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THE FORCING GEODETIC NUMBER OF A GRAPH

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Dedicated to Frank Harary
on the occasion of the 50th anniversary of his Ph.D.

Abstract

For two vertices u and v of a graph G, the set $I(u, v)$ consists of all vertices lying on some $u - v$ geodesic in G. If S is a set of vertices of G, then $I(S)$ is the union of all sets $I(u, v)$ for $u, v \in S$. A set S is a geodetic set if $I(S) = V(G)$. A minimum geodetic set is a geodetic set of minimum cardinality and this cardinality is the geodetic number $g(G)$. A subset T of a minimum geodetic set S is called a forcing subset for S if S is the unique minimum geodetic set containing T. The forcing geodetic number $f_G(S)$ of S is the minimum cardinality among the forcing subsets of S, and the forcing geodetic number $f(G)$ of G is the minimum forcing geodetic number among all minimum geodetic sets of G. The forcing geodetic numbers of several classes of graphs are determined. For every graph G, $f(G) \leq g(G)$. It is shown that for all integers a, b with $0 \leq a \leq b$, a connected graph G such that $f(G) = a$ and $g(G) = b$ exists if and only if $(a, b) \notin \{(1, 1), (2, 2)\}$.

Keywords: geodetic set, geodetic number, forcing geodetic number.

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1. Introduction

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. It is well known that this distance is a metric on the vertex set $V(G)$. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, denoted by *rad* G, of G and the maximum eccentricity is its *diameter*, denoted by *diam* G. A

$u - v$ path of length $d(u,v)$ is called a $u - v$ *geodesic*. We define $I(u,v)$ (interval) as the set of all vertices lying on some $u - v$ geodesic of G, and for a nonempty subset S of $V(G)$, we define

$$I(S) = \bigcup_{u,v \in S} I(u,v).$$

A set S of vertices of G is defined in [1] to be a *geodetic set* in G if $I(S) = V(G)$, and a geodetic set of minimum cardinality is a *minimum geodetic set*. The cardinality of a minimum geodetic set in G is called the *geodetic number* $g(G)$.

For a minimum geodetic set S of G, a subset T of S with the property that S is the unique minimum geodetic set containing T is called a *forcing subset* of S. The *forcing geodetic number* $f_G(S,g)$ of S in G is the minimum cardinality of a forcing subset for S, while the *forcing geodetic number* $f(G,g)$ of G is the smallest forcing number among all minimum geodetic sets of G. Since the parameter g is understood in this context, we write $f_G(S)$ for $f_G(S,g)$ and $f(G)$ for $f(G,g)$. Hence if G is a graph with $f(G) = a$ and $g(G) = b$, then $0 \le a \le b$ and there exists a minimum geodetic set S of cardinality b containing a forcing subset T of cardinality a. For the graph G of Figure 1, $g(G) = 3$. There are four minimum geodetic sets in G, namely $S_1 = \{u,x,z\}$, $S_2 = \{v,y,w\}$, $S_3 = \{x,y,w\}$, and $S_4 = \{v,y,z\}$. Since S_1 is the only minimum geodetic set containing u, it follows that $f_G(S_1) = 1$. No other vertex of G belongs to only one minimum geodetic set, so $f_G(S_i) \ge 2$ for $i = 2,3,4$. (In fact, $f_G(S_i) = 2$ for $i = 2,3,4$.) Therefore, $f(G) = 1$.

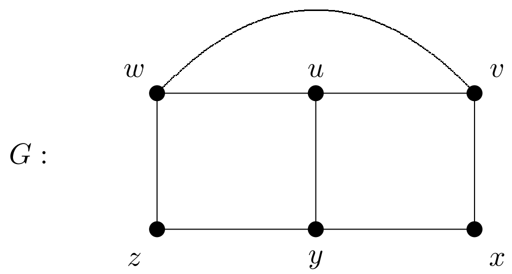

Figure 1. A graph with forcing geodetic number 1

It is immediate that $f(G) = 0$ if and only if G has a unique minimum geodetic set. If G has no unique minimum geodetic set but contains a vertex belonging to only one minimum geodetic set, then $f(G) = 1$. We summarize these observations below.

Lemma 1.1. *For a graph G, the forcing geodetic number $f(G) = 0$ if and only if G has a unique minimum geodetic set. Moreover, $f(G) = 1$ if and only if G has at least two distinct minimum geodetic sets but some vertex of G belongs to exactly one minimum geodetic set.*

The following result is a direct consequence of Lemma 1.1.

Corollary 1.2. *For a graph G, the forcing geodetic number $f(G) \geq 2$ if and only if every vertex of each minimum geodetic set belongs to at least two minimum geodetic sets.*

2. FORCING GEODETIC NUMBERS OF CERTAIN GRAPHS

In this section, we determine the forcing geodetic numbers of some well known graphs. We begin by determining the forcing geodetic number of the famous Petersen graph P of Figure 2. For a set S of vertices in a graph G, we write $N(S)$ for the *neighborhood* of S, that is, the set of all vertices that are neighbors of at least one vertex in S, while the *closed neighborhood* of S is defined by $N[S] = N(S) \cup S$.

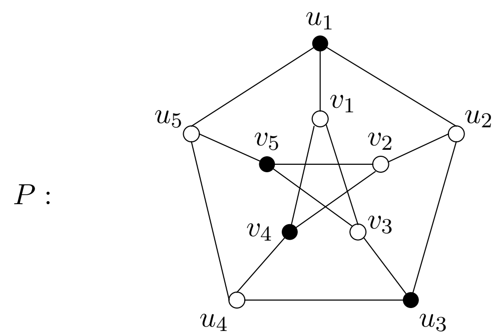

Figure 2. The Petersen graph P

It can be verified that the geodetic number of P is 4 and that every set of four independent vertices of P is a minimum geodetic set. Since all independent sets of cardinality 4 are similar in P, we consider the independent set $S = \{u_1, u_3, v_4, v_5\}$. For every $w \in S$, there exists a minimum geodetic set containing w that is distinct from S. For example, $\{u_1, u_4, v_2, v_3\}$ is another minimum geodetic set containing u_1. Therefore, every vertex of each minimum geodetic set of P belongs to at least two minimum geodetic sets and so, by Corollary 1.2, $f_P(S) \geq 2$. Moreover, if $S_0 = \{x, y\} \subset S$, then $V(P) - N[S_0]$ consists of three vertices exactly two of which are nonadjacent. For example, let $S_0 = \{v_4, v_5\}$. Then $V(P) - N[S_0] = \{u_1, u_2, u_3\}$, where only u_1 and u_3 are nonadjacent. Therefore, S is the unique minimum geodetic set containing S_0. This implies that $f_P(S) = 2$ and that $f(P) = 2$.

The following two observations appeared in [2].

Theorem A. *If v is a vertex of a graph G such that $\langle N(v) \rangle$ is complete, then v belongs to every geodetic set of G.*

Corollary B. *Each end-vertex of a graph G belongs to every geodetic set of G.*

Since the set of all end-vertices of T is the unique minimum geodetic set of T (see [1]), we have the following result.

Theorem 2.1. *For a tree T, the forcing geodetic number $f(T)$ is 0.*

The *corona* $G \circ K_1$ of a graph G of order n is that graph obtained from G by joining one new vertex to each vertex of G. Thus the order of $G \circ K_1$ is $2n$. For a connected graph G of order at least 2, we show that the set S of end-vertices of $G \circ K_1$ is a unique minimum geodetic set of $G \circ K_1$, implying that $f(G \circ K_1) = 0$. By Corollary B, every geodetic set of $G \circ K_1$ contains S. It suffices to show that S is a geodetic set of G. For $v \in V(G)$, there exists $u \in V(G)$ that is adjacent to v. Let $v', u' \in V(G \circ K_1)$ be the end-vertices joined to v and u, respectively. Then v lies on the $v' - u'$ geodesic v', v, u, u' in $G \circ K_1$. This implies that the set S is a geodetic set of $G \circ K_1$. Therefore, $f(G \circ K_1) = 0$ for all connected graphs G.

Now we determine the forcing geodetic numbers of cycles.

Theorem 2.2. *The forcing geodetic number of C_n, $n \geq 3$, is*

$$f(C_n) = \begin{cases} 1, & \textit{if } n \textit{ is even}, \\ 2, & \textit{otherwise.} \end{cases}$$

Proof. If n is even, then $g(C_n) = 2$ and every pair of antipodal vertices of C_n forms a minimum geodetic set of C_n. Consequently, C_n does not have a unique minimum geodetic set but every vertex of C_n has a unique vertex antipodal to it, so $f(C_n) = 1$. If n is odd, then $g(C_n) = 3$. Again C_n has more than one minimum geodetic set. Moreover, every vertex of C_n belongs to at least two distinct minimum geodetic sets. By Corollary 1.2, $f(C_n) \geq 2$. On the other hand, for every pair u, v of adjacent vertices in C_n, there is a unique vertex w in C_n with $d(u, w) = d(v, w)$. Then $\{u, v, w\}$ is the unique minimum geodetic set containing $\{u, v\}$, implying that $f(C_n) = 2$. ■

It was shown in [1] that $g(C_n) = g(C_n \times K_2)$, where $C_n \times K_2$ is the Cartesian product of C_n and K_2. A proof similar to that of Theorem 2.2 gives $f(C_n) = f(C_n \times K_2)$. It is easy to verify that for $n \geq 4$ the geodetic number of the wheel $W_n = C_n + K_1$ of order $n + 1$ is $\lceil \frac{n}{2} \rceil$ and that $f(W_n) = f(C_n)$.

We have determined the forcing geodetic numbers of trees and cycles. A closely related class of graphs is the unicyclic graphs. A graph is *unicyclic* if it is connected and contains exactly one cycle.

Theorem 2.3. *Let G be a unicyclic graph that is not a cycle. If the cycle C of G has length k, and ℓ is the greatest order of a path on C every vertex of which has degree 2 in G, then*

$$f(G) = \begin{cases} 0\,, & \text{if } \ell \leq (k-2)/2 \text{ or if } \ell = k-1 \text{ is odd,} \\ 1\,, & \text{otherwise.} \end{cases}$$

Proof. Let $C : v_1, v_2, \cdots, v_k, v_1$ and let W be the set of all end-vertices of G. Assume, without loss of generality, that $P : v_1, v_2, \cdots, v_\ell$, where $\deg v_i = 2$ for $i = 1, 2, \cdots, \ell$. So $\deg v_k \geq 3$ and $\deg v_{\ell+1} \geq 3$. If $\ell \leq (k-2)/2$, then W is the unique minimum geodetic set of G and so $f(G) = 0$. Therefore, we assume that $\ell \geq (k-1)/2$. Since $I(W) = V(G) - V(P)$, it follows that W is not a geodetic set of G, so $g(G) \geq |W| + 1$. Assume first that $\ell = k - 1$. If ℓ is odd, then $W \cup \left\{v_{\frac{\ell+1}{2}}\right\}$ is the unique minimum geodetic set of G and so $f(G) = 0$. On the other hand, if ℓ is even, then there are pairs v_i, v_j of vertices with $1 \leq i < j \leq \ell$ such that $W \cup \{v_i, v_j\}$ is a minimum geodetic set. However, there is only one such set containing v_ℓ, namely, $W \cup \left\{v_{\frac{\ell}{2}}, v_\ell\right\}$ and so $f(G) = 1$. Hence we assume that $(k-1)/2 \leq \ell \leq k-2$. If ℓ is odd, then both $W \cup \left\{v_{\frac{\ell+1}{2}}\right\}$ and $W \cup \left\{v_{\frac{\ell-1}{2}}\right\}$ are minimum geodetic sets of G; while if ℓ is even, then $W \cup \left\{v_{\frac{\ell}{2}}\right\}$ and $W \cup \left\{v_{\frac{\ell+2}{2}}\right\}$ are minimum geodetic sets of G. In either case, $f(G) = 1$. ■

Next we determine the geodetic and forcing geodetic numbers of the hypercubes Q_n, where $n \geq 2$. The hypercube Q_n can be considered as that graph whose vertices are labeled by the binary n-tuple $(a_1, a_2, \cdots, a_n)$ (that is a_i is 0 or 1 for $1 \leq i \leq n$) and such that two vertices are adjacent if and only if their corresponding n-tuples differ in precisely one position.

Theorem 2.4. *For $n \geq 2$, $g(Q_n) = 2$ and $f(Q_n) = 1$.*

Proof. Since every minimum geodetic set of Q_n is of the form

$$S = \{(a_1, a_2, \cdots, a_n), (1 - a_1, 1 - a_2, \cdots, 1 - a_n)\}$$

where $a_1, a_2, \cdots, a_n \in \{0, 1\}$, it follows that $g(Q_n) = 2$. Certainly, Q_n has more than one minimum geodetic set and every minimum geodetic set S is the unique minimum geodetic set containing $(a_1, a_2, \cdots, a_n)$. Therefore, $f(Q_n) = 1$. ■

Next, we study the forcing numbers of complete bipartite graphs. Let $1 \leq r \leq s$ be two integers. It was shown in [1] that $g(K_{r,s}) = s$ if $r = 1$; while $g(K_{r,s}) = \min\{r, 4\}$, if $r \geq 2$.

Theorem 2.5. *Let $K_{r,s}$ be a complete bipartite graph with $r + s \geq 2$ and $1 \leq r \leq s$. Then*

$$f(K_{r,s}) = \begin{cases} 0\,, & r = 1 \quad \text{or} \quad r = 2, 3 \text{ and } r < s, \\ 1\,, & r = 2, 3 \quad \text{and} \quad r = s, \\ 3\,, & r = 4, \\ 4\,, & r \geq 5. \end{cases}$$

Proof. Let $V_1 = \{u_1, u_2, \cdots, u_r\}$ and $V_2 = \{v_1, v_2, \cdots, v_s\}$ be the partite sets of $G = K_{r,s}$. If $r = 1$, then G is a tree and its forcing geodetic number is 0.

We first assume that $r \in \{2, 3\}$. If $r < s$, then G has a unique minimum geodetic set, namely V_1, and so the forcing geodetic number is 0 by Lemma 1.1. If $r = s$, then G has two distinct minimum geodetic sets, namely V_1 and V_2. So the forcing geodetic number is at least 1 by Lemma 1.1. Certainly, for each $v \in V_i$ with $i = 1, 2$, the set V_i is the unique minimum geodetic set containing v. So the forcing geodetic number is 1.

Next we assume that $r = 4$ and so $g(G) = 4$. If $r = s$, then V_1, V_2, and $S = \{u_1, u_2, v_1, v_2\}$ are minimum geodetic sets of G. All other minimum geodetic sets are similar to S. Since V_1 is the unique minimum geodetic set containing $\{u_1, u_2, u_3\}$ and V_1 is not the unique minimum geodetic set containing any 2-element or 1-element subset of V_1, it follows that $f_G(V_1) = 3$. Similarly, $f_G(V_2) = 3$. Moreover, since S is not the unique minimum geodetic set containing any of its proper subsets, $f_G(S) = 4$. Therefore, $f(G) = 3$. If $r < s$, then V_1 and $S = \{u_1, u_2, v_1, v_2\}$ are minimum geodetic sets of G. All other minimum geodetic sets are similar to S. Since $f_G(V_1) = 3$ and $f_G(S) = 4$, it follows that $f(G) = 3$.

Finally, we assume that $r \geq 5$. Then every minimum geodetic set S of G is of the form $S = \{u_{i_1}, u_{i_2}, v_{j_1}, v_{j_2}\}$, where $1 \leq i_1 < i_2 \leq r$ and $1 \leq j_1 < j_2 \leq s$. Since S is not the unique minimum geodetic set containing any of its proper subsets, $f(G) = 4$. ■

3. Graphs with Prescribed Geodetic and Forcing Geodetic Numbers

We have already noted that if G is a graph with $f(G) = a$ and $g(G) = b$, then $0 \leq a \leq b$. We now establish a converse result beginning with the case where $a \neq b$.

Theorem 3.1. *Every pair a, b of integers with $0 \leq a < b$ can be realized as the forcing geodetic number and geodetic number, respectively, of some graph.*

Proof. We have already seen that if $G = K_b$, then $f(G) = 0$ and $g(G) = b$. Thus, we assume that $0 < a < b$. We consider two cases.

Case 1. $a = 1$.
If $b = 2$, then every even cycle has forcing geodetic number 1 and geodetic number 2. So we assume that $b \geq 3$. Consider the graph G of Figure 3.

We first show that $g(G) = b$. Let $U = \{u_1, u_2, \cdots, u_{b-2}\}$ be the set of end-vertices of G. Since $U \cup \{v_3, x\}$ is a geodetic set of G, it follows that $g(G) \leq b$. On the other hand, by Corollary B, every minimum geodetic set of G includes U. Moreover, it is routine to verify that for each $z \in V(G) - U$, the set $U \cup \{z\}$ is not a geodetic set of G, implying that $g(G) \geq b$. Therefore, $g(G) = b$.

Next we show that $f(G) = 1$. Since $U \cup \{v_3, x\}$ and $U \cup \{v_3, y\}$ are two distinct minimum geodetic sets of G, it follows that $f(G) \geq 1$ by Lemma 1.1. Moreover, $S = U \cup \{v_3, x\}$ is the unique minimum geodetic set containing $\{x\}$. This implies that $f_G(S) = 1$. Therefore, $f(G) = 1$.

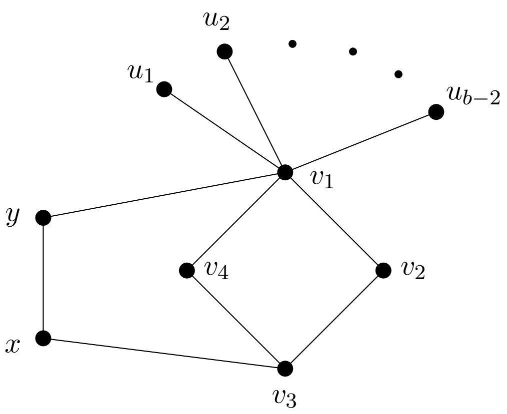

Figure 3. The graph G with $g(G) = b$ and $f(G) = 1$

Case 2. $a \geq 2$.
First assume that $b = a+1$. We consider the graph G of order $2a+4$ shown in Figure 4.

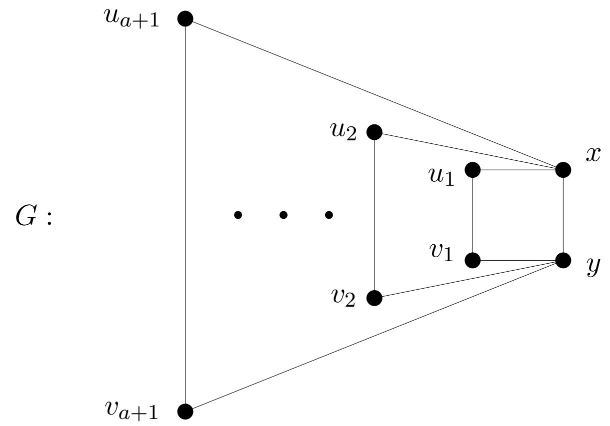

Figure 4. The graph G with $g(G) = f(G) + 1$

We first show that $g(G) = a + 1$. Since $\{u_2, u_3, \cdots, u_{a+1}, v_1\}$ is a geodetic set of G, it follows that $g(G) \leq a+1$. Next we show that $g(G) \geq a+1$. For every i with $1 \leq i \leq a+1$, each of u_i and v_i lies only on geodesics with initial or terminal vertex u_i or v_i. This implies that if W is a minimum geodetic set of G, then W contains at least one vertex from each set $\{u_i, v_i\}$, where $1 \leq i \leq a+1$, implying that $g(G) \geq a+1$. Therefore, $g(G) = a+1 = b$

and $W \subseteq V(G) - \{x, y\}$. Furthermore, $W \neq \{u_1, u_2, \cdots, u_{a+1}\}$ and $W \neq \{v_1, v_2, \cdots, v_{a+1}\}$.

Next we show that $f(G) = a$. Again, let W be a minimum geodetic set of G. Since $W \cap \{u_1, u_2, \cdots, u_{a+1}\} \neq \emptyset$ and $W \cap \{v_1, v_2, \cdots, v_{a+1}\} \neq \emptyset$, it follows that W is not the unique minimum geodetic set containing any of its subsets W' with $|W'| < a$. So $f_G(W) \geq a$ for every minimum geodetic set W of G. Therefore, $f(G) \geq a$. On the other hand, $W = \{u_2, u_3, \cdots, u_{a+1}, v_1\}$ is the unique minimum geodetic set containing $\{u_2, u_3, \cdots, u_{a+1}\}$, implying that $f_G(W) = a$. Therefore, $f(G) = a$.

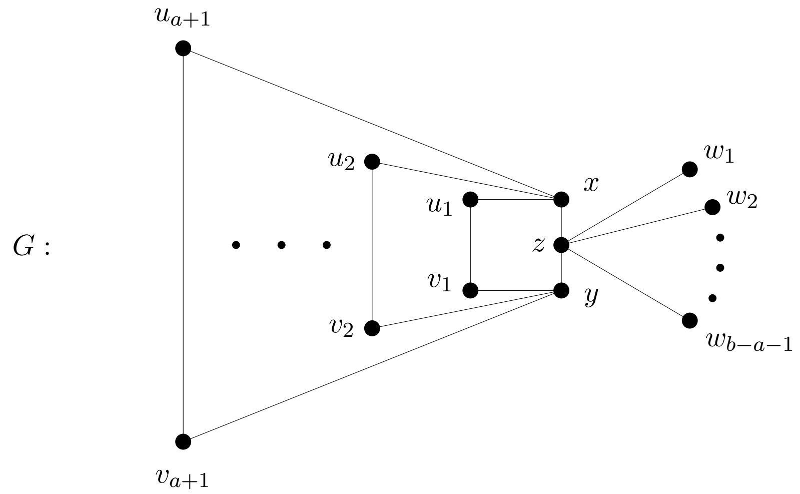

Figure 5. The graph G with $g(G) = b$ and $f(G) = a$

We now assume that $b \geq a + 2$. Consider the graph of Figure 5. We first show that $g(G) = b$. Since

$$S = \{u_2, u_3, \cdots, u_{a+1}, v_1, w_1, w_2, \cdots, w_{b-a-1}\}$$

is a geodetic set, $g(G) \leq b$. Next we show that $g(G) \geq b$. By Corollary B, the end-vertices $w_1, w_2, \cdots, w_{b-a-1}$ belong to every geodetic set of G. Moreover, as above, for every i with $1 \leq i \leq a+1$, each of u_i and v_i lies only on geodesics with initial or terminal vertex u_i or v_i. This implies that every minimum geodetic set of G must contain at least one vertex from each set $\{u_i, v_i\}$, where $1 \leq i \leq a + 1$. Therefore, $g(G) \geq (b - a - 1) + (a + 1) = b$.

Next we show that $f(G) = a$. Every minimum geodetic set S of G has the form

$$S = U \cup V \cup W$$

where $U \subseteq \{u_1, u_2, \cdots, u_{a+1}\}$, $V \subseteq \{v_1, v_2, \cdots, v_{a+1}\}$ and $W = \{w_1, w_2, \cdots, w_{b-a-1}\}$ with $U \neq \emptyset$ and $V \neq \emptyset$. This implies that if S is a minimum geodetic set of G and $S' \subseteq S$ with $|S'| < a$, then S is not the unique minimum geodetic set containing S'. Therefore, $f(G) \geq a$. On the other hand, the set

$$S = \{u_2, u_3, \cdots, u_{a+1}, v_1, w_1, w_2, \cdots, w_{b-a-1}\}$$

is the unique minimum geodetic set containing $\{u_2, u_3, \cdots, u_{a+1}\}$, implying that $f_G(S) = a$. Therefore, $f(G) = a$. ∎

Next we show that if a connected graph of order at least 2 has its geodetic number equal to its forcing geodetic number, then this number exceeds 2. First observe that $2 \leq g(G) \leq n$ for every connected graph of order $n \geq 2$. Therefore, the case $f(G) = g(G) = 1$ is impossible for any connected graph of order at least 2.

Theorem 3.2. *If G is a connected graph with geodetic number 2, then $f(G) < 2$.*

Proof. Let $\{u, v\}$ be a minimum geodetic set of G. Then $d(u, v) = diam\ G$ and every vertex of G lies on some $u - v$ geodesic of G. If $f(G) = 2$, then there exists $x \neq v$ such that $\{u, x\}$ is also a minimum geodetic set of G. However, the fact that x lies on some $u - v$ geodesic of G implies that $d(u, x) < d(u, v) = diam\ G$, which is a contradiction. ∎

We have seen in Theorem 3.2 that $f(G) < g(G)$ if $g(G) = 2$. Next we show that every integer $a \geq 3$ is simultaneously realizable as both the geodetic number and forcing geodetic number of some connected graph.

Theorem 3.3. *For every integer $a \geq 3$, there exists a connected graph G such that*

$$g(G) = f(G) = a.$$

Proof. For each integer $a \geq 3$, we construct a graph G_a with $f(G_a) = g(G_a) = a$. We consider two cases, according to whether a is even or a is odd.

Case 1. $a = 2k$, *where* $k \geq 2$.

We have seen in Theorem 2.5 that if $5 \leq r \leq s$, then $f(K_{r,s}) = g(K_{r,s}) = 4$. So we may assume that $k \geq 3$. For $k = 3$, let F_1 and F_2 be two copies of

$K_{5,5}$ with $V(F_1) = X \cup Y$ and $V(F_2) = U \cup V$, where $X = \{x_1, x_2, \cdots, x_5\}$, $Y = \{y_1, y_2, \cdots, y_5\}$, $U = \{u_1, u_2, \cdots, u_5\}$, and $V = \{v_1, v_2, \cdots, v_5\}$ are the respective partite sets of F_1 and F_2. Then the graph G_6 is formed from F_1 and F_2 by adding the edge x_5u_1. The graph G_6 is shown in Figure 6.

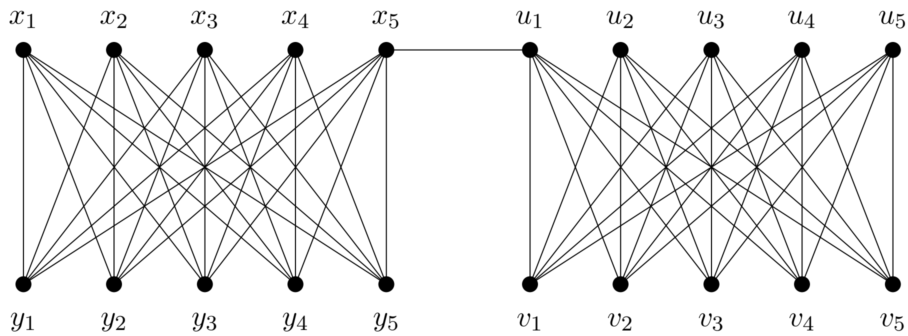

Figure 6. The graph G_6

We first show that $g(G_6) = 6$. Since

$$S = \{x_1, y_1, y_2, u_5, v_4, v_5\}$$

is a geodetic set of G_6, it follows that $g(G_6) \leq 6$. Assume, to the contrary, that $g(G_6) < 6$. Let W be a geodetic set of G_6 with $|W| = 5$. We claim that W must contain at least one vertex from each set of X, Y, U, and V. Assume, to the contrary, that this is not the case. There are two subcases.

Subcase 1.1. $W \cap X = \emptyset$.
Observe that in this subcase, each vertex y_i $(1 \leq i \leq 5)$ only lies on those geodesics with initial or terminal vertex y_i. Then $W = Y$ as $|W| = |Y|$. However, $I(Y) = V(F_1) \neq V(G_6)$, contradicting the fact that W is a geodetic set of G_6.

Subcase 1.2. $W \cap Y = \emptyset$.
Since each vertex x_i $(1 \leq i \leq 4)$ lies only on those geodesics with initial or terminal vertex x_i, it follows that $\{x_1, x_2, x_3, x_4\} \subset W$. So $W = \{x_1, x_2, x_3, x_4, v\}$ for some $v \in V(G_6) - \{x_1, x_2, x_3, x_4\}$. It is routine to verify that W is not a geodetic set of G_6 for each vertex v of G_6.

Therefore, W contains at least one vertex from each of X, Y, U, and V. Without loss of generality, let $W = \{u, v_1, x, y_1, w\}$, where $x \in X$, $u \in U$,

$w \in V(G_6) - \{u, v_1, x, y_1\}$. There are only three possible choices for u and x that are not equivalent, namely (1) $u = u_1$ and $x = x_5$, (2) $u \neq u_1$ and $x = x_5$, and (3) $u \neq u_1$ and $x \neq x_5$. It can be verified that W is not a geodetic set of G_6 in any of these cases, implying that $g(G_6) \geq 6$ and so $g(G_6) = 6$.

Next we show that $f(G_6) = 6$. Vertices of $U - \{u_1\}$ (of $X - \{x_5\}$) are interior vertices only for geodesics with initial and terminal vertex in V (in Y). Vertices of V (of Y) are interior vertices only for geodesics with an end-vertex in U (in X). From this it is clear that $|S \cap (U - \{u_1\})| = |S \cap (X - \{x_5\})| = 1$ and $|S \cap V| = |S \cap Y| = 2$, implying that S is not the unique minimum geodetic set containing any of its proper subsets. Therefore, $f(G_6) = 6$.

We can now extend G_6 to construct a graph G_{2k} for all $k \geq 4$. Let $F_1, F_2, \cdots, F_{k-1}$ be $k-1$ copies of $K_{5,5}$. For each i with $1 \leq i \leq k-1$, assume that the partite sets of F_i are $U_i = \{u_{i1}, u_{i2}, \cdots, u_{i5}\}$ and $V_i = \{v_{i1}, v_{i2}, \cdots, v_{i5}\}$. Then the graph G_{2k} is formed from $F_1, F_2, \cdots, F_{k-1}$ by adding the $k-2$ new edges $u_{i5}u_{i+1,1}$ between F_i and F_{i+1}, where $1 \leq i \leq k-2$. Then every minimum geodetic set of G_{2k} contains exactly one vertex from each of $U_1 - \{u_{15}\}$ and $U_{k-1} - \{u_{k-1,1}\}$ and exactly two vertices from each set V_i, $1 \leq i \leq k-1$. Therefore, $g(G_{2k}) = f(G_{2k}, g) = 2k$.

Case 2. $a = 2k+1$, *where* $k \geq 1$.
For $k = 1$, let the graph G_3 be obtained from the cycle $C_9 : u_1, u_2, x_1, v_1, v_2, y_1, w_1, w_2, z_1, u_1$ by adding three new vertices x_2, y_2, z_2 and the six edges x_2u_2, x_2v_1, y_2v_2, y_2w_1, z_2w_2, and z_2u_1. The graph G_3 is shown in Figure 7.

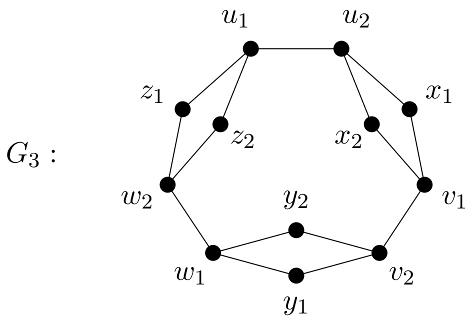

Figure 7. The graph G_3 with $g(G_3) = f(G_3) = 3$

Since the diameter of G_3 is 4 and there are exactly two geodesics between every two vertices at distance 4 in G_3, there is no 2-element geodetic set in G_3. Hence $g(G_3) \geq 3$. On the other hand, $\{u_1, v_1, w_1\}$ is a geodetic set of G_3 and so $g(G_3) = 3$. Since the minimum geodetic sets of G_3 are precisely those that have the form $\{u, v, w\}$, where $u \in \{u_1, u_2\}$, $v \in \{v_1, v_2\}$ and $w \in \{w_1, w_2\}$, it follows that $f(G_3) = g(G_3) = 3$.

We now assume that $k = 2$ and construct a connected graph G_5. Define the vertex set of G_5 to consist of five mutually disjoint sets $V_i = \{v_{i1}, v_{i2}, \cdots, v_{i5}\}$, $1 \leq i \leq 5$, such that v_{ip} is adjacent v_{jq} if and only if $j - i = \pm 1 (\text{mod } 5)$. Thus the graph G_5 is actually the composition $C_5[5K_1]$.

We first show that $g(G_5) = 5$. Since $\{v_{11}, v_{21}, \cdots, v_{51}\}$ is a geodetic set, $g(G_5) \leq 5$. We show now that every minimum geodetic set W contains exactly one vertex from each set V_i $(1 \leq i \leq 5)$. Assume, to the contrary, that this is not the case. We consider two subcases.

Subcase 2.1. *There exists exactly one set* V_i*,* $1 \leq i \leq 5$*, such that* $V_i \cap W = \emptyset$*, say* $V_i = V_1$. Since either $|V_3 \cap W| = 1$ or $|V_4 \cap W| = 1$, it follows that either $V_2 \not\subseteq I(W)$ or $V_5 \not\subseteq I(W)$, which is a contradiction.

Subcase 2.2. *There are two distinct sets* V_i*,* V_j*,* $1 \leq i, j \leq 5$*, such that* $V_i \cap W = \emptyset$ *and* $V_j \cap W = \emptyset$. We first assume that $j - i = \pm 1 (\text{mod } 5)$, say $V_i = V_1$ and $V_j = V_2$. Since $V_i \subseteq I(W)$ for $i = 1, 2$, we must have $|W \cap V_5| \geq 2$ and $|W \cap V_3| \geq 2$. But then $V_3 \cup V_5 \not\subseteq I(W)$, a contradiction. Otherwise, we may assume that $V_1 \cap W = V_3 \cap W = \emptyset$. However, then, either $W = V_2$ or $V_2 \not\subseteq I(W)$, both of which are impossible.

Hence $g(G_5) = 5$ and every minimum geodetic set of G_5 contains exactly one vertex from each set V_i $(1 \leq i \leq 5)$. Therefore, $f(G_5) = 5$.

We now assume that $k \geq 3$. Let $U_1 = \{u_{11}, u_{12}, \cdots, u_{15}\}$ and $V_1 = \{v_{11}, v_{12}, v_{13}, v_{14}\}$ be the partite sets of the graph $K_{4,5}$ and let

$$M = \{u_{11}v_{11}, u_{12}v_{12}, u_{13}v_{13}, u_{14}v_{14}\}$$

be a maximum matching of $K_{4,5}$. Now let $H = K_{4,5} - M$. We construct the graph G_{2k+1} from the graph G_{2k} in Case 1 by replacing F_1 by H, but the construction is otherwise the same. The graph G_7 is shown in Figure 8.

It can be verified that any subset of $V(G_{2k+1})$, consisting of one vertex of $U_{k-1} - \{u_{k-1,1}\}$, two vertices of each set $V_1, V_2, \ldots, V_{k-1}$, and two vertices of $U_1 - \{u_{15}\}$, is a minimum geodetic set of G_{2k+1}. Therefore, $g(G) = f(G) = 1 + 2(k-1) + 2 = 2k + 1$. This completes the proof. ■

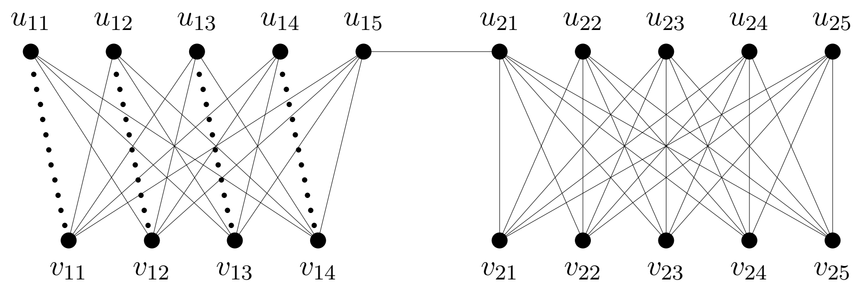

Figure 8. The graph G_7

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