,10) Ramsey graphs in the characterisation of (4,7,15) graphs.

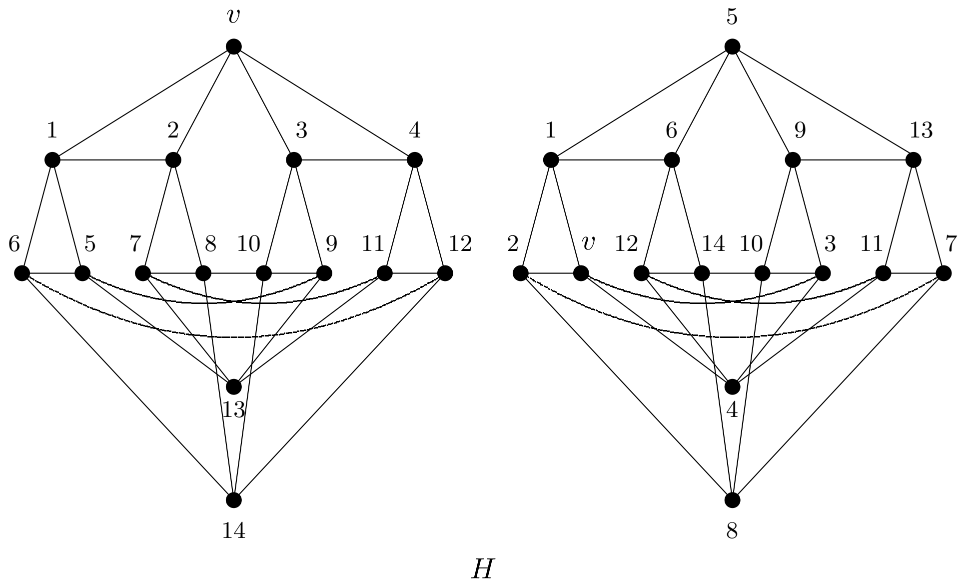

Figure 3. The only $(3,7,15)$ graph

Theorem 17. *The only* $(4,7,15)$ *Ramsey graph is the graph* H *in Figure* 3.

Proof. Consider any $t_1\,(4,7)$ Ramsey colouring (R,B) of K_{15}. By Propositions 5 and 6,

$$4 \leq \deg v \leq 5$$

for each $v \in V\,(R)$. Consider v with $|R_v| = 5$. Since $\Delta\,(\langle R_v\rangle) \leq 1$, R_v has an isolated vertex x. By Proposition 4(iv), $|S_x| \leq 2$. But then $\deg x \leq 3$, a contradiction and it follows that R is 4-regular. Therefore for each vertex v and each $x \in R_v$, $|S_x| \leq 3$. Since $\Delta\,(\langle R_v\rangle) \leq 1$, degree conditions imply that

$$2 \leq |S_x| \leq 3$$

for each $x \in R_v$. A counting argument shows that $|S_x| = 2$ for at least two vertices $x \in R_v$; necessarily each such vertex is adjacent to some other vertex in R_v.

Say $R_v = \{1,2,3,4\}$, where 1-2 is red, $S_1 = \{5,6\}$, $S_2 = \{7,8\}$, $\{9,10\} \subseteq S_3$, $\{11,12\} \subseteq S_4$ and $T_v \subseteq \{13,14\}$. For $i = 1,2$, let $X_i = B_v - S_i$. By Proposition 7 $\langle X_i\rangle \cong F$. By Proposition 14 vertices 3 and 4 are adjacent to at most two vertices, and hence exactly two vertices, in each of X_1 and X_2, so that $S_3 = \{9,10\}$, $S_4 = \{11,12\}$ and $T_v = \{13,14\}$. To satisfy the degree requirements, 3-4 is red. By repeating the above argument for the vertices 1,2,3 and 4, we find that 5-6, 7-8, 9-10 and 11-12 are red. Now $\langle B_v \rangle$ is a (4,6,10) graph such that S_i, $i = 1, ..., 4$, are pairwise disjoint subsets of B_v. Since R is 4-regular, $\langle B_v \rangle$ has eight vertices of degree 3 and two of degree 4, and the only possibility is $\langle B_v \rangle \cong G_2$, where $\{13, 14\} = \{p, q\}$. Without losing generality we may assume that $R_{13} = \{5, 7, 9, 11\}$ and $R_{14} = \{6, 8, 10, 12\}$, with the edges 5-9, 7-11, 6-12 and 8-10 red. This is the 4-regular graph H in Figure 3.

By redrawing H with other vertices in the place of v, as for example indicated in Figure 3, it can be seen that H is vertex transitive. If H has a $d7$, say X, then $\langle X \rangle$ has an isolated vertex. Without loss of generality suppose this is v. Then $V(H) - N[v]$ contains a $d6$ – but by construction, $\langle V(H) - N[v] \rangle \cong G_2$. ■

Corollary 18. $t_1(4, 7) = 16$.

5 Amazingly, $t_1(4, 8) = 17$

Let J be any graph obtained from H by adding a vertex w to H and joining w to vertices of H in any way without creating a 4-cycle. (For example, let $J \cong H \cup K_1$.) Then J does not contain a $d8$ and $\overline{J}$ does not contain a $d4$ and so J is a (4,8,16) Ramsey graph. It follows that

$$t_1(4, 8) \geq 17.$$

To show that $t_1(4, 8) = 17$ we first prove the following lemma.

Lemma 19. *If R is a $(4, 8, 17)$ graph, then R does not have*

(i) *two vertices with total degree at most 4,*
(ii) *two vertices with a common neighbour with total degree at most 5,*
(iii) *two adjacent vertices with total degree at most 6 and*
(iv) *two vertices with total degree at most 7 which lie on a common triangle.*

Proof. In each of the above cases, if u and v are vertices with the stated properties, then $|V(R) - N[\{u, v\}]| \geq 11$, contradicting Theorem 15 and Proposition 7. ■

Theorem 20. $t_1(4, 8) = 17$.

Proof. Suppose to the contrary that (R, B) is a $t_1(4,8)$ Ramsey colouring of K_{17}. Then

$$1 \leq |R_v| \leq 7$$

for each vertex v. If $|R_v| = 7$, then $|S_x| \leq 1$ for each $x \in R_v$. Since $\Delta(\langle R_v \rangle) \leq 1$ there exists a vertex x which is isolated in $\langle R_v \rangle$. Hence $\deg x = 2$ and $\deg y \leq 3$ for all other $y \in R_v$, contradicting Lemma19(ii). Hence for each vertex v,

$$1 \leq |R_v| \leq 6.$$

Consider v with $|R_v| = 6$. Then $|S_x| \leq 2$ for each $x \in R_v$ and so $\deg x \leq 3$ if x is isolated in $\langle R_v \rangle$ and $\deg x \leq 4$ otherwise. Thus if x is isolated in $\langle R_v \rangle$, then some $x' \neq x$ is also isolated in $\langle R_v \rangle$ and so $|S_x| = |S_{x'}| = 2$ by Lemma19(ii). On the other hand, if x is not isolated in $\langle R_v \rangle$, then by Lemma19(iv) $|S_x| = 2$ as well. But then R has more than 17 vertices, which is impossible. Thus

$$1 \leq |R_v| \leq 5$$

for each vertex v. However, if $\deg v = 1$, then v is adjacent to vertices of degree at least 6 only (Lemma19(iii)) and so we also have

$$2 \leq |R_v| \leq 5$$

for each v.

Consider v with $|R_v| = 2$; say $R_v = \{1, 2\}$. Then v is adjacent to vertices of degree 5 only and so for each $i = 1, 2$, $|S_i| = 4$ by Lemma19(iv). Let $X = V(K_{17}) - N[\{1, v\}]$. Then $|X| = 10$ and so by Proposition 7, $\langle X \rangle \cong G_1$, G_2 or G_3. But then vertex 2 is adjacent to four vertices of G_i, at least two of which are joined by a path of length two (Proposition 14), thus forming a 4-cycle in R, a contradiction. We have now proved that

$$3 \leq |R_v| \leq 5$$

for each v.

Suppose $|R_v| = 3$; say $R_v = \{1, 2, 3\}$. By Lemma19 $3 \leq |S_i| \leq 4$ for each $i = 1, 2, 3$. If (say) $|S_1| = 3$ and $X = V(K_{17}) - N[\{1, v\}]$, then $|X| = 10$ and thus $\langle X \rangle \cong G_1$, G_2 or G_3. But then S_2 and S_3 are disjoint subsets of X of size at least three and so by Proposition 14, vertex 2 or 3 is adjacent to two vertices of X which are joined by a path of length two. This forms a 4-cycle in R, a contradiction. Thus $|S_i| = 4$ for each $i = 1, 2, 3$. By Proposition 4(v), $\langle R_v \rangle \cong 3K_1$ and by Proposition 4(vi), $\langle S_1 \cup S_2 \rangle$ is a union of paths or an 8-cycle. In the former case $\langle S_1 \cup S_2 \rangle$ contains a 1-dependent set of size $\lceil 16/3 \rceil = 6$, which together with $\{v, 3\}$ forms a $d8$. Hence $\langle S_1 \cup S_2 \rangle \cong C_8$. Since no vertex in S_1 is adjacent to more than one vertex in S_2 and *vice versa*, the 8-cycle in $\langle S_1 \cup S_2 \rangle$ is of the form $x_1x_2y_1y_2x_3x_4y_3y_4$ with $x_j \in S_1$, $y_j \in S_2$. Let the vertex sequence of this 8-cycle be 4-5-8-9-6-7-10-11 with $S_1 = \{4, 5, 6, 7\}$ and $S_2 = \{8, 9, 10, 11\}$. Now $u \in T_v \neq \emptyset$ is adjacent to at most one, and thus to exactly one vertex in each S_i, $i = 1, 2, 3$. Without loss of generality say u-4 is red. To avoid the 4-cycles u-4-5-8 and u-4-11-10, u-8 and u-10 are blue. Therefore u-9 or u-11 is red. In the former case $\{v, 3, u, 5, 8, 10, 11, 6\}$ is a $d8$ and in the latter case $\{v, 3, u, 4, 8, 9, 7, 10\}$ is a $c8$. We conclude that for each vertex v,

$$4 \leq |R_v| \leq 5$$

and since R is not 5-regular, R has a vertex v of degree 4.

If $|R_v| = 5$, say $R_v = \{1, 2, 3, 4, 5\}$, then the degree conditions and a counting argument show that $|S_i| = 2$ for four values of i, say $i = 1, ..., 4$, and $2 \leq |S_5| \leq 3$. But if $|S_5| = 2$, then since at least one of the vertices in R_v is isolated in $\langle R_v \rangle$, $\delta\,(R) = 3$, a contradiction. Hence $|S_5| = 3$. Since $\delta(R) = 4$ we may assume that 1-2 and 3-4 are red. Let $X = V(K_{17}) - N\,[\{1, 2\}]$ and note that $|X| = 10$; hence by Proposition 7 $\langle X \rangle \cong G_1$, G_2 or G_3. Then $v \in N\,[\{1, 2\}]$ is adjacent to three vertices in X and so by Proposition 14, $\langle X \rangle \cong G_3$. Since 3-4 is red, the only possibility is that without losing generality, vertex 3 corresponds to $t \in V(G_3)$ and vertex 4 to k. Since $|S_5| = 3$, the other neighbour of v in G_3 has degree three. But the only possible choice of a vertex together with t and k is l, and $\deg l = 2$. We conclude that R is 4-regular.

Let v with $R_v = \{1, 2, 3, 4\}$ be a vertex of R and suppose firstly that v lies on a triangle; say 1-2 is red. If 3-4 is blue, then $|S_3| = |S_4| = 3$. But then for $X = V(K_{17}) - N\,[\{v, 1\}]$, $\langle X \rangle \cong G_i$ for some $i = 1, 2, 3$ and vertices 3 and 4 are adjacent to two disjoint sets of three vertices in G_i, which is impossible by Proposition 14. Hence 3-4 is red and $|S_3| = |S_4| = 2$ so that $|T_v| = 4$. Repeating this argument for any vertex of R which lies on a triangle, such as vertices 1-4, we obtain that $\langle S_i \rangle \cong K_2$ for $i = 1, ..., 4$. Hence for $i = 2, 3, 4$, $S_i \subseteq X$ such that the vertices of each S_i are adjacent in $\langle X \rangle$. Since the S_i are moreover disjoint, it follows from Proposition 14 (and its proof) that $\langle X \rangle \cong G_2$ and each vertex i, for $i = 2, 3, 4$, is adjacent to one vertex in $N\,(p)$ and one vertex in $N\,(q)$. Further, say $S_1 = \{5, 6\}$. Vertices 5 and 6 are also adjacent to two vertices each in X. However, by considering the degrees of the vertices of G_2 we see that there are exactly 8 edges from $N[\{v,1\}]$ to X, a contradiction.

Hence R is triangle-free and $\langle S_i \rangle \cong 3K_1$ for each $i = 1, ..., 4$. Then (for example) $\langle S_1 \cup S_2 \rangle$ is a subgraph of $3K_2$ (the only possible edges join vertices in S_1 to vertices in S_2 and no vertex in S_i is adjacent to more than one vertex in S_j) and so $S_1 \cup S_2 \cup \{3,4\}$ is a $d8$, a contradiction. This completes the proof that there is no $t_1(4,8)$ Ramsey colouring of K_{17} and hence $t_1(4,8) = 17$. ■

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