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THE CROSSING NUMBERS OF PRODUCTS OF A 5-VERTEX GRAPH WITH PATHS AND CYCLES

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Abstract

There are several known exact results on the crossing numbers of Cartesian products of paths, cycles or stars with "small" graphs. Let H be the 5-vertex graph defined from K_5 by removing three edges incident with a common vertex. In this paper, we extend the earlier results to the Cartesian products of $H \times P_n$ and $H \times C_n$, showing that in the general case the corresponding crossing numbers are $3n - 1$, and $3n$ for even n or $3n + 1$ if n is odd.

Keywords: graph, drawing, crossing number, path, cycle, Cartesian product.

1991 Mathematics Subject Classification: 05C10, 05C38.

1. Introduction

Let G be a simple graph with vertex set V and edge set E. The *crossing number* $\nu(G)$ of a graph G is the smallest number of pairs of nonadjacent edges that intersect in any drawing of G in the plane. It is implicit that the edges in a drawing are Jordan arcs (hence, nonselfintersecting), and it is easy to see that a drawing with the minimum number of crossings (an optimal drawing) must be a *good* drawing; that is, each two edges have at most one point in common, which is either a common end-vertex or a crossing. For a detailed account concerning this topic, the reader is referred to [3] and [10]. Let D be a good drawing of the graph G. We denote the number of crossings in D by $\nu_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote

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by $\nu_D(G_i, G_j)$ the number of crossings between edges of G_i and edges of G_j, and by $\nu_D(G_i)$ the number of crossings among edges of G_i in D.

The *Cartesian product* $G_1 \times G_2$ of graphs G_1 and G_2 has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set

$$E(G_1 \times G_2) = \{\{(u_i, v_j), (u_h, v_k)\} : \ (u_i = u_h \text{ and } \{v_j, v_k\} \in E(G_2))$$

$$\text{or } (\{u_i, u_h\} \in E(G_1) \text{ and } v_j = v_k)\}.$$

Let C_n and P_n be the *cycle* and the *path* with n edges, and S_n the *star* $K_{1,n}$. In [2] and [4] are determined the crossing numbers of the Cartesian products of all 4-vertex graphs with cycles and in [5] and [6] with paths and stars. It thus seems natural to inquire about the crossing numbers of the products of 5-vertex graphs with cycles, paths or stars. In [5], [8], and [9] it is shown that $\nu(S_4 \times P_n) = 2(n-1)$, $\nu(S_4 \times C_n) = 2n$, $\nu(K_{2,3} \times P_n) = 2n$, $\nu(K_{2,3} \times S_n) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n$, and $\nu(C_5 \times C_n) = 3n$. Let G_1, G_2, and G_3 be the three graphs of order five defined by removing from K_5 the edges of an elementary subdivision of $K_{1,3}$, the edges of K_3, and the edges of $K_{1,2}$, respectively. In [7] it is shown that $\nu(G_1 \times P_n) = 2n - 2$ and $\nu(G_2 \times P_n) = \nu(G_3 \times P_n) = 3n - 1$. Let H be the 5-vertex graph defined from K_5 by removing three edges incident with a common vertex. In this paper, we extend the earlier results to the products of H with a path P_n and a cycle C_n, showing that in the general case the corresponding crossing numbers are $3n-1$, and $3n$ for even n or $3n+1$ if n is odd.

2. The Crossing Number of $H \times P_n$

We assume $n \geq 1$ and find it convenient to consider the graph $H \times P_n$ in the following way. It has $5(n+1)$ vertices, which we denote x_i for $x = a, b, c, d, e$ and $i = 0, 1, \ldots, n$, and $12n + 7$ edges that are the edges in the $n+1$ copies H^i and the five paths $x_0 x_1 ... x_n$ (see Figure 1). Furthermore, we call the former edges red and the latter ones blue.

For $i = 0, 1, \ldots, n$, let d_i and e_i be the vertices of H^i of degree four and degree one, respectively. We denote by K_4^i the subgraph of H^i induced by the vertices a_i, b_i, c_i, and d_i. Let T^i, $i = 1, 2, \ldots, n$, be the subgraph of the graph $H \times P_n$ with the vertices of K_4^{i-1} and K_4^i and the blue edges joining K_4^{i-1} to K_4^i. For $i = 1, 2, \ldots, n-1$, let Q^i denote the subgraph of $H \times P_n$ induced by the vertices in K_4^{i-1}, K_4^i, and K_4^{i+1}. Thus, Q^i has 18 red edges in K_4^{i-1}, K_4^i, and K_4^{i+1} and 8 blue edges in T^i and T^{i+1}. Clearly, Q^i is isomorphic to $K_4 \times P_2$. In a good drawing of $K_4 \times P_n$, we define the *force* $f(Q^i)$ of Q^i to be the total number of crossings of the following types:

(1) a crossing of a blue edge in $T^i \cup T^{i+1}$ with an edge in K_4^i;

(2) a crossing of a blue edge in T^i with a blue edge in T^{i+1}; and

(3) an internal crossing in K_4^i (a crossing among red edges of K_4^i).

A moment's thought shows that no crossing counted in $f(Q^i)$ is counted in $f(Q^j)$ if $i \neq j$. The *totalforce* of the drawing of $K_4 \times P_n$ is the sum of these forces.

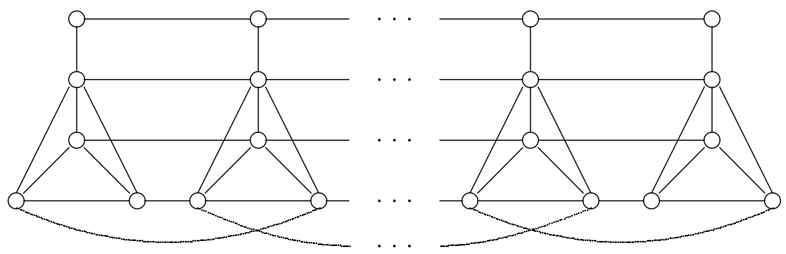

Figure 1

We say that a good drawing of $K_4 \times P_n$ is *coherent* if each K_4^i (whether or not it has an internal crossing) has the property that all the other vertices of the graph lie in the same "region" in the view of the subdrawing of K_4^i. (The possible crossings are considered to be vertices of the map.)

The graph K_4 is 3-connected. In this paper, we will often use the following facts: If two different K_4^i and K_4^j cross each other, then in any good drawing they cross at least three times. Consider a good and coherent drawing of $K_4 \times P_n$. In such a drawing red edges of two different K_4^i and K_4^j cannot cross each other. As K_4 is not outerplanar, either every K_4^i has an internal crossing or it is crossed by a blue edge incident with K_4^i. Moreover, for $i = 1, 2, \dots, n-1$, if K_4^i has no internal crossing, then the edges of $T^i \cup T^{i+1}$ cross K_4^i at least twice. If some K_4^i is crossed by a blue edge not incident with K_4^i, then this edge crosses K_4^i at least three times.

Lemma 1. *Let D be a good and coherent drawing of $K_4 \times P_2$. If $\nu_D(K_4^0, T^2) = 0$ and $\nu_D(K_4^2, T^1) = 0$, then D has force at least three. Moreover, if in D there are two adjacent edges of $T^1 \cup T^2$ without crossings, then D has force at least four or $\nu_D(K_4^1, T^1 \cup T^2) = 2$ and $\nu_D(T^1, T^2) = 1$.*

Proof. The graph $K_4 \times P_2$ we can denote by Q^1. Let us denote by Q_c^1 the graph obtained from Q^1 by contracting K_4^0 to the vertex k_4^0 and K_4^2

to the vertex k_4^2. Thus, $Q_c^1 = K_4^1 \cup T_c^1 \cup T_c^2$, where T_c^1 (T_c^2) consists of four edges incident with the vertex k_4^0 (k_4^2). Let D_c be the good drawing of Q_c^1 induced by D. The drawing D is coherent, so in D_c the vertices k_4^0 and k_4^2 lie in the same region in the view of the subdrawing of K_4^1. As $\nu_D(K_4^0, T^2) = \nu_D(K_4^2, T^1) = 0$ and all crossings in the good drawing D_c are counted in $f(Q^1)$ in the drawing D, then $f(Q^1) \geq \nu_{D_c}(Q_c^1)$. Thus, it remains to show that $\nu_{D_c}(Q_c^1) \geq 3$. The subgraph $K_4^1 \cup T_c^1$ of Q_c^1 is isomorphic to K_5, and in [1] it is shown that every good drawing of K_5 has an odd number of crossings. Consider the subdrawing of $K_4^1 \cup T_c^1$ induced by D_c. If it has more than two crossings, we are done. Suppose now that it has one crossing. Since the optimal drawing of K_5 is unique within isomorphism, this subdrawing creates the map with eight regions in such a way that there are at most three vertices of $K_4^1 \cup T_c^1$ on the boundary of every region. In D_c the vertex k_4^2 lies in the region with the vertex k_4^0 on its boundary in the view of the subdrawing of $K_4^1 \cup T_c^1$. Therefore, at most two vertices of K_4^1 are on the boundary of this region, and, in D_c, the edges of T_c^2 cross the edges of $K_4^1 \cup T_c^1$ at least twice. So, in D_c there are at least three crossings.

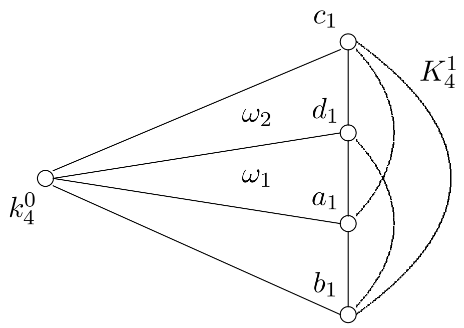

Figure 2

Now we show that if D has force three and two adjacent edges of $T^1 \cup T^2$ incident with the same vertex of K_4^1 are not crossed, then $\nu_D(K_4^1, T^1 \cup T^2) = 2$ and $\nu_D(T^1, T^2) = 1$. Without loss of generality, assume the edges $\{d_0, d_1\}$ and $\{d_1, d_2\}$ are not crossed in D. Thus, in D_c, the edges $\{k_4^0, d_1\}$ and $\{d_1, k_4^2\}$ are not crossed. First, we suppose that the edges of K_4^1 cross each other in D_c. (In a good drawing they cannot cross more than once.) The subgraphs $K_4^1 \cup T_c^1$ and $K_4^1 \cup T_c^2$ are isomorphic to K_5. As there is no good drawing of K_5 with two crossings, the condition $f(Q^1) = \nu_D(Q_c^1) = 3$ implies that for some i, $i \in \{1, 2\}$, $\nu_{D_c}(K_4^1, T_c^i) = 0$. Suppose $\nu_{D_c}(K_4^1, T_c^1) = 0$.

The subdrawing of $K_4^1 \cup T_c^1$ induced by D_c divides the plane as shown in Figure 2, and, in D_c, the vertex k_4^2 lies in the region ω_1 (ω_2) of the subdrawing. The edges of T_c^2 cannot cross the edge $\{k_4^0, d_1\}$ and it is easy to see that in D_c the edge $\{k_4^2, b_1\}$ crosses an edge of the triangle $k_4^0 d_1 a_1$ ($k_4^0 d_1 c_1$) and the edge $\{k_4^2, c_1\}$ ($\{k_4^2, a_1\}$) crosses edges of triangles $k_4^0 d_1 a_1$ and $k_4^0 d_1 b_1$ ($k_4^0 d_1 c_1$ and $k_4^0 d_1 b_1$). This contradicts the assumption $\nu_{D_c}(Q_c^1) = 3$. Therefore, $\nu_{D_c}(K_4^1) = 0$, and since K_4 is not an outerplanar graph, both T_c^1 and T_c^2 cross K_4^1 in D_c. Moreover, neither T^1 nor T^2 crosses K_4^1 twice, since otherwise we obtain as a subdrawing the complete graph K_5 with two crossings. Hence $\nu_{D_c}(K_4^1, T_c^1) = \nu_{D_c}(K_4^1, T_c^2) = 1$, and from the condition $\nu_{D_c}(Q_c^1) = 3$, it follows that $\nu_{D_c}(T_c^1, T_c^2) = 1$. This implies that in the drawing D, $\nu_D(K_4^1, T^1 \cup T^2) = 2$ and $\nu_D(T^1, T^2) = 1$, as claimed. ∎

Lemma 2. *If D is a good drawing of $H \times P_n$, $n \geq 2$, in which every H^i, $i = 0, 1, \ldots, n$, has at most 2 crossings on its edges, then D has at least $3n - 1$ crossings.*

Proof. First we show that the subdrawing of $K_4 \times P_n$ induced by D is coherent. Clearly, $\nu_D(K_4^i, K_4^j) = 0$ for all $i \neq j$, otherwise $\nu_D(K_4^i, K_4^j) \geq 3$ and H^i (H^j) has at least three crossings. Suppose that in D the subgraphs K_4^j and K_4^l are in different regions in the wiev of the subdrawing of K_4^i. The subdrawing of K_4^i divides the plane into several regions in such a way that no three vertices of K_4^i are on the common boundary of two regions. Thus, in D, K_4^i has at least five common points with the five paths joining K_4^j to K_4^l, and at most two of these points are vertices. This contradicts the hypothesis.

Therefore, the subdrawing of $K_4 \times P_n$ is coherent. As K_4^i has at most two crossings on its edges, no blue edge not incident with K_4^i crosses K_4^i, and, by Lemma 1, $f(Q^i) \geq 3$ for every $i = 1, 2, \ldots, n-1$. Every good drawing of $K_4 \times P_n$, $n \geq 2$, has at least one of the edges of K_4^0 and also K_4^n crossed. These two crossings are not counted in the total force of the drawing D, so the number of crossings in D is at least $2 + \Sigma_{i=1}^{n-1} f(Q^i) \geq 3n - 1$. ∎

Theorem 1. $\nu(H \times P_n) = 3n - 1$ *for* $n \geq 1$.

Proof. The drawing in Figure 1 shows that $\nu(H \times P_n) \leq 3n - 1$ for $n \geq 1$. We prove the reverse inequality by induction on n. In [2] it is shown that $\nu(K_4 \times P_1) = 2$, and since $H \times P_1$ contains $K_4 \times P_1$, the result is true for $n = 1$. Assume it is true for $n = k$, $k \geq 1$, and suppose that there is a good drawing of $H \times P_{k+1}$ with fewer than $3(k+1) - 1$ crossings. By Lemma 2, some H^i must then have at least three crossings on its edges. By the removal of all edges of this H^i, we obtain a graph homeomorphic to $H \times P_k$ or one that contains the subgraph $H \times P_k$ and has a drawing with fewer than $3k-1$ crossings. This contradicts the induction hypothesis. ■

3. The Crossing Number of $H \times C_n$

The graph $H \times C_n$ consists of the subgraph $H \times P_{n-1}$ and of five edges $\{x_0, x_{n-1}\}$ for $x = a, b, c, d$, and e. Thus, the graph $H \times C_n$ has $7n$ red edges in n copies H^i for $i = 0, 1, \dots, n-1$, and $5n$ blue edges in five n-cycles C_n^x for $x = a, b, c, d$, and e. Clearly, $K_4 \times C_n$ is a subgraph of $H \times C_n$. We denote by T^i the subgraph of $K_4 \times C_n$ induced by the edges joining K_4^{i-1} to K_4^i for $i = 0, 1, \dots, n-1$, i taken modulo n. We say that a good drawing of $K_4 \times C_n$ is *coherent* if each subdrawing of its subgraph isomorphic to $K_4 \times P_{n-1}$ is coherent.

Lemma 3. *Let D be a good and coherent drawing of $K_4 \times C_3$ in which every T^i, $i = 0, 1, 2$, has its edges crossed at most four times. If for some i, $\nu_D(K_4^{i+1}, T^i) \neq 0$, i taken modulo 3, then D has at least 10 crossings.*

Proof. First we show that if in D some blue 3-cycle crosses some K_4^i, $i \in \{0, 1, 2\}$, then it crosses this K_4^i at least twice. Suppose that C_3^x, $x \in \{a, b, c, d\}$, crosses K_4^i. As in the good drawing D, two adjacent edges cannot cross each other, C_3^x crosses the red 3-cycle of K_4^i created by three edges of K_4^i not incident with the common vertex of K_4^i and C_3^x. This red 3-cycle and C_3^x are vertex-disjoint cycles and such cycles cannot cross each other only once.

Without loss of generality, assume that in D an edge of T^1 crosses K_4^2 and assume D has fewer than ten crossings. Since T^1 and K_4^2 are vertex-disjoint subgraphs of $K_4 \times C_3$ and since K_4^2 is 3-connected, this edge of T^1 crosses K_4^2 at least three times, and on the edges of T^1 there is at most one other crossing. Thus, $\nu_D(T^1, K_4^0 \cup T^0) = 0$ or $\nu_D(T^1, K_4^1 \cup T^2) = 0$.

Suppose that $\nu_D(T^1, K_4^1 \cup T^2) = 0$ and that in D two blue edges cross each other. Since in the good drawing no two edges of a 3-cycle cross each other, in D there are at least two crossings between two different blue 3-cycles. This implies that in D there are at most seven crossings on the edges of K_4^0, K_4^1, and K_4^2. As we assumed above, $\nu_D(K_4^1, T^1) = 0$. The graph K_4 is not outerplanar, so in the coherent drawing D the subgraph K_4^1 has an internal crossing. In this case the subdrawing of $K_4^0 \cup T^1 \cup K_4^1$

induced by D divides the plane in such a way that on the boundaries of two neighboring regions outside K_4^1 there are at most three vertices of K_4^1 (see Figure 3). Since only one edge of T^1 crosses K_4^2, the vertices of K_4^2 lie, in D, in at most two neighboring regions outside K_4^1, and, since $\nu_D(T^1, T^2) = 0$, in D at least one edge of T^2 crosses K_4^1 or K_4^0. If $\nu_D(K_4^1, T^2) \neq 0$, then $\nu_D(K_4^1, T^2) \geq 2$, and in D there are at least eight crossings on the edges of K_4^0, K_4^1, and K_4^2 (at least three on K_4^1, at least four on K_4^2, and at least one crossing on K_4^0). If $\nu_D(K_4^0, T^2) \neq 0$, then $\nu_D(K_4^0, T^2) \geq 3$, and there are at least four crossings on the edges of K_4^0. Thus, in D there are more than seven crossings on the edges of K_4^0, K_4^1, and K_4^2 again.

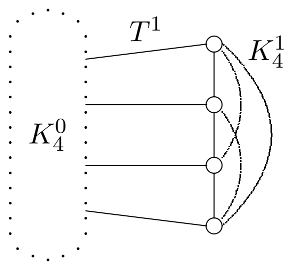

Figure 3

Therefore, in D, no two blue edges of 3-cycles cross each other. As the vertices of K_4^2 lie, in D, in at most two neighbouring regions outside K_4^1, in D one blue 3-cycle must then have crossed its edges at least four times by the edges of K_4^0 and K_4^1. Since $\nu_D(K_4^1) = 1$ and either $\nu_D(K_4^0) = 1$ or $\nu_D(K_4^0, T^1) = 1$, together with at least four crossings on the edges of K_4^2, in D there are more than nine crossings. This contradicts our assumption.

For $\nu_D(T^1, K_4^0 \cup T^0) = 0$, we can use the same arguments. Hence, D has at least ten crossings. ■

Lemma 4. *$\nu(H \times C_3) \geq 10$.*

Proof. Beineke and Ringeisen [2] showed that $\nu(K_4 \times C_3) = 9$, and therefore, $\nu(H \times C_3) \geq 9$. Assume that there is a drawing of $H \times C_3$ with nine crossings and let D be such a drawing. Then the drawing D is optimal and no edge not belonging to its subgraph $K_4 \times C_3$ is crossed. As the drawing D is good, none of the 3-cycles C_3^x, $x = a, b, c, d, e$, has an internal crossing. Thus the subdrawing D^* induced by D by the edges of C_3^d and C_3^e and the edges joining these two 3-cycles induces the map in the plane with two triangular and three quadrangular regions. In D the other vertices of the graph must lie in the triangular region of D^* bounded by C_3^d; otherwise the edges of D not belonging to D^* cross only the edges of C_3^d and these can be redrawn to give a drawing with fewer crossings. Moreover, in D, no edge of C_3^d is crossed because of good drawing.

Since $\nu(K_4 \times P_1) = 2$, in D there are at most seven crossings on the edges of any subgraph K_4^i of $H \times C_3$, $\quad i = 0, 1, 2$. First, we show that no K_4^i has more than six crossings on its edges. Without loss of generality, suppose that K_4^1 has seven crossings in D. Then in the subdrawing D^{**} obtained from D by deleting the edges of K_4^1 there are two crossings, and therefore, the edges of K_4^0 and K_4^2 cannot cross each other. This implies that in D^{**} there are only two internal crossings of K_4^0 and K_4^2 and D^{**} divides the plane as shown in Figure 4. It is easy to see that in D^{**} there are eight possibilities to draw the other three vertices of K_4^1 and that in each case in D the edges of K_4^1 cross the edges of D^{**} at least eight times. This contradicts the assumption of the optimal drawing. Therefore, in D, every K_4^i has at most six crossings on its edges.

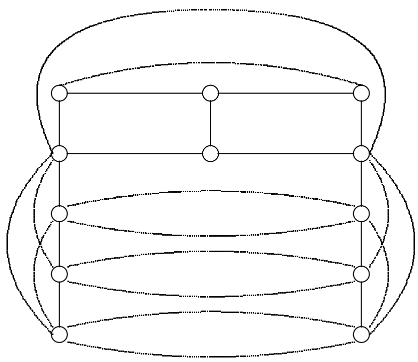

Figure 4

In the proof of Lemma 5 in [2], it is shown that any non-coherent drawing of $K_4 \times C_3$ with at most six crossings on the edges of any K_4^i is not optimal. This implies that the subdrawing of $K_4 \times C_3$ of our drawing D is coherent.

We note that in the next part of the proof i is taken modulo 3. In D there are at most four crossings on the edges of any T^i, $i \in \{0, 1, 2\}$; otherwise, by deleting these edges and the fifth edge joining e_{i-1} to e_i we obtain a subdrawing of $H \times P_2$ with at most four crossings. Since the subdrawing of $K_4 \times C_3$ induced by D is coherent and has nine crossings, by Lemma 3, $\nu_D(K_4^{i+1}, T^i) = 0$ for $i = 0, 1, 2$, and from the property of a good and coherent drawing it follows that no two different K_4^i and K_4^j cross each other in D. As we mentioned above, C_3^d has no crossing on its edges and, by Lemma 1, every subdrawing of the subgraph isomorphic to $K_4 \times P_2$ has force exactly three. Moreover, $\nu_D(K_4^i, T^i \cup T^{i+1}) = 2$ and $\nu_D(T^i, T^{i+1}) = 1$ for each $i = 0, 1, 2$. Therefore, there are six crossings between the edges of K_4^i, $i = 0, 1, 2$, and the edges of C_3^a, C_3^b, and C_3^c, and there are three crossings among the edges of C_3^a, C_3^b, and C_3^c. Since all these 3-cycles are vertex-disjoint and none of them has an internal crossing, this is impossible. This completes the proof. ■

Lemma 5. *If D is a good drawing of $K_4 \times C_n$, $n \geq 3$, in which no K_4^i, $i = 0, 1, \ldots, n-1$, has more than three crossings on its edges, then D is coherent.*

Proof. Suppose D is not coherent and assume, without loss of generality, that K_4^0 has vertices of $K_4^1, \ldots, K_4^{n-1}$ in more than one of its regions in the drawing. Then the edges of K_4^0 are crossed at least four times since the subgraph induced by the vertices in $K_4^1, \ldots, K_4^{n-1}$ is 4-connected. This contradiction completes the proof. ■

Lemma 6. *Let $n \geq 5$ be odd and let D be a good and coherent drawing of $H \times C_n$. If in D every K_4^i, $i = 0, 1, \ldots, n-1$, has at most three crossings on its edges and if no edge of the subgraph induced by the vertices d_i and e_i, $i = 0, 1 \ldots, n-1$, is crossed, then D has at least $3n + 1$ crossings.*

Proof. First, we note that i is taken modulo n in the proof. By hypothesis, no two different H^i and H^j cross each other, thus $\nu_D(H^i, H^j) = 0$ if $i \neq j$. Moreover, for $i = 0, 1, \ldots, n-1$, $\nu_D(K_4^i, T^r) = 0$ if $r \neq i$, $i+1$. If not, K_4^i and T^r cross each other at least three times and K_4^i either has an internal crossing or at least two crossings with one blue n-cycle. Hence, by Lemma 1, $f(Q^i) \geq 3$ for every subdrawing of Q^i. Suppose that for every i, $i = 0, 1, \ldots, n-1$, $f(Q^i) = 3$, since otherwise we are done. By Lemma 1, for every i, $\nu_D(K_4^i, T^i \cup T^{i+1}) = 2$ and $\nu_D(T^i, T^{i+1}) = 1$. In a good drawing no two adjacent edges cross each other and so every crossing between T^i and T^{i+1} is a crossing between two different blue n-cycles. Since two disjoint n-cycles can cross each other only an even number of times, in D there are at least $n+1$ crossings among the blue n-cycles and at least $2n$ crossings between K_4^i and $T^i \cup T^{i+1}$ for all $i = 0, 1, ..., n-1$. This completes the proof. ■

Theorem 2. *For* $n \geq 3$, $\nu(H \times C_n) = \begin{cases} 3n & \textit{if } n \textit{ is even}, \\ 3n+1 & \textit{if } n \textit{ is odd}. \end{cases}$

Figure 5

Proof. In Figure 5 there are drawings of $H \times C_3$ and $H \times C_4$ with 10 and 12 crossings, respectively. By appropriately inserting H's in pairs into the cycles of these drawings, as suggested by the arrangement in Figure 6, we obtain $\nu(H \times C_n) \leq 3n$ for even n, and $\nu(H \times C_n) \leq 3n + 1$ if n is odd. The graph $H \times C_n$ contains a subgraph $K_4 \times C_n$ whose crossing number is $3n$, see [2]. Hence, for even n we are done. By Lemma 4, the result is true for $n = 3$. It remains to show the reverse inequality for odd n, $n \geq 5$. Therefore, we assume that for odd n, $n \geq 5$, an optimal drawing of $H \times C_n$ has fewer than $3n + 1$ crossings and let D be such a drawing.

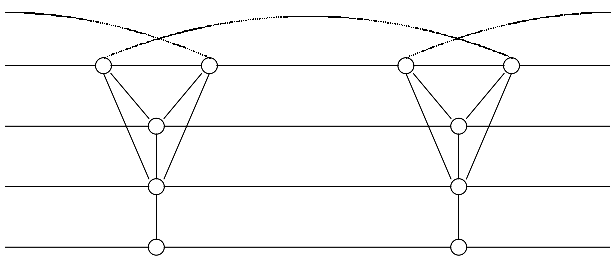

Figure 6

Then in D no edge not belonging to the subgraph $K_4 \times C_n$ is crossed and, since D is optimal, no edge of C_n^d is crossed in D. Moreover, no K_4^i has more than three crossings. Otherwise by deleting suitable edges from D we obtain either $K_4 \times C_n$ with fewer than $3n$ crossings or $H \times C_{n-1}$ with fewer than $3(n-1)$ crossings, a contradiction. Thus, by Lemma 5, D is coherent and, by Lemma 6, D has at least $3n+1$ crossings. This contradiction completes the proof. ■

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