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THE SUM NUMBER OF d-PARTITE COMPLETE HYPERGRAPHS

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Abstract

A d-uniform hypergraph $\mathcal{H}$ is a *sum hypergraph* iff there is a finite $S \subseteq \mathbb{N}^+$ such that $\mathcal{H}$ is isomorphic to the hypergraph $\mathcal{H}_d^+(S) = (V, \mathcal{E})$, where $V = S$ and $\mathcal{E} = \{\{v_1, \ldots, v_d\} : (i \neq j \Rightarrow v_i \neq v_j) \wedge \sum_{i=1}^{d} v_i \in S\}$. For an arbitrary d-uniform hypergraph $\mathcal{H}$ the *sum number* $\sigma = \sigma(\mathcal{H})$ is defined to be the minimum number of isolated vertices $w_1, \ldots, w_\sigma \notin V$ such that $\mathcal{H} \cup \{w_1, \ldots, w_\sigma\}$ is a sum hypergraph.

In this paper, we prove

$$\sigma(\mathcal{K}^d_{n_1,\ldots,n_d}) = 1 + \sum_{i=1}^{d}(n_i - 1) + \min\left\{0, \left\lceil \frac{1}{2}\left(\sum_{i=1}^{d-1}(n_i - 1) - n_d\right)\right\rceil\right\},$$

where $\mathcal{K}^d_{n_1,\ldots,n_d}$ denotes the d-partite complete hypergraph; this generalizes the corresponding result of Hartsfield and Smyth [8] for complete bipartite graphs.

Keywords: sum number, sum hypergraphs, d-partite complete hypergraph.

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1. Introduction and Definitions

The concept of sum graphs and integral sum graphs was introduced by Harary ([6], [7]). Many results for these kinds of graphs have been obtained in recent years, for a brief summary see for instance Sonntag and Teichert [12].

The graph theoretic concept mentioned above can be generalized to uniform hypergraphs as follows. All hypergraphs considered here are supposed

to be nonempty and finite without loops and multiple edges. In standard terminology we follow Berge [1].

A hypergraph $\mathcal{H} = (V, \mathcal{E})$ with vertex set V and edge set $\mathcal{E} \subseteq \mathcal{P}(V) - \{\emptyset\}$ is *d-uniform* iff $2 \leq d \in \mathbb{N}$ and $|e| = d$ $(\forall e \in \mathcal{E})$. Let $S \subseteq \mathbb{N}^+$ be finite. $\mathcal{H}_d^+(S) = (V, \mathcal{E})$ is called the *d-uniform sum hypergraph of* S iff $V = S$ and

$$\mathcal{E} = \left\{ \{v_1, v_2, \ldots, v_d\} : (i \neq j \Rightarrow v_i \neq v_j) \wedge \sum_{i=1}^{d} v_i \in S \right\}.$$

The d-uniform hypergraph $\mathcal{H}$ is a *sum hypergraph* iff there exists a set $S \subseteq \mathbb{N}^+$ such that $\mathcal{H} \cong \mathcal{H}_d^+(S)$. For $d = 2$ we obtain the known concept of sum graphs. For an arbitrary d-uniform hypergraph $\mathcal{H}$ the *sum number* $\sigma = \sigma(\mathcal{H})$ is defined to be the minimum number of isolated vertices $w_1, \ldots, w_\sigma \notin V$ such that $\mathcal{H} \cup \{w_1, \ldots, w_\sigma\}$ is a sum hypergraph. If also nonpositive integers are allowed as elements of S, i.e. $S \subseteq \mathbb{Z}$, we obtain the definitions of *integral sum hypergraphs* and the *integral sum number* $\zeta = \zeta(\mathcal{H})$ in the same manner.

As for graphs, the determination of the sum number (integral sum number) for certain classes of hypergraphs is an interesting question. The following results are known:

- If $\mathcal{T}_d$ denotes a d-uniform hypertree, then $d \geq 3$ implies $\sigma(\mathcal{T}_d) = 1$ and $\zeta(\mathcal{T}_d) = 0$ (Sonntag and Teichert ([12], [13])). Ellingham [5] proved for nontrivial trees $\mathcal{T}_2 = T$ that $\sigma(T) = 1$. Sharary [11] showed that all caterpillars are integral sum graphs and Chen [4] proved that the generalized stars and the trees in which any two distinct forks have distance at least four are integral sum graphs; both authors conjecture $\zeta(T) = 0$ for all trees but this problem remains still open.
- For the d-uniform complete hypergraph on n vertices $\mathcal{K}_n^d$ we obtain the sum number $\sigma(\mathcal{K}_n^d) = d(n-d) + 1$ for $n - 2 \geq d \geq 2$; this was shown by Sonntag and Teichert [13] for $d \geq 3$ and by Bergstrand et al. [2] for graphs $\mathcal{K}_n^2 = K_n$. Chen [3] as well as Sharary [10] showed that for complete graphs $\zeta(K_n) = \sigma(K_n)$ if $n \geq 4$. For $n-2 \geq d \geq 3$ Sonntag and Teichert [13] found bounds for $\zeta(\mathcal{K}_n^d)$ and conjectured that $\zeta(\mathcal{K}_n^d) = \sigma(\mathcal{K}_n^d)$ is true for hypergraphs too.

In this paper, we determine the sum number for a third class of uniform hypergraphs. As a generalization of complete bipartite graphs Berge [1] defined the *d-partite complete hypergraph* $\mathcal{K}_{n_1,\ldots,n_d}^d$ as follows: Let $X_1, X_2, \ldots, X_d$ be pairwise disjoint sets of cardinalities $n_1 \leq n_2 \leq \ldots \leq n_d$. The vertices of $\mathcal{K}_{n_1,\ldots,n_d}^d$ are the elements of $\bigcup_{i=1}^d X_i$ and the edges are all $\{v_1, v_2, \ldots, v_d\}$ with $v_i \in X_i$ for $i = 1, \ldots, d$.

For $x \in \mathbb{R}$ let $\lceil x \rceil$ denote the smallest integer $\geq x$. Hartsfield and Smyth [8] proved for complete bipartite graphs K_{n_1,n_2}

Theorem 1. *For given integers $n_1 \geq 2$ and $n_2 \geq n_1$ holds*

$$\sigma(K_{n_1,n_2}) = \left\lceil \frac{1}{2}(3n_1 + n_2 - 3) \right\rceil .$$

■

For the symmetric bipartite graph $K_{n,n}$ Miller et al. [9] showed

Theorem 2. $\zeta(K_{n,n}) = \sigma(K_{n,n}) \quad for \quad n \geq 2.$ ■

The problem to determine $\zeta(K_{n_1,n_2})$ for $n_1 \neq n_2$ remains still open.

In the following, we generalize Theorem 1. In Section 2, we prove several lemmata; for the determination of the sum number we distinguish two cases concerning the cardinality of the maximum vertex subset X_d of $\mathcal{K}^d_{n_1,\ldots,n_d}$. Summarizing these results, we give in Section 3 a general formula for the sum number of d-partite complete hypergraphs.

2. Two Cases for the Determination of the Sum Number

We use the following notations:
$X_1, \ldots, X_d$ is the vertex partition of the complete d-partite hypergraph $\mathcal{K}^d_{n_1,\ldots,n_d}$, where n_i denotes the cardinality of X_i and $n_1 \leq n_2 \leq \ldots \leq n_d$ is fulfilled. $\mathcal{E}$ is the set of edges of $\mathcal{K}^d_{n_1,\ldots,n_d}$ and Y is a set of isolated vertices such that for some labelling $\mathcal{K}^d_{n_1,\ldots,n_d} \cup Y$ can be recognized as a sum hypergraph. All vertices of $\bigcup_{i=1}^d X_i \cup Y$ are referenced by their labels.

Lemma 3. *There are $1+\sum_{i=1}^d (n_i-1)$ pairwise different sums $v_1+v_2+\ldots+v_d$ of vertices $v_i \in X_i,\ i = 1, \ldots, d$.*

Proof. Using the notation $X_i = \{v_i^1, \ldots, v_i^{n_i}\}$ with $v_i^k < v_i^l$ if $k < l$ for $i = 1, \ldots, d$ we consider the following sets of sums:

$$S_1 = \{v_1^j + v_2^1 + \ldots + v_d^1 : j \in \{1, \ldots, n_1\}\},$$

$$S_i = \{v_1^{n_1} + \ldots + v_{i-1}^{n_{i-1}} + v_i^j + v_{i+1}^1 + \ldots + v_d^1 : j \in \{2, \ldots, n_i\}\},\ i = 2, \ldots, d.$$

Clearly, $S_i \cap S_j = \emptyset$ for $i \neq j$ and $|S_1| = n_1,\ |S_i| = n_i - 1$ for $i = 2, \ldots, d$. Hence there are $1 + \sum_{i=1}^d (n_i - 1)$ pairwise different sums. ■

Lemma 4. *Consider a labelling of* $V' = \bigcup_{i=1}^{d} X_i \cup Y$ *such that* $\mathcal{K}^d_{n_1,\ldots,n_d} \cup Y$ *is a sum hypergraph and let* $v_d^* \in X_d$ *be arbitrarily chosen. Then*

$$(1) \qquad \left(\left\{ \sum_{i=1}^{d-1} v_i + v_d^* : v_i \in X_i \right\} \subseteq X_d \right) \vee \left(\left\{ \sum_{i=1}^{d-1} v_i + v_d^* : v_i \in X_i \right\} \subseteq Y \right)$$

Proof. 1. Suppose there are $v_i^* \in X_i,\ i = 1, \ldots, d-1$, such that $\sum_{i=1}^{d} v_i^* = \tilde{v}_d \in X_d$ and let $v_i \in X_i\ i = 1, \ldots, d-1$, be arbitrarily chosen. In part 1, we show that this implies $v_d' := \sum_{i=1}^{d-1} v_i + v_d^* \in X_d$.

Because of $\{v_1, \ldots, v_{d-1}, \tilde{v}_d\} \in \mathcal{E}$ there is a vertex

$$v'' = \sum_{i=1}^{d-1} v_i + \tilde{v}_d = \sum_{i=1}^{d-1} v_i + \sum_{i=1}^{d} v_i^* = \sum_{i=1}^{d-1} v_i^* + v_d'.$$

Hence $\{v_1^*, \ldots, v_{d-1}^*, v_d'\} \in \mathcal{E}$ and by the definition of the d-partite complete hypergraph it follows $v_d' \in X_d$.

2. Suppose there are $v_i^* \in X_i,\ i = 1, \ldots, d-1$, such that $\sum\limits_{i=1}^{d} v_i^* = \tilde{v}_j \in X_j$ for $j \in \{1, \ldots, d-1\}$. It follows analogously to part 1 that

$$\forall\ i \in \{1, \ldots, j-1, j+1, \ldots, d\}\ \forall\ v_i \in X_i : \sum_{\substack{i=1 \\ i \neq j}}^{d} v_i + v_j^* \in X_j.$$

By Lemma 3 there are at least $1 + \sum_{\substack{i=1 \\ i \neq j}}^{d} (n_i - 1)$ pairwise different sums each containing v_j^*. With $v_j^* \in X_j$ we obtain

$$n_j = |X_j| \geq 1 + \left(1 + \sum_{\substack{i=1 \\ i \neq j}}^{d} (n_i - 1) \right) \geq 1 + n_d,$$

which contradicts the supposition $n_j \leq n_d$ made in the beginning of this chapter. Hence this case is impossible.

3. Suppose there are $v_i^* \in X_i,\ i = 1, \ldots, d-1$, such that $\sum_{i=1}^{d} v_i^* \in Y$. With parts 1, 2 we obtain $\sum_{i=1}^{d-1} v_i + v_d^* \notin \bigcup_{i=1}^{d} X_i$ for arbitrary vertices $v_i \in X_i$, $i = 1, \ldots, d-1$. By the sum hypergraph property we obtain $\sum_{i=1}^{d-1} v_i + v_d^* \in Y$ and this proves (1). ∎

If $\hat{V} \subseteq \bigcup_{i=1}^{d} X_i \cup Y$ denotes the subset of vertices representing sums $\sum_{i=1}^{d} v_i$ where $\{v_1, \ldots, v_d\} \in \mathcal{E}$ it follows from Lemma 4 that only the cases $\hat{V} = Y$ (case 1) or $\hat{V} \cap X_d \neq \emptyset$ (case 2) are possible. Clearly, it depends on the cardinality n_d of X_d whether the first case appears or the second one is possible.

Case 1. $n_d \leq 1 + \sum_{i=1}^{d-1}(n_i - 1)$ (i.e., $\hat{V} = Y$).

Lemma 5. *If* $n_d \leq 1 + \sum_{i=1}^{d-1}(n_i - 1)$ *then*

$$\sigma(\mathcal{K}^d_{n_1,\ldots,n_d}) \geq 1 + \sum_{i=1}^{d}(n_i - 1). \tag{2}$$

Proof. Let $v_d^* \in X_d$ be arbitrarily chosen and assume $\sum_{i=1}^{d-1} v_i + v_i^* \in X_d$ for some $v_i \in X_i,\ i = 1, \ldots, d-1$. Then Lemma 4 implies that each sum containing v_d^* belongs to X_d and from Lemma 3 follows $n_d \geq 1 + \left(1 + \sum_{i=1}^{d-1}(n_i - 1)\right)$, a contradiction. Thus $\hat{V} \subseteq Y$ and with Lemma 3 we obtain (2). ∎

Our aim is to show that equality is fulfilled in (2). For this purpose we describe an appropriate labelling:

Lemma 6. *Let* $\bigcup_{i=1}^{d} X_i \cup Y$ *be labelled as follows:*

$$\begin{aligned} X_i &= \{x_i + 1, \ldots, x_i + n_i\}, \quad i = 1, \ldots, d; \\ Y &= \left\{\sum_{i=1}^{d}(x_i + 1), \ldots, \sum_{i=1}^{d}(x_i + n_i)\right\} \\ &\quad \text{with } x_i = 10^{t+i};\ i = 1, \ldots, d, \text{ where } t \geq \lg\,(dn_d). \end{aligned} \tag{3}$$

Then the resulting sum hypergraph consists of $\mathcal{K}^d_{n_1,\ldots,n_d}$ with vertex set $\bigcup_{i=1}^{d} X_i$ and $|Y| = 1 + \sum_{i=1}^{d}(n_i - 1)$ isolated vertices.

Proof. Clearly, (3) implies $Y = \hat{V} = \left\{\sum_{i=1}^{d} v_i : v_i \in X_i,\ i = 1, \ldots, d\right\}$ and $|Y| = 1 + \sum_{i=1}^{d}(n_i - 1)$.

It remains to show that the labelling does not induce too many edges by the sum hypergraph property, i.e., if $\sum_{i=1}^{d} v^i \in Y$ then $\{v^1, \ldots, v^d\} \in \mathcal{E}$ follows. Let $\alpha_1, \ldots, \alpha_{d+t+1}$ be the digits of the label $\alpha_1\alpha_2 \ldots \alpha_{d+t+1}$ of a vertex v. The condition $t \geq \lg\,(dn_d)$ implies

(4)
- if $v \in X_i$ $(i \in \{1, \ldots, d\})$, then
  $\alpha_{d-i+1} = 1;\ \alpha_j = 0 \quad \text{for} \quad j \in \{1, \ldots, d\} - \{d - i + 1\},$
- if $v \in Y$, then
  $\alpha_j = 1 \quad \text{for} \quad j \in \{1, \ldots, d\}.$

Thus we have for arbitrary pairwise different vertices $v^1, \ldots, v^d \in \bigcup_{i=1}^{d} X_i \cup Y$ that $\sum_{i=1}^{d} v^i \notin X_i,\ i = 1, \ldots, d$. Further in case of $\sum_{i=1}^{d} v^i \in Y$, neither $v^k, v^l \in X_i$ nor $v^k \in Y$ for some $i, k, l \in \{1, \ldots, d\}, k \neq l$ is possible, because this would imply $\alpha_j \neq 1$ for at least one $j \in \{1, \ldots, d\}$ in both of the cases, a contradiction to (4). ∎

Summarizing the results of Lemmata 5, 6, we have shown

Theorem 7. *If* $n_d \leq 1 + \sum_{i=1}^{d-1}(n_i - 1)$, *then*

$$\sigma(\mathcal{K}^d_{n_1,\ldots,n_d}) = 1 + \sum_{i=1}^{d}(n_i - 1); \tag{5}$$

especially for $n_1 = \ldots = n_d = n$ *we have*

$$\sigma(\mathcal{K}^d_{n,\ldots,n}) = 1 + d(n - 1). \tag{6}$$

∎

Notice that for complete bipartite graphs K_{n_1,n_2} because of $n_1 \leq n_2$ the supposition of Theorem 7 can only be true for $n_1 = n_2 = n$ and this leads to $\sigma(K_{n,n}) = 2n - 1$ as a special case of (6) which corresponds to the value given in Theorem 1.

Case 2. $n_d \geq 2 + \sum_{i=1}^{d-1}(n_i - 1)$ (i.e. $\hat{V} \cap X_d \neq \emptyset$).

We introduce the notations

$$X'_d = \{v'_d \in X_d \ \forall\ i \in \{1, \ldots, d-1\}\ \exists\ v'_i \in X_i : \sum_{i=1}^{d} v'_i \in X_d\}, \tag{7}$$

$$X''_d = \{v''_d \in X_d \ \forall\ i \in \{1, \ldots, d\}\ \exists\ v'_i \in X_i \ : v''_d = \sum_{i=1}^{d} v'_i\}. \tag{8}$$

The aim of this section is the construction of a labelling that reduces the cardinality of Y (below the value given in (5)) to a minimum. Due to Lemma 4 this is only possible by the maximization of $|X'_d|$.

In the following, we suppose that at least two vertex subsets X_i contain two or more vertices, i.e.,

$$2 \leq n_{d-1} \leq n_d \tag{9}$$

(otherwise we obtain $\sigma(\mathcal{K}^d_{n_1,\ldots,n_d}) = 1$ immediately).

Lemma 8. *If a labelling of $\mathcal{K}^d_{n_1,\ldots,n_d}$ generates a maximum cardinality $|X'_d|$, then*

$$X'_d \cap X''_d = \emptyset. \tag{10}$$

Proof. We prove (10) by induction on $n_d = |X_d|$. For $n_d \leq 1+\sum_{i=1}^{d-1}(n_i-1)$ we have shown $X'_d = \emptyset$ in Case 1, hence (10) is true. Now suppose that (10) is valid for some $n_d \geq 2 + \sum_{i=1}^{d-1}(n_i - 1)$ and form $\tilde{X}_d$ by adding one new vertex to X_d. Then $\tilde{n}_d = |\tilde{X}_d| = n_d + 1$, further we denote by $\tilde{X}'_d \subseteq \tilde{X}_d$ and $\tilde{X}''_d \subseteq \tilde{X}_d$ the sets corresponding to (7) and (8), respectively. In the following, we have to show that $\tilde{X}'_d \cap \tilde{X}''_d = \emptyset$.

1. We show

$$|X'_d| \leq |\tilde{X}'_d| \leq |X'_d| + 1. \tag{11}$$

The first inequality is obvious. Let $v_d^{\min} = \min\{v'_d : v'_d \in \tilde{X}'_d\}$, then

$$v_d^{\min} \notin \tilde{X}''_d. \tag{12}$$

Now assume that $|\tilde{X}'_d| \geq |X'_d|+2$ is fulfilled and consider $\tilde{X}'_d - \{v_d^{\min}\}$. Then $|\tilde{X}_d - \{v_d^{\min}\}| = n_d$ and with (12) we obtain $|\tilde{X}'_d - \{v_d^{\min}\}| \geq |X'_d| + 1$, a contradiction to the maximality of $|X'_d|$, hence (11) is true.

2. Now we prove

$$|\tilde{X}'_d| = |X'_d| \Rightarrow \tilde{X}'_d \cap \tilde{X}''_d = \emptyset. \tag{13}$$

With $v_d^{\max} = \max\{v''_d : v''_d \in \tilde{X}''_d\}$ we obtain

$$v_d^{\max} \notin \tilde{X}'_d. \tag{14}$$

Consider $\hat{X}_d := \tilde{X}_d - \{v_d^{\max}\}$ and form the sets $\hat{X}'_d$ and $\hat{X}''_d$ corresponding to (7) and (8) respectively. Then $\hat{X}'_d \subseteq \tilde{X}'_d$ and next we show that even equality is fulfilled:

Consider an arbitrary $\hat{v}_d \in \tilde{X}'_d$. Then there are $\hat{v}_i \in X_i,\ i = 1, \ldots, d-1$, such that $\sum_{i=1}^{d} \hat{v}_i = \bar{v}_d \in \tilde{X}''_d$. If $\bar{v}_d \neq v_d^{\max}$ then $\bar{v}_d \in \hat{X}''_d$ and hence $\hat{v}_d \in \hat{X}'_d$. If $\bar{v}_d = v_d^{\max}$, we choose $\hat{\hat{v}}_i \in X_i,\ i = 1, \ldots, d-1$, and because of (9) we can suppose that $\hat{v}_j \neq \hat{\hat{v}}_j$ for exactly one $j \in \{1, \ldots, d-1\}$. Lemma 4 implies $\sum_{i=1}^{d-1} \hat{\hat{v}}_i + \hat{v}_d = \bar{\bar{v}}_d \in \tilde{X}''_d$ and because of $\bar{\bar{v}}_d \neq \bar{v}_d = v_d^{\max}$ we have $\bar{\bar{v}}_d \in \hat{X}''_d$, hence $\hat{v}_d \in \hat{X}'_d$ again.
Together this yields

$$\hat{X}'_d = \tilde{X}'_d. \tag{15}$$

From the left side of (13) follows $|\hat{X}'_d| = |X'_d|$. Because of $|\tilde{X}_d - \{v_d^{\max}\}| = n_d$ we obtain by the induction hypothesis $\hat{X}'_d \cap \hat{X}''_d = \emptyset$. Using (15) and (14) this leads to

$$\emptyset = \hat{X}'_d \cap \hat{X}''_d = \tilde{X}'_d \cap \hat{X}''_d = \tilde{X}'_d \cap (\hat{X}''_d \cup v_d^{\max}) = \tilde{X}'_d \cap \tilde{X}''_d.$$

3. Because of (11) and (13) we can suppose

$$|\tilde{X}'_d| = |X'_d| + 1 \tag{16}$$

in the following. Next we prove that

$$|\tilde{X}'_d \cap \tilde{X}''_d| \leq 1 \tag{17}$$

must be true.

Assume $|\tilde{X}'_d \cap \tilde{X}''_d| \geq 2$ and let $v_d^1, v_d^2 \in \tilde{X}'_d \cap \tilde{X}''_d$ be two distinct vertices. Further let $v_d^{\min} \in \tilde{X}'_d - \tilde{X}''_d$ be defined as in part 1 of the proof and consider $\check{X}_d = \tilde{X}_d - \{v_d^{\min}\}$ with the subsets $\check{X}'_d$ and $\check{X}''_d$ formed corresponding to (7) and (8), respectively. Then $|\check{X}_d| = n_d$ and (16) implies $|\check{X}'_d| = |X'_d|$. Using the induction hypothesis we obtain $\check{X}'_d \cap \check{X}''_d = \emptyset$, i.e., the deletion of $v_d^{\min}$ in $\tilde{X}'_d$ causes that $v_d^1, v_d^2 \notin \check{X}''_d$. Hence

$$\forall\ i \in \{1, \ldots, d-1\}\ \exists\ v_i^1, v_i^2 \in X_i :$$

$$\sum_{i=1}^{d-1} v_i^1 + v_d^{\min} = v_d^1 \ \wedge\ \sum_{i=1}^{d-1} v_i^2 + v_d^{\min} = v_d^2 \tag{18}$$

and

$$\forall\, i \in \{1,\dots,d-1\}\ \forall\, v_i' \in X_i\ \forall v_d' \in \tilde{X}_d' :$$

$$v_d' \neq v_d^{\min} \Rightarrow \left(\sum_{i=1}^{d} v_i' \neq v_d^1\ \wedge\ \sum_{i=1}^{d} v_i' \neq v_d^2\right). \tag{19}$$

These facts are useful for the consideration of $\underline{X}_d = \tilde{X}_d - \{v_d^2\}$ with the subsets $\underline{X}_d'$ and $\underline{X}_d''$ formed corresponding to (7) and (8), respectively: Using (18) and (19) we obtain $v_d^1 \in \underline{X}_d''$ and $v_d^1 \in \underline{X}_d'$, respectively. Hence

$$v_d^1 \in \underline{X}_d' \cap \underline{X}_d''. \tag{20}$$

On the other hand, we obtain from (18) and (19) that $v_d^{\min} \in \underline{X}_d'$ and $\tilde{X}_d' - \{v_d^2\} \subseteq \underline{X}_d'$, respectively, i.e., $\underline{X}_d' = \tilde{X}_d' - \{v_d^2\}$ and with (16) we obtain $|\underline{X}_d'| = |X_d'|$. Because of $|\underline{X}_d| = |X_d| = n_d$ the induction hypothesis yields $\underline{X}_d' \cap \underline{X}_d'' = \emptyset$; a contradiction to (20) which implies the validity of (17).

4. We conclude the proof by constructing a contradiction to (17) in case of $\tilde{X}_d' \cap \tilde{X}_d'' \neq \emptyset$. Suppose $\hat{v}_d \in \tilde{X}_d' \cap \tilde{X}_d''$, then

$$\forall\, i \in \{1,\dots,d-1\}\ \exists\, v_i' \in X_i\ \exists\, v_d' \in \tilde{X}_d' : \hat{v}_d = \sum_{i=1}^{d} v_i'. \tag{21}$$

Consider vertices $v_i^1, v_i^2 \in X_i$ for $i = 1,\dots,d$. By (9) we can suppose $v_k^1 \neq v_k^2$ for at least one $k \in \{1,\dots,d-1\}$. Using Lemma 4 and $\hat{v}_d \in \tilde{X}_d'$ we obtain $\sum_{i=1}^{d-1} v_i^j + \hat{v}_d \in \tilde{X}_d''$, $j = 1,2$ and with (21) this yields

$$\sum_{i=1}^{d-1} v_i^j + \sum_{i=1}^{d} v_i' = \left(\sum_{i=1}^{d-1} v_i^j + v_d'\right) + \sum_{i=1}^{d-1} v_i' \in \tilde{X}_d'' \quad \text{for} \quad j = 1,2.$$

Hence

$$\sum_{i=1}^{d-1} v_i^j + v_d' \in \tilde{X}_d',\ j = 1,2. \tag{22}$$

On the other hand, $v_d' \in \tilde{X}_d'$ implies $\sum_{i=1}^{d-1} v_i^j + v_d' \in \tilde{X}_d''$ for $j = 1,2$ and together with (22) it follows $|\tilde{X}_d' \cap \tilde{X}_d''| \geq 2$; a contradiction to (17). Hence $\tilde{X}_d' \cap \tilde{X}_d'' = \emptyset$ and the proof of Lemma 8 is completed. ∎

Lemma 9. *If* $n_d \geq 2 + \sum_{i=1}^{d-1}(n_i - 1)$, *then*

$$\sigma(\mathcal{K}^d_{n_1,\ldots,n_d}) \geq \left\lceil \frac{1}{2}\left(3\sum_{i=1}^{d-1}(n_i - 1) + n_d\right)\right\rceil . \tag{23}$$

Proof. As mentioned before, we have to maximize $n'_d = |X'_d|$ to obtain a minimum number $|Y|$ of isolated vertices. Lemma 8 yields $X''_d \subseteq X_d - X'_d$ and with Lemmata 3, 4 we obtain $n_d - n'_d \geq n'_d + \sum_{i=1}^{d-1}(n_i - 1)$, i.e.,

$$n'_d \leq \frac{1}{2}\left(n_d - \sum_{i=1}^{d-1}(n_i - 1)\right). \tag{24}$$

Again using Lemma 3 we obtain with (24) a bound for the number of different sums containing elements of $X_1, \ldots, X_{d-1}, X_d - X'_d$:

$$|Y| \geq 1 + \sum_{i=1}^{d-1}(n_i - 1) + (n_d - n'_d - 1) \geq \frac{1}{2}\left(3\sum_{i=1}^{d-1}(n_i - 1) + n_d\right),$$

hence (23) is fulfilled. ∎

As in case 1 we next describe a labelling that provides equality in (23).

First of all suppose that $n_d - \sum_{i=1}^{d-1}(n_i - 1)$ is even and choose $n'_d = \frac{1}{2}(n_d - \sum_{i=1}^{d-1}(n_i - 1))$ which is by (24) the maximum possible value. Now let $\bigcup_{i=1}^{d} X_i \cup Y$ be labelled as follows:

$$\begin{aligned} X_i &= \{x_i + 1, \ldots, x_i + n_i\}\ ,\ i = 1, \ldots, d-1, \\ X'_d &= \{x_d + 1, \ldots, x_d + n'_d\}, \\ X''_d &= \left\{\sum_{i=1}^{d}(x_i + 1), \ldots, \sum_{i=1}^{d-1}(x_i + n_i) + x_d + n'_d\right\}, \end{aligned} \tag{25}$$

$$Y = \left\{\sum_{i=1}^{d-1}(x_i + 1) + \sum_{i=1}^{d}(x_i + 1), \ldots, \sum_{i=1}^{d-1}(x_i + n_i) + \sum_{i=1}^{d-1}(x_i + n_i) + x_d + n'_d\right\}$$

with $x_i = 10^{t+i},\ i = 1, \ldots, d$, where $t \geq \lg\,((2d-1)n_d)$.

A simple calculation yields $|X'_d| + |X''_d| = |X_d|$ and $|Y| = \frac{1}{2}(3\sum_{i=1}^{d-1}(n_i - 1) + n_d)$. The labelling for odd values of $n_d - \sum_{i=1}^{d-1}(n_i - 1)$ can be obtained

from (25) (constructed for n_d+1 vertices) by deleting any vertex of X'_d. For this case follows $|Y| = \frac{1}{2}(3\sum_{i=1}^{d-1}(n_i-1)+n_d+1)$.

Summarizing the results we obtain:

$$|Y| = \left\lceil \frac{1}{2}\left(3\sum_{i=1}^{d-1}(n_i-1)+n_d\right)\right\rceil . \tag{26}$$

Similarly to case 1, Lemma 6, we will show

Lemma 10. *If $\bigcup_{i=1}^{d} X_i \cup Y$ is labelled according to* (25) *the resulting sum hypergraph consists of $\mathcal{K}^d_{n_1,\ldots,n_d}$ with vertex set $\bigcup_{i=1}^{d} X_i$ and $|Y|$ isolated vertices.*

Proof. Clearly the labelling generates the edge set $\mathcal{E}$ of $\mathcal{K}^d_{n_1,\ldots,n_d}$ and we have to show that (25) does not induce too many edges by the sum hypergraph property. Again let $\alpha_1\alpha_2\ldots\alpha_{d+t+1}$ be a vertex with digits $\alpha_1,\ldots,\alpha_{d+t+1}$. Because of $t \geq \lg((2d-1)n_d)$ we obtain three types of vertices:

- for any vertex of

$$\tilde{X}_i = \begin{cases} X_i\,, & \text{for} \quad i \in \{1,\ldots,d-1\}, \\ X'_d\,, & \text{for} \quad i = d: \end{cases}$$

$$\alpha_{d-i+1} = 1\ ;\ \alpha_j = 0 \quad \text{for} \quad j \in \{1,\ldots,d\} - \{d-i+1\}, \tag{27}$$

- for any vertex of $X''_d : \alpha_j = 1 \quad \text{for} \quad j \in \{1,\ldots,d\}$,
- for any vertex of $Y : \alpha_1 = 1\ ;\ \alpha_j = 2 \quad \text{for} \quad j \in \{2,\ldots,d\}$.

Let $s = \sum_{i=1}^{d} v^i$ be a sum of pairwise disjoint vertices of $\bigcup_{i=1}^{d} X_i \cup Y$. We use the notation $S = \{v^1,\ldots,v^d\}$ and distinguish the following cases:

(A) $|S \cap Y| = 1$,
(B) $|S \cap Y| \geq 2$,
(C) $S \cap Y = \emptyset$,
  (C1) $|S \cap X_d| \geq 2$,
  (C2) $|S \cap X_d| \leq 1 \wedge \exists\ m \in \{1,\ldots,d-1\} : |S \cap X_m| \geq 2$,
    (C21) $S \cap X''_d \neq \emptyset$,
    (C22) $S \cap X''_d = \emptyset$.

Using (27) it can easily be seen that for each of these cases the following holds:

$$\begin{array}{ll} & \alpha_1 \geq 2 \\ \text{or} & \alpha_i \geq 3 \quad \text{for at least one} \quad i \in \{2,\ldots,d\} \\ \text{or} & \alpha_i = 0,\ \alpha_j \geq 2 \quad \text{for some} \quad i,j \in \{2,\ldots,d\}, \end{array}$$

i.e., $s \notin \bigcup_{i=1}^{d} X_i \cup Y$. Hence $s \in \bigcup_{i=1}^{d} X_i \cup Y$ iff $\{v^1,\ldots,v^d\} \in \mathcal{E}$. ∎

Summarizing the results of Lemmata 8, 9 and (26), we have shown

Theorem 11. *If* $n_d \geq 2 + \sum_{i=1}^{d-1}(n_i - 1)$, *then*

$$\sigma(\mathcal{K}^d_{n_1,\ldots,n_d}) = \left\lceil \frac{1}{2}\left(3\sum_{i=1}^{d-1}(n_i-1) + n_d\right)\right\rceil . \tag{28}$$

∎

3. Main Result

In case of $n_d \geq 2 + \sum_{i=1}^{d-1}(n_i - 1)$ we obtain with (28)

$$\sigma(\mathcal{K}^d_{n_1,\ldots,n_d}) = \left(1 + \sum_{i=1}^{d}(n_i-1)\right) + \left\lceil \frac{1}{2}\left(\sum_{i=1}^{d-1}(n_i-1) - n_d\right)\right\rceil .$$

Here the first summand is the sum number (5) for the case $n_d \leq 1 + \sum_{i=1}^{d-1}(n_i - 1)$ given in Theorem 7 and the second one is negative for $n_d \geq 2 + \sum_{i=1}^{d-1}(n_i - 1)$ but nonnegative for $n_d \leq 1 + \sum_{i=1}^{d-1}(n_i - 1)$. This yields the main result:

Theorem 12. *For* $d \geq 2$ *and* $2 \leq n_{d-1} \leq n_d$ *the sum number of the d-partite complete hypergraph is given by*

$$\sigma(\mathcal{K}^d_{n_1,\ldots,n_d}) = 1 + \sum_{i=1}^{d}(n_i-1) + \min\left\{0, \left\lceil \frac{1}{2}\left(\sum_{i=1}^{d-1}(n_i-1) - n_d\right)\right\rceil\right\}.$$

∎

Obviously, for $d = 2$, because of $(n_1 - 1) - n_2 \leq -1$ we obtain the value given in Theorem 1.

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