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DISTANCE PERFECTNESS OF GRAPHS

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Abstract

In this paper, we propose a generalization of well known kinds of perfectness of graphs in terms of distances between vertices. We introduce generalizations of α-perfect, χ-perfect, strongly perfect graphs and we establish the relations between them. Moreover, we give sufficient conditions for graphs to be perfect in generalized sense. Other generalizations of perfectness are given in papers [3] and [7].

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1. Introduction

We consider only simple graphs which have no loops and multiple edges and we generally follow the standard terminology of Berge [1]. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. By a *path* joining vertices x_1 and x_n in the graph G we mean a sequence of vertices $x_1, \ldots, x_n$ such that $(x_i, x_{i+1}) \in E(G)$, for $i = 1, \ldots, n-1$, $n \geq 2$. We will denote it by $(x_1, \ldots, x_n)$ or sometimes by $P_{x_1 x_n}$. A path with $x_1 = x_n$ is called a *cycle*. Recall that the *distance* $d_G(x, y)$ of two vertices x and y in G is meant as the length (i.e., the number of edges) of the shortest path joining x and y in G. If G is connected, then $d_G(x, y)$ is finite. By C_n, $n \geq 3$ we denote a graph called an *n-cycle*, if $V(C_n)$ can be so arranged as a sequence $x_1, \ldots, x_n$, then $E(C_n) = \{(x_i, x_{i+1}); i = 1, \ldots, n \text{ and } x_{n+1} = x_1\}$. Analogously we define a graph called an *n-path*, $n \geq 2$ and we denote it by P_n. By $\langle V_1 \rangle_G$ we will denote a subgraph of G induced by $V_1 \subset V(G)$. If H is a subgraph of G induced by some subset, then we shall briefly write $H \leq G$. We say

that G is *R-free* if it does not contain induced subgraphs isomorphic to a given graph R. Note also that the words maximum and minimum refer to the cardinality of a set with a prescribed property. Also as usual the word maximal refers to set-inclusion. Let k be a fixed positive integer such that $k \geq 1$. We say that a subset $Q \subseteq V(G)$ is a *k-distance* clique of (or in) G if

(1) for any $x, y \in Q$, $d_G(x, y) \leq k$ and
(2) $\langle Q \rangle_G$ is connected.

Denote by $\mathcal{C}_k(G)$ the family of all maximal k-distance cliques of G. Note that 1-distance clique is a clique of G. We say that $S \subset V(G)$ is a *k-stable transversal* of G if for each $Q \in \mathcal{C}_k(G)$, $|S \cap Q| = 1$. This implies that for every $x, y \in S$, $d_G(x, y) > k$. A subset of vertices which has such a property we will call a *k-distance stable set* in G. The *k-distance chromatic number* $\chi_k(G)$ of G is the smallest cardinality among partitions of the $V(G)$ into k-distance stable sets. The minimum number of k-distance cliques which cover $V(G)$ we denote $\Theta_k(G)$. Moreover, $\omega_k(G)$ denotes the cardinality of the maximum k-distance clique and $\alpha_k(G)$ is the cardinality of the maximum k-distance stable set.

Note that $\chi_1(G)$ is the chromatic number $\chi(G)$, 1-stable set is a stable set in a graph, $\Theta_1(G)$ is the minimum number of cliques which cover $V(G)$, $\omega_1(G)$ is the maximum cardinality of a clique and $\alpha_1(G)$ is the stability number of the graph G.

We say that a graph G is a *clique-tree* if

(3) G is connected and
(4) G is C_n-free, for $n \geq 4$ and
(5) for any two cliques $Q_i, Q_j \in \mathcal{C}_1(G)$, $|Q_i \cap Q_j| \leq 1$ and
(6) for an arbitrary clique $Q \in \mathcal{C}_1(G)$, $1 \leq |Q \cap \bigcup_{Q' \in \mathcal{C}_1(G), Q' \neq Q} Q'| \leq 2$.

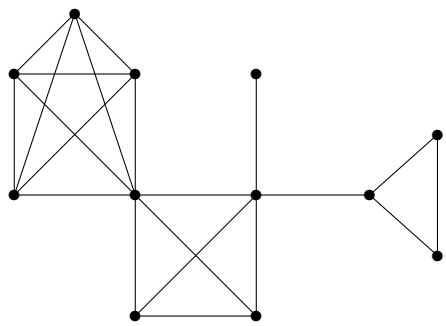

Figure 1

Note that trees are clique-trees. Another example of a clique-tree is on Figure 1.

For $k \geq 1$ by a k-th *power* of a graph G we mean a graph G^k such that $V(G^k) = V(G)$ and $(x, y) \in E(G^k)$ if and only if $d_G(x, y) \leq k$.
Note that G^1 is isomorphic to G.

For $k \geq 1$ we define three classes of graphs $\mathcal{P}_{\chi_k}$, $\mathcal{P}_{\alpha_k}$ and $\mathcal{P}_{kS}$ in the following way:

(7) $G \in \mathcal{P}_{\chi_k}$ if and only if for each $H \leq G$, $\chi_k(H) = \omega_k(H)$,
(8) $G \in \mathcal{P}_{\alpha_k}$ if and only if for each $H \leq G$, $\alpha_k(H) = \Theta_k(H)$,
(9) $G \in \mathcal{P}_{kS}$ if and only if for each $H \leq G$, H has a k-stable transversal.

Note that if $k = 1$, then we obtain the well-known classes of graphs, namely $\mathcal{P}_{\chi_1}$ is a class of χ-perfect graphs, $\mathcal{P}_{\alpha_1}$ is a class of α-perfect graphs and $\mathcal{P}_{1S}$ is a class of strongly perfect graphs. For more information about α-perfect, χ-perfect and strongly perfect graphs the reader is refered to [1], [2], [5], [6], [8].

The dependencies between these classes are known.

Theorem 1 [5]**.** *A graph is α-perfect if and only if it is χ-perfect.*

As a consequence: a graph which is α-perfect and χ-perfect was called *perfect.*

Theorem 2 [2]**.** *A strongly perfect graph is perfect.*

In other words, $\mathcal{P}_{\chi_1} = \mathcal{P}_{\alpha_1} \supset \mathcal{P}_{1S}$.

At the beginning we recall some classical results concerning perfect and strongly perfect graphs which will be used in our further investigations. For convenience, we put $\mathcal{P} = \mathcal{P}_{\chi_1} = \mathcal{P}_{\alpha_1}$ and $\mathcal{SP} = \mathcal{P}_{1S}$.

Theorem 3 [2]**.** *If G is P_4-free, then $G \in \mathcal{SP}$.*

Theorem 4 [2]**.** *If G is triangulated, then $G \in \mathcal{SP}$.*

Theorem 5 [1]**.** *If $G \in \mathcal{P}$, then G is C_{2n+1}-free, for $n \geq 2$.*

2. Results

Throughout this section, we assume that $k \geq 1$. The aim of this section is to formulate the dependencies between classes $\mathcal{P}_{\chi_k}$, $\mathcal{P}_{\alpha_k}$ and $\mathcal{P}_{kS}$, for $k \geq 1$. We also give some examples of graphs belonging to these classes.

Theorem 6. *If G is connected and $|V(G)| \leq k+1$, then $G \in \mathcal{P}_{\chi_k} \cap \mathcal{P}_{\alpha_k} \cap \mathcal{P}_{kS}$.*

Theorem 7. *For any $t \geq 2$, $P_t \in \mathcal{P}_{\chi_k} \cap \mathcal{P}_{\alpha_k} \cap \mathcal{P}_{kS}$.*

A necessary condition for a graph to belong to $\mathcal{P}_{\chi_k} \cup \mathcal{P}_{\alpha_k} \cup \mathcal{P}_{kS}$ in the next theorem is given.

Theorem 8. *If $G \in \mathcal{P}_{\chi_k} \cup \mathcal{P}_{\alpha_k} \cup \mathcal{P}_{kS}$, then G is $C_{n(k+1)+r}$-free, for $k \geq 1$, $n \geq 2$ and $0 < r \leq k$. Moreover, the graph $C_{n(k+1)+r}$ is a minimal (with respect to the number of edges by a fixed number of vertices) forbidden subgraph.*

Proof. For $k = 1$ the result follows from Theorem 2 and 5.

Let $k \geq 2$ and assume that G has $C_{n(k+1)+r}$, $k \geq 1$, $n \geq 2$, $0 < r \leq k$, as induced subgraph.

1) It is evident that $\chi_k(C_{n(k+1)+r}) = k + r + 1$ and $\omega_k(C_{n(k+1)+r}) = k + 1$, so $C_{n(k+1)+r} \notin \mathcal{P}_{\chi_k}$.

2) Analogously we show that $C_{n(k+1)+r} \notin \mathcal{P}_{\alpha_k}$, because $\alpha_k(C_{n(k+1)+r}) = n$ and $\Theta_k(C_{n(k+1)+r}) = n + 1$.

3) Suppose to the contrary that $C_{n(k+1)+r} \in \mathcal{P}_{kS}$. This implies that $C_{n(k+1)+r}$ has a k-stable transversal S. Let $x, y \in S$, then $d_{C_{n(k+1)+r}}(x, y) \geq k + 1$. Suppose that there exist vertices $x_i, x_j \in S$ such that $d_{C_{n(k+1)+r}}(x_i, x_j) = l$, where $k + 1 < l < 2(k + 1)$. This means that there exists a path joining vertices x_i and x_j whose inner vertices $x_{i+1}, x_{i+2}, \ldots x_{i+l-1}$ do not belong to S. Moreover, it may be noted that the vertices $x_{i+1}, x_{i+2}, \ldots x_{i+k+1}$ constitute a k-distance clique Q of $C_{n(k+1)+r}$ which has no common vertex with the set S. This contradicts the assumption that S is a k-stable transversal of $C_{n(k+1)+r}$.

In conclusion: for any two vertices belonging to any k-stable transversal of $C_{n(k+1)+r}$ holds either $d_{C_{n(k+1)+r}}(x_i, x_j) = k + 1$ or $d_{C_{n(k+1)+r}}(x_i, x_j) \geq 2(k + 1)$ and there exists in the shortest path joining x_i, x_j at least one vertex which belongs to S.

As a consequence: if $x_t \in S$, where $1 \leq t \leq n(k+1)+r$, then $x_{t+p(k+1)} \in S$ for $p = 1, 2, \ldots, n - 1$. If $t + p(k + 1) > n(k + 1) + r$, then $x_{t+p(k+1)} = x_{t+p(k+1)-(n(k+1)+r)}$. Note that

$$\begin{aligned} d_{C_{n(k+1)+r}}(x_t, x_{t+(n-1)(k+1)}) &= \min\{k + r + 1, (n - 1)(k + 1)\} \\ &= \begin{cases} k + 1, & if \quad n = 2, \\ k + r + 1, & if \quad n > 2. \end{cases} \end{aligned}$$

In other words, in both the cases there exists a path $(x_{t+(n-1)(k+1)}, x_{t+(n-1)(k+1)+1}, \ldots, x_{t+n(k+1)+r} = x_t)$ of length $k+r+1$ not containing inner vertices from the set S. Choose $k+1$ vertices from this path and form a k-distance clique $Q' = \{x_{t+(n-1)(k+1)+1}, \ldots, x_{t+(n-1)(k+1)+k+1} = x_{t+n(k+1)}\}$. Because of $S \cap Q' = \emptyset$ we obtain a contradiction to the assumption that S is a k-stable transversal of $C_{n(k+1)+r}$. Finally, $C_{n(k+1)+r} \notin \mathcal{P}_{kS}$.

Of course, $C_{n(k+1)+r}$ is a minimal forbidden subgraph for classes $\mathcal{P}_{\chi_k}$, $\mathcal{P}_{\alpha_k}$ and $\mathcal{P}_{kS}$ because every its induced proper connected subgraph $H \leq C_{n(k+1)+r}$ is a path or an isolated vertex. So $H \in \mathcal{P}_{\chi_k}$, $\mathcal{P}_{\alpha_k}$ and $H \in \mathcal{P}_{kS}$, which completes the proof.

Theorem 9. *Let m be an integer, $m \geq 3$ and $k \geq 1$. $C_m \in \mathcal{P}_{\chi_k} \cap \mathcal{P}_{\alpha_k} \cap \mathcal{P}_{kS}$ if and only if $m \leq 2k+1$ or there exists an integer $n \geq 2$ such that $m = n(k+1)$.*

Proof. If $m = n(k+1) + r$, $n \geq 2$, $0 < r \leq k$, then $C_m \notin \mathcal{P}_{\chi_k} \cap \mathcal{P}_{\alpha_k} \cap \mathcal{P}_{kS}$ from Theorem 8. Now, suppose that $m \leq 2k+1$ or $m = n(k+1)$ and $H \leq C_m$. Consider two cases:

Case 1. H is a proper subgraph of C_m.
If H is isomorphic to P_t, where $1 < t < m-1$, then from Theorem 7 we have that $\chi_k(H) = \omega_k(H)$, $\alpha_k(H) = \Theta_k(H)$ and H has a k-stable transversal. For disconnected subgraphs we prove analogously for each component. If H is totaly disconnected, then $\chi_k(H) = \omega_k(H) = 1$, $\alpha_k(H) = \Theta_k(H) = |V(H)|$ and $V(H)$ is a k-stable transversal.

Case 2. H is isomorphic to C_m.
If $m \leq 2k+1$, then C_m is a k-distance clique. From this fact it follows that $\chi_k(H) = |V(H)| = \omega_k(H)$, $\alpha_k(H) = 1 = \Theta_k(H)$ and an arbitrary vertex from $V(H)$ is a k-stable transversal of H.

If $m = n(k+1)$ for $n \geq 2$, then $\chi_k(H) = k+1 = \omega_k(H)$, $\alpha_k(H) = n = \Theta_k(H)$ and $S_t = \{x_t, x_{t+(k+1)}, \ldots, x_{t+(n-1)(k+1)}\}$ for $t = 1, \ldots, k$.

Thus the theorem is proved.

Theorem 10 [4]**.** *If G is connected, then for $k \geq 2$, $\chi_k(G) = k+1$ if and only if*

(a) $|V(G)| = k+1$ *or*
(b) *G is isomorphic to P_m, for $m \geq k+1$ or*
(c) *G is isomorphic to $C_{n(k+1)}$, for $n \geq 1$.*

As a consequence of Theorems 6, 7, 9 and 10 is the following statement.

Corollary 1. If $\chi_k(G) = k+1$, then $G \in \mathcal{P}_{\chi_k} \cap \mathcal{P}_{\alpha_k} \cap \mathcal{P}_{kS}$.

Now we establish dependencies between distance perfectness of G and perfectness of G^k, for $k \geq 1$.

Considering G and G^k immediately gives:

(10) $S \subset V(G)$ is a k-distance stable set in G if and only if S is a stable set in G^k.

(11) If Q is a (maximal) k-distance clique of G, then Q is a (maximal) clique of G^k.

For (11) the opposite implication is not true because if Q is a clique of G^k, then $\langle Q \rangle_G$ is not connected in the general case. For example (see Figure 2): $\{x_1, x_3, x_5\}$ is a clique in G^2, but it is not 2-distance clique in G because $\langle \{x_1, x_3, x_5\} \rangle_G$ is disconnected.

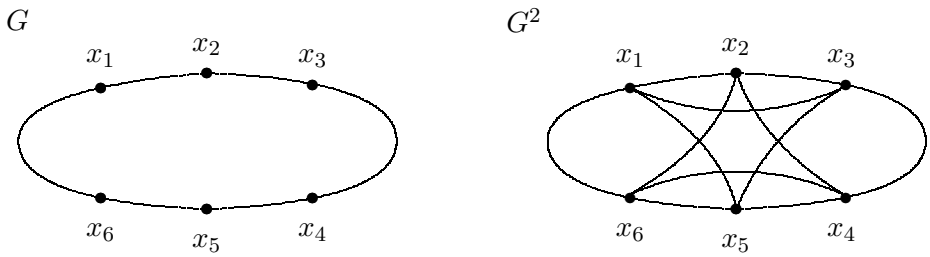

Figure 2

The converse implication holds only for special classes of graphs. All this together yields that distance perfectness of G is not equivalent to perfectness of G^k.

Theorem 11. *For $n \geq 1$, $C^k_{n(k+1)} \in \mathcal{SP}$ if and only if $n = 1, 2$.*

Proof. For $k = 1$ the result follows from Theorem 9.

Suppose that $k \geq 2$ and $n = 1, 2$. Then by the definition of the k-th power of a graph we obtain that $C^k_{n(k+1)}$ does not have the graph P_4 as induced subgraph. Hence, by Theorem 3 $C^k_{n(k+1)} \in \mathcal{SP}$. Now assume that $k \geq 2$, $n \geq 3$ and let $V(C_{n(k+1)}) = \{x_1, x_2, \ldots, x_{n(k+1)}\}$. There are integers t, r such that $t \geq n$, $0 \leq r < k$ and $n(k+1) = tk + r$. It is not difficult to observe that the graph $C^k_{n(k+1)}$ has an induced subgraph C isomorphic to an m-cycle C_m, namely:

1) if t is odd and $r = 0$ (i.e., $n(k+1) = tk$, so $3 \leq n < t$), then $V(C) = \{x_1, x_{1+k}, \ldots, x_{1+tk} = x_1\}$ and C is isomorphic to C_m for $m = t \geq 5$, or

2) if t is odd and $r = 1$, then $V(C) = \{x_1, x_{1+k}, \ldots, x_{1+(t-3)k}, x_{1+(t-3)k+1}, x_{1+(t-2)k+1}, x_{1+(t-2)k+2}, x_{1+(t-1)k+2}, x_{1+tk+1} = x_1\}$ and C is isomorphic to C_m for $m = t + 2 \geq 5$, or

3) if t is odd and $1 < r < k$, then $V(C) = \{x_1, x_{1+k}, \ldots, x_{1+(t-1)k}, x_{1+(t-1)k+1}, x_{1+tk+1}, x_{1+tk+r} = x_1\}$ and C is isomorphic to C_m for $m = t + 2 \geq 5$.

4) if t is even and $r = 0$, then $V(C) = \{x_1, x_{1+k}, \ldots, x_{1+(t-2)k}, x_{1+(t-2)k+1}, x_{1+(t-1)k+1}, x_{1+tk} = x_1\}$ and C is isomorphic to C_m for $m = t + 1 \geq 5$, or

5) if t is even and $r \neq 0$, then $V(C) = \{x_1, x_{1+k}, \ldots, x_{1+tk}, x_{1+tk+r} = x_1\}$ and C is isomorphic to C_m for $m = t + 1 \geq 5$.

Consequently, $C^k_{n(k+1)}$ has C_m, $m \geq 5$ as induced subgraph, hence $C^k_{n(k+1)} \notin \mathcal{SP}$ by Theorem 2 and 5. Thus the theorem is proved.

In the same way we can prove the following theorem.

Theorem 12. *For $k \geq 2$ a graph $C^k_{n(k+1)} \in \mathcal{P}$ if and only if $n = 1, 2$.*

Corollary 2. The following equivalence is true only for $n = 1, 2$
$C_{n(k+1)} \in \mathcal{P}_{\chi_k} \cap \mathcal{P}_{\alpha_k} \cap \mathcal{P}_{kS}$ if and only if $C^k_{n(k+1)} \in \mathcal{P} \cap \mathcal{P}_{1S} = \mathcal{SP}$.

Theorem 13. *Let T be a tree. A subset $Q \subset V(T)$ is a maximal k-distance clique of T if and only if Q is a maximal clique of T^k, for $k \geq 1$.*

Proof. For $k = 1$ the theorem is obvious.

Let $k \geq 2$. If Q is a maximal k-distance clique of T, then Q is a maximal clique of T^k by (11).

Conversely we assume that Q is a maximal clique in T^k. We shall show that Q is a maximal k-distance clique of T. From the definition of the k-th power of a graph it follows that if $x, y \in Q$, then $d_T(x, y) \leq k$. Moreover, if Q is a maximal subset with this property in T^k, then it is maximal in T. Hence it remains to show that the induced subgraph $\langle Q \rangle_T$ is connected. Assume that $|Q| \geq 2$ and $\langle Q \rangle_T$ is disconnected. Let $Q = \bigcup_{i=1}^n Q_i$, where $\langle Q_i \rangle_T$ is connected and $\langle Q_i \cup \{v\} \rangle_T$ is disconnected for $v \in Q_j$, $j \neq i$. In consequence, there exist two vertices, say $x \in Q_i$ and $y \in Q_j$ not joined by a path in $\langle Q \rangle_T$. But there exists a path of length at most k joined x and y in T. Then we deduce that there is a vertex $z \in P_{xy}$ and $z \in V(T) \setminus Q$. From the fact that $z \notin Q$ it follows that there exists $u \in Q$ such that $d_T(z, u) > k$, because Q is maximal. This implies the existence of a path P_{xu} in T of length greater than k. Because in a tree every pair of vertices is joined by

exactly one path, so $d_T(x,u) > k$ but this cannot occur since $x, u \in Q$, so $d_T(x,u) < k$.

Theorem 14. *Let T be a tree. Then $T \in \mathcal{P}_{\chi_k} \cap \mathcal{P}_{\alpha_k} \cap \mathcal{P}_{kS}$ if and only if $T^k \in \mathcal{SP}$, for $k \geq 1$.*

The proof is straightforward using Theorem 13 and applying (10).

It is well known, see [1], that for an arbitrary $k \geq 1$, T^k is triangulated and by Theorem 4, $T^k \in \mathcal{SP} \subset \mathcal{P}$.

All this together gives

Corollary 3. For $k \geq 1$, $T \in \mathcal{P}_{\chi_k} \cap \mathcal{P}_{\alpha_k} \cap \mathcal{P}_{kS}$.

Theorem 15. *If G is a clique-tree, then $G \in \mathcal{P}_{kS}$.*

Proof. If G is a clique, then of course $G \in \mathcal{P}_{kS}$.

If G is a tree, then the result follows from Corollary 3. Assuming that G is not a tree and G is not a clique, we consider the following contraction of G to a tree. Create a subset $R \subset V(G)$ in the following way. Let $Q \in \mathcal{C}_1(G)$ be an arbitrary maximal clique of G. There are two cases to establish.

Case 1. If $\langle Q \rangle_G$ is isomorphic to K_2, then $Q \subset R$.

Case 2. Suppose that $\langle Q \rangle_G$ is not isomorphic to K_2. Then by the definition of a clique-tree it follows that for any $Q_i \in \mathcal{C}_1(G)$, $Q_i \neq Q$ we must have $|Q \cap Q_i| \leq 1$. Let $Q_1, Q_2, \ldots, Q_t$, $t \geq 1$ be cliques such that $|Q \cap Q_i| = 1$ (Since G is connected and G is not a clique, there exists at least one such clique). Moreover, $1 \leq |Q \cap (Q_1 \cup \ldots \cup Q_t)| \leq 2$ by (6).

If $|Q \cap (Q_1 \cup \ldots \cup Q_t)| = 2$, then $Q \cap (Q_1 \cup \ldots \cup Q_t) \subset R$. If $|Q \cap (Q_1 \cup \ldots \cup Q_t)| = 1$, then $(Q \cap (Q_1 \cup \ldots \cup Q_t) \cup \{x\}) \subset R$ where x is an arbitrary vertex of Q such that $x \notin Q \cap (Q_1 \cup \ldots \cup Q_t)$. Of course, a subgraph induced by a subset R is connected and it does not contain cycles, i.e., $\langle R \rangle_G$ is a tree. So $\langle R \rangle_G$ has a k-stable transversal S, it follows from Corollary 3. From this fact it follows that every maximal k-distance clique Q' of $\langle R \rangle_G$ meets S. Moreover, for every maximal k-distance clique $Q^* \subset V(G)$ of a clique-tree there exists a maximal k-distance clique $Q' \subset R$ such that $Q' \subseteq Q^*$. So, for each $Q^* \subset V(G)$, $Q^* \cap S \neq \emptyset$. Hence S is a k-stable transversal of a clique-tree. Since any induced proper subgraph H of a clique-tree is a clique-tree, the existence of k-distance stable transversal of H is assured.

Thus $G \in \mathcal{P}_{kS}$ which proves the theorem.

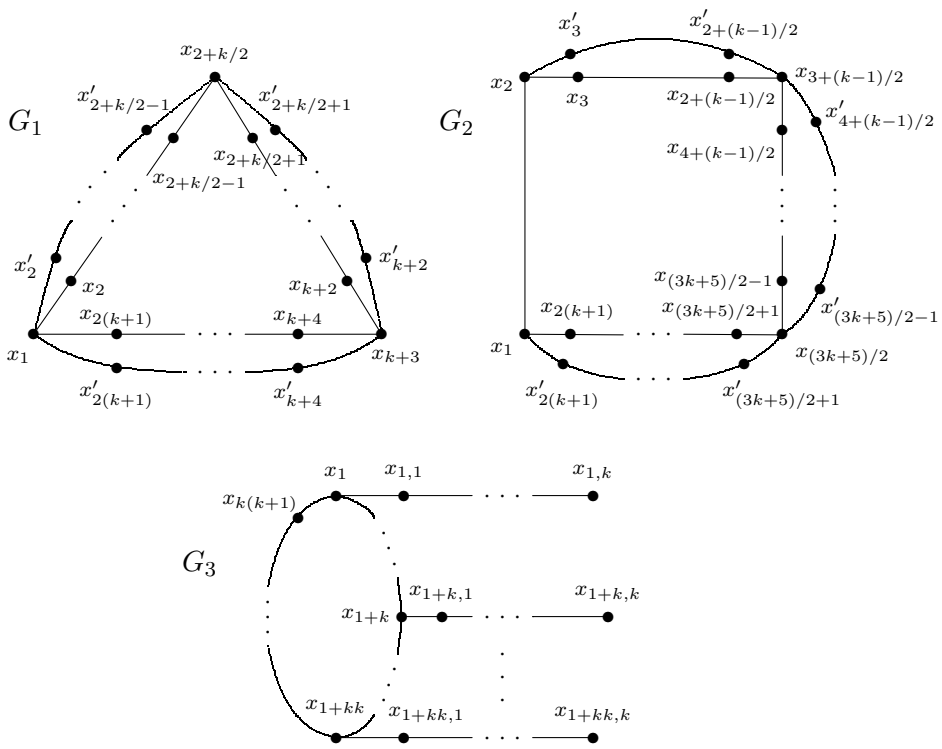

Figure 3

For a fixed positive integer $k \geq 2$ we construct special graphs G_1, G_2 and G_3.

Let G and R be two disjoint copies of a $2(k+1)$-cycle $C_{2(k+1)}$, on the vertex sets $V(G) = \{x_1, \ldots, x_{2(k+1)}\}$ and $V(R) = \{x'_1, \ldots, x'_{2(k+1)}\}$ and the edge sets $E(G) = \{(x_i, x_{i+1}); i = 1, 2, \ldots, 2(k+1)$ and $x_{2(k+1)+1} = x_1\}$ and $E(R) = \{(x'_i, x'_{i+1}); i = 1, 2, \ldots, 2(k+1)$ and $x'_{2(k+1)+1} = x'_1\}$.

Consider two cases:

(12) If k is even, then we identify vertices x_i and x'_i for $i = 1, 2 + \frac{k}{2}, k+3$. The resulting graph we denote by G_1.

(13) If k is odd, then we identify vertices x_i and x'_i for $i = 1, 2, 3 + \frac{k-1}{2}, k+3$ and replace multiple edges with end vertices x_1 and x_2 by an edge. The resulting graph we denote by G_2.

(14) By the graph G_3 we will mean a graph obtained from $k(k+1)$-cycle $C_{k(k+1)}$ by adding $k+1$ vertex disjoint paths of length k such that each path starts in a vertex x_{1+tk} for $t = 0, 1, \ldots k$.

Theorem 16. *$\mathcal{P}_{kS} \subset \mathcal{P}_{\chi_k}$ if and only if $k = 1$.*

Proof. If $k = 1$, then $\mathcal{SP} \subset \mathcal{P}_{\chi_1} = \mathcal{P}$ by Theorem 2. Assuming that k is fixed and $k > 1$, we shall show that there is a graph G such that
(15) $G \in \mathcal{P}_{kS}$ and $G \notin \mathcal{P}_{\chi_k}$.

It turns out if k is even, then the graph G_1 (see Figure 3) constructed above satisfies condition (15).

Let H be an arbitrary induced subgraph of G_1. Consider the following cases:

Case 1. H has exactly two $2(k+1)$-cycles as induced subgraphs. Then H is isomorphic to G_1, hence it is easy to see that $S = \{x_{2+\frac{k}{2}}, x_{3+\frac{3k}{2}}\}$ is a k-stable transversal of G_1.

Case 2. H has exactly one $2(k+1)$-cycle as induced subgraph.

In this case we have the following possibilities:

Subcase 2.1. H has exactly two $(k+2)$-cycles as induced subgraphs. Then H does not have any $2k$-cycle (in otherwise there exist two $2(k+1)$-cycles). Then $S = \{x_1, x_{2+k}\}$.

Subcase 2.2. H has less than two $(k+2)$-cycles as induced subgraphs. Then either $S = \{x_1, x_{2+k}\}$ or $S = \{x_1, x'_{2+k}\}$.

Case 3. H has no induced $2(k+1)$-cycle.
In this case we have the following possibilities:

Subcase 3.1. H has exactly two $(k+2)$-cycles as induced subgraphs. Analogously as in 2.1.

Subcase 3.2. H has exactly one $(k+2)$-cycle as induced subgraph. Denote by x_iPx_j subgraph induced by subset $\{x_i, x_{i+1}, \dots x_j\}$ of vertices placed on the n-cycle. Suppose that H has an induced $(k+2)$-cycle and an induced $2k$-cycle. $V(C_{2k})$ and $V(C_{k+2})$ are k-distance cliques of H, so k-stable transversal of x_1Px_{3+k} is a k-stable transversal of C_{2k}. Analogously k-stable transversal of $x_1Px_{2+\frac{k}{2}}$ is a k-stable transversal of C_{k+2}. We can consider a subgraph $H' \leq H$ such that $V(H') = (V(H) \setminus V(x_1Px_{2+\frac{k}{2}} \cup x_1Px_{3+k})) \cup \{x_1, x_{2+\frac{k}{2}}, x_{3+k}\}$. Of course H' is an acyclic graph because we delete a part of each cycle in H. So by Corollary 3, H' has a k-stable transversal S. The same set S is a k-stable transversal of H.

If H does not have an induced $2k$-cycle, than we prove analogously.

Subcase 3.3. H has no induced $(k+2)$-cycle.
If H has $2k$-cycle, then we prove analogously as in Subcase 3.2.
If H has no cycle, then H is acyclic. So from Corollary 3, H has a k-stable transversal.

All this together gives that $G_1 \in \mathcal{P}_{kS}$ for even k.
To prove the remaining part of the theorem we observe that the maximum k-distance clique Q of G_1 is in the form $Q = \{x_2, x_2', x_3, x_3', \ldots, x_{(2+\frac{k}{2})-1}, x'_{(2+\frac{k}{2})-1}, x_{2+\frac{k}{2}}, x_{(2+\frac{k}{2})+1}, x'_{(2+\frac{k}{2})+1}, \ldots, x_{(3+k)-1}, x'_{(3+k)-1}\}$ and $\omega_k(G_1) = 2k+1$.

Of course, Q is coloured by $2k+1$ distinct colours and assume that vertex $x_{2+\frac{k}{2}} \in Q$ has a colour φ. Consider vertices $x_{3+\frac{3k}{2}}$ and $x'_{3+\frac{3k}{2}}$. It should be noted that the distance between an arbitrary vertex from Q and $x_{3+\frac{3k}{2}}$ or $x'_{3+\frac{3k}{2}}$ is less than or equal to $k+1$. Moreover, there exists only one vertex $x_{2+\frac{k}{2}}$, such that $d_{G_1}(x_{2+\frac{k}{2}}, x_{3+\frac{3k}{2}}) = k+1$ and $d_{G_1}(x_{2+\frac{k}{2}}, x'_{3+\frac{3k}{2}}) = k+1$. So we can colour the vertex $x_{3+\frac{3k}{2}}$ also by φ. But for $x'_{3+\frac{3k}{2}}$ we have used a new colour not used for colouring Q. From this fact it follows that $\chi_k(G_1) > 2k+1$. Hence $G_1 \notin \mathcal{P}_{\chi_k}$.

Now assume that k is odd and consider graph G_2 (see Figure 3). To prove that $G_2 \in \mathcal{P}_{kS}$ we use the same method as for the graph G_1. As the maximum k-distance clique which realizes $\omega_k(G_2) = 2k$ we can take $Q = \{x_2, x_3, x_3', \ldots, x_{(3+\frac{k-1}{2})-1}, x'_{(3+\frac{k-1}{2})-1}, x_{3+\frac{k-1}{2}}, x_{(3+\frac{k-1}{2})+1}, x'_{(3+\frac{k-1}{2})+1}, \ldots, x_{k+2}, x'_{k+2}\}$. Evidently Q is coloured by $2k$ distinct colours and let $x_{3+\frac{k-1}{2}}$ has a colour φ, then proving analogously as for G_1 we obtain that either $x_{4+\frac{3k-1}{2}}$ or $x'_{4+\frac{3k-1}{2}}$ have to be coloured by a new colour not used for Q. Consequently, $\chi_k(G_2) > 2k$, so $G_2 \notin \mathcal{P}_{\chi_k}$ for an odd k.

Thus the theorem is proved.

Theorem 17. *$\mathcal{P}_{kS} \subset \mathcal{P}_{\alpha_k}$ if and only if $k = 1$.*

Proof. If $k = 1$, then $\mathcal{P}_{kS} \subset \mathcal{P}_{\alpha_k} = \mathcal{P}$ by Theorem 2.

We shall show that there exists a graph G such that $G \in \mathcal{P}_{kS}$ and $G \notin \mathcal{P}_{\alpha_k}$. As the graph G we can take a graph G_3 (see Figure 3). First, we shall show that $G_3 \in \mathcal{P}_{kS}$.

Let $H \leq G_3$ be an induced subgraph of G_3. Consider two cases:

Case 1. H has an induced $k(k+1)$-cycle.

If H is isomorphic to G_3, then H has a k-stable transversal of the form $S = \{x \in V(G_3); x = x_{1+t(k+1)}$ for $t = 0, 1, \ldots, (k-1)\} \cup \{x \in V(G_3); x = x_{t,m}$ for $t = 1, 2, \ldots, k$ and $m = k + 1 - d_{G_3}(x_1, x_{1+tk})/k\}$.

If H is connected and not isomorphic to G_3, then the k-stable transversal of H is $S' = S \cap V(H)$.

Case 2. H has no induced $k(k+1)$-cycle.
In this case H is acyclic and by Carollary 3 it has a k-stable transversal. All this together gives that $G_3 \in \mathcal{P}_{kS}$.

Now we prove that $G_3 \notin \mathcal{P}_{\alpha_k}$. The k-stable transversal S obtained in Case 1 is the maximum k-stable set in G_3, so $\alpha_k(G_3) = 2k$. The number $\theta_k(G_3) = 2k + 1$ is realized by a family of k-distance cliques $Q_i = \{x_{1+i(k+1)}, \ldots, x_{1+i(k+1)+k}\}$ for $i = 0, 1, \ldots, k-1$ and $Q'_i = \{x_{1+i(k+1)}, x_{1+i(k+1),1}, \ldots x_{1+i(k+1),k}\}$, $i = 0, 1, \ldots, k$.

All this together leads to $\alpha_k(G_3) < \theta_k(G_3)$ and this shows that $G_3 \notin \mathcal{P}_{\alpha_k}$. Thus the theorem is proved.

Theorem 18. *$\mathcal{P}_{\alpha_k} = \mathcal{P}_{\chi_k}$ if and only if $k = 1$.*

Proof. If $k = 1$, then $\mathcal{P}_{\alpha_k} = \mathcal{P}_{\chi_k} = \mathcal{P}$ from Theorem 1.

We shall show that if $k > 1$, then $\mathcal{P}_{\chi_k} \not\subset \mathcal{P}_{\alpha_k}$. In other words, it suffices to show that for $k > 1$ there exists a graph G such that $G \in \mathcal{P}_{\chi_k}$ and $G \notin \mathcal{P}_{\alpha_k}$.

From the proof of Theorem 17 we have that $G_3 \notin \mathcal{P}_{\alpha_k}$. It remains to prove that $G_3 \in \mathcal{P}_{\chi_k}$. Let Q be a k-distance clique of the form $Q = \{x_{1+k-a}, \ldots, x_{1+k}, \ldots, x_{1+k+b}\} \cup \{x_{1+k,1}, \ldots, x_{1+k,a}\}$ where $a = b = \frac{k}{2}$ for an even k or $a = \frac{k-1}{2}$, $b = \frac{k+1}{2}$ for an odd k (see Figure 3). Then each maximal k-distance clique of G_3 induces a subgraph isomorphic to $\langle Q \rangle_{G_3}$. To prove that for an arbitrary $H \leq G_3$, $\chi_k(H) = \omega_k(H)$ we consider the following cases.

Case 1. H is isomorphic to G_3.
Then for an even k, $\chi_k(H) = \frac{3k+2}{2} = \omega_k(H)$ and $\chi_k(H) = \frac{3k+1}{2} = \omega_k(H)$ for an odd k. The function which realizes the colouring of H is of the form $f : V(H) \to \{0, 1, \ldots, \chi_k(H) - 1\}$ and

$f(x_n) = i$, for $n = i(\mathrm{mod}(k+1))$, $i = 0, 1, \ldots, k$,
$f(x_{i+tk,m}) = k + m$, for $m = 1, \ldots, a$, $t = 0, \ldots, k$,
$f(x_{i+tk,m}) = f(x_{1+tk+a+1})$, for $m = a+1, \ldots, k$, $t = 0, \ldots, k$.

Case 2. H has one induced $k(k+1)$-cycle and H is not isomorphic to G_3.

Let P be a longest induced path in H such that $E(P) \cap E(C_{k(k+1)}) = \emptyset$ and the length of this path is p.

For $p \geq a$, we prove analogously as in Case 1.

If $0 \leq p < a$, then $\chi_k(H) = k + 1 + p = \omega_k(H)$ and the colouring realizes function $g : V(H) \to \{0, 1, \ldots, \chi_k(H) - 1\}$ where $g(v) = f(v)$ for $v \in V(H) \subset V(G_3)$.

Case 3. H is acyclic.
From Corollary 3 we have that $\chi_k(H) = \omega_k(H)$.

Thus the theorem is proved.

From the proof of Theorem 18 it follows

Corollary 4. $\mathcal{P}_{\chi_k} \not\subset \mathcal{P}_{\alpha_k}$ for $k > 1$.

References

[1] C. Berge, Graphs and Hypergraphs, North-Holland, Amsterdam, 1973.

[2] C. Berge and P. Duchet, *Strongly perfect graphs*, Ann. Discrete Math. **21** (1984) 57–61.

[3] A.L. Cai and D. Corneil, *A generalization of perfect graphs — i-perfect graphs*, J. Graph Theory **23** (1996) 87–103.

[4] F. Kramer and H. Kramer, *Un Probléme de coloration des sommets d'un gráphe*, C.R. Acad. Sc. Paris, **268** Sĕrie A (1969) 46–48.

[5] L. Lovász, *Normal hypergraphs and the perfect graph conjecture*, Discrete Math. **2** (1972) 253–267.

[6] F. Maffray and M. Preissmann, *Perfect graphs with no P_5 and no K_5*, Graphs and Combin. **10** (1994) 173–184.

[7] H. Müller, *On edge perfectness and class of bipartite graphs*, Discrete Math. **148** (1996) 159–187.

[8] G. Ravindra, *Meyniel's graphs are strongly perfect*, Ann. Discrete Math. **21** (1984) 145–148.

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