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EXTREMAL PROBLEMS FOR FORBIDDEN PAIRS THAT IMPLY HAMILTONICITY

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Abstract

Let C denote the claw $K_{1,3}$, N the net (a graph obtained from a K_3 by attaching a disjoint edge to each vertex of the K_3), W the wounded (a graph obtained from a K_3 by attaching an edge to one vertex and a disjoint path P_3 to a second vertex), and Z_i the graph consisting of a K_3 with a path of length i attached to one vertex. For k a fixed positive integer and n a sufficiently large integer, the minimal number of edges and the smallest clique in a k-connected graph G of order n that is CY-free (does not contain an induced copy of C or of Y) will be determined for Y a connected subgraph of either P_6, N, W, or Z_3. It should be noted that the pairs of graphs CY are precisely those forbidden pairs that imply that any 2-connected graph of order at least 10 is hamiltonian. These extremal numbers give one measure of the relative strengths of the forbidden subgraph conditions that imply a graph is hamiltonian.

1 Introduction

We will deal only with finite graphs without loops or multiple edges. Notation will be standard, and we will generally follow the notation of Chartrand and Lesniak in [3] and Bondy and Murty in [2]. Given a graph F, a graph G

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is said to be F-*free* if there is no induced subgraph of G that is isomorphic to F. We will denote by $F \leq G$ that F is an induced subgraph of G. The graph F is generally called a *forbidden subgraph* of G. In the case of forbidden pairs of graphs, say F and H, we will simply say the graph is FH-free, as opposed to $\{F, H\}$-free. The degree of a vertex v in a graph G will be denoted by $d(v)$, and the minimum and maximum degree of vertices in G will be denoted by $\delta(G)$ and $\Delta(G)$ respectively. The independence number of G will be denoted by $\alpha(G)$.

Singletons and forbidden pairs of connected graphs that imply that a 2-connected graph is hamiltonian have been characterized. Also, similar characterizations have been given for other hamiltonian properties such as traceable, pancyclic, cycle extendable, etc. A collection of graphs that are frequently used as forbidden in results of this type are pictured in Figure 1.

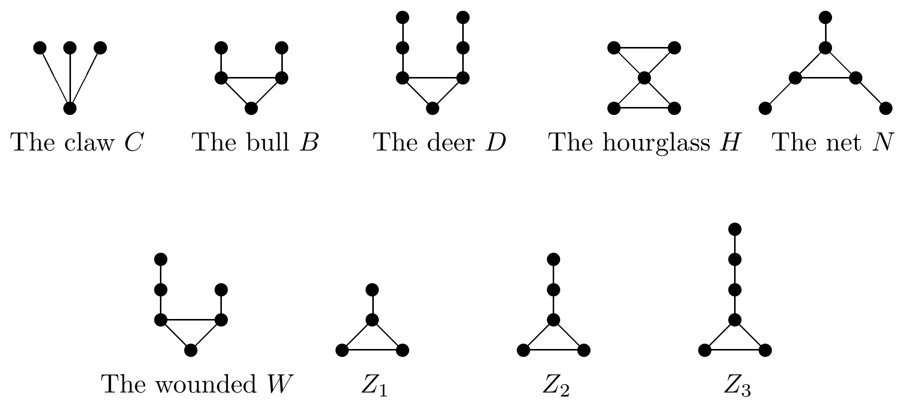

Figure 1

The following result, which extends the results of Bedrossian in [1], gives all forbidden pairs that imply hamiltonicity in 2-connected graphs. A survey of results of this kind for other hamiltonian type properties can be found in [6], and a more general survey on claw-free graphs can be found in [7].

Theorem 1 [8]**.** *Let X and Y be connected graphs with $X, Y \not\leq P_3$, and let G be a 2-connected graph of order $n \geq 10$. Then, G being XY-free implies that G is hamiltonian if, and only if, up to the order of the pairs, $X = C$ and Y is a subgraph of either P_6, N, W, or Z_3.*

The well known degree type conditions that imply that a graph is hamiltonian, such as Dirac's in [4], Ore's in [10], or many of the other degree conditions that followed these two conditions, also imply that the graph is very dense. One motivation, among many others, to look at forbidden subgraph conditions is that they do not, at least on the surface, require that the graph be so dense. Thus, it is natural to examine the number of edges in a graph and the clique size of the graph forced by the forbidden subgraph conditions that imply hamiltonicity, or other hamiltonian type properties. This is the objective of this paper.

The number of edges in a graph G will be denoted by $e(G)$, and the clique number will be denoted $\omega(G)$. The number of edges $e(G)$ or the clique size $\omega(G)$ implied in the case of forbidden pairs X and Y (neither of which is a P_3) such that XY-free implies that a 2-connected graph G is hamiltonian, varies significantly. We will see that for some forbidden pairs the graph G can be very sparse, but some forbidden pairs imply that the graph has many edges.

In the next sections we will investigate the number of edges $e(G)$ and the clique number $\omega(G)$ implied by the pairs of forbidden subgraphs that imply a graph is hamiltonian (see Theorem 1). For some forbidden subgraph pair conditions on a graph G exact bounds will be given, and in other cases we will give reasonable bounds on $e(G)$ and $\omega(G)$. More specifically, in Section 2 we will deal with forbidden subgraphs that imply the graph is relatively dense, in Section 3 forbidden pairs that imply only a moderate number of edges will be considered, and in Section 4 forbidden subgraph pairs that place minimal density conditions on a graph G will be investigated.

Actually the extremal numbers for larger classes of forbidden subgraphs can be considered. For $i \geq 1$ the graph Z_i will denote the graph obtained by identifying the endvertex of a path of length i with one of the vertices of a triangle. For $i, j \geq 1$, the generalized bull $B_{i,j}$ is the graph obtained by attaching two vertex disjoint paths of lengths i and j to distinct vertices of a triangle. Thus $B_{1,1}$ is the Bull and $B_{1,2}$ is the Wounded W. Likewise, the generalized net $N_{i,j,k}$ can be defined for $i, j, k \geq 1$.

The forbidden claw C does not imply the existence of many edges or large cliques, even in the presence of a connectivity condition. The following result makes this precise.

Theorem 2. *Let G be a k-connected C-free graph of order n. If n is sufficiently large, then $\omega(G) \geq \lceil (k+2)/2 \rceil$ and $e(G) \geq kn/2$. These results are sharp for k even, and nearly sharp for k odd.*

Proof. Since G is k-connected, each vertex must have degree at least k, which implies that $e(G) \geq kn/2$. If $k \leq 4$, then the C-free property implies that G has a clique with at least $(k+2)/2$ vertices, so we assume that $k \geq 5$. If G has a vertex v of degree at least k^2, then the neighborhood $N(v)$ of v contains a K_k, since $\alpha(N) < 3$ from the C-free property and $r(K_3, K_k) \leq k^2$ (see [9]). Thus, we can assume that $\Delta(G) < k^2$. Thus, G has large diameter, say d, since at most k^{2i} vertices can be within a distance i of any vertex of G and n the order of G is large.

Select a diameter path $P = (x_0, x_1, \cdots, x_d)$. No vertex $y \notin P$ can be adjacent to more than 3 vertices of P, otherwise, P would not be a diameter path. If y is adjacent to precisely one vertex on P and that vertex is x_i for some $1 \leq i < d$, then there would be a claw centered at x_i. Also, if y is adjacent to precisely x_i and x_{i+2} with $1 \leq i \leq d-2$, then there is a claw centered at x_i. We can assume that y is adjacent to two or three consecutive vertices on the path P, if it has at least one adjacency on P.

Choose a vertex x_i near the middle of the path P, and let N denote the neighborhood of x_i off of the path P. Partition N into three sets: N^0 are those vertices adjacent to each of x_{i-1}, x_i, x_{i+1}, N^- are those vertices adjacent to x_i, x_{i-1} and possibly x_{i-2}, and N^+ are those vertices adjacent to x_i, x_{i+1} and possibly x_{i+2}. Note that to avoid a claw, each of the sets N^-, N^0, N^+ induces a complete graph. Thus, if $N^0 = \emptyset$, we can assume that with no loss of generality that $|N^-| \geq (k-2)/2$, and so $N^- \cup \{x_{i-1}, x_i\}$ induces a complete graph with at least $(k+2)/2$ vertices. Thus, we can assume that $N^0 \neq \emptyset$. If some vertex $y \in N^0$ is not adjacent to some vertex $y^- \in N^-$ and also to some vertex $y^+ \in N^+$, then there would be a claw centered at x_i unless y^- and y^+ are adjacent. However, in this case there would be a claw centered at x_{i-1} unless y^- is adjacent to x_{i-2}. Likewise, y^+ must be adjacent to x_{i+2}. This gives a contradiction to the fact that P is a distance path, because the path $x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}$ could be replaced by the shorter path $x_{i-2}, y^-, y^+, x_{i+2}$. Therefore, we can assume that N^0 can be partitioned into $N^0_- \cup N^0_+$, where each vertex in N^0_- is adjacent to each vertex in N^-, and correspondingly each vertex in N^0_+ is adjacent to each vertex in N^+. Hence, one of the sets $N^- \cup N^0_- \cup \{x_{i-1}, x_i\}$ or $N^+ \cup N^0_+ \cup \{x_i, x_{i+1}\}$ induces a complete graph with at least $(k+2)/2$ vertices.

To see the sharpness consider the graph $H = C_n^{\lceil k/2 \rceil}$. It is easily checked that H is a $2\lceil k/2 \rceil$-connected C-free graph with $e(H) = 2\lceil k/2 \rceil n$ and $\omega(H) = \lceil (k+2)/2 \rceil$. This completes the proof of Theorem 2. ∎

2 Forbidden Subgraphs that Imply Dense Graphs

The only single forbidden graph F that implies that a 2-connected F-free G is hamiltonian, or has any of the other common hamiltonian type properties, is $F = P_3$. Clearly any connected P_3-free graph G must be complete, so this is an example of a forbidden subgraph condition that forces extreme density.

Another well known example that implies the graph is dense is the case of CZ_1-free graphs. This fact is part of the folklore of the discipline, but for sake of completeness, we give the result and its short proof here.

Theorem 3. *If G is a connected CZ_1-free graph of order n with $\Delta(G) \geq 3$, then $G = K_n - M$, when M is a matching (possibly empty) in G.*

Proof. Consider a maximum clique H in G, which has $m \geq 3$ vertices, since $\Delta(G) \geq 3$, and G is C-free. Each vertex of G adjacent to a vertex in H must be adjacent to precisely $m - 1$ vertices; otherwise, there is either a larger clique or an induced Z_1. A vertex a distance 2 from H immediately gives a Z_1, so each vertex of G not in H is adjacent to $m - 1$ vertices of H. Likewise, to avoid a Z_1 or a C, distinct vertices not in H must avoid different vertices of H and must be adjacent. This completes the proof of Theorem 3. ■

As a consequence of Theorem 3 we have that any 2-connected CZ_1-free graph G must have $e(G) \geq n(n-2)/2$ and $\omega(G) \geq n/2$. The next theorem is another example of a forbidden pair of subgraphs that implies the graph is dense.

Theorem 4. *Let G be a k-connected graph $(k \geq 1)$ of order $n \geq k + 1$ that is CP_4-free. Then, $\omega(G) \geq \lceil n/2 \rceil$, and $e(G) \geq \lceil (n^2 + (2k-2)n - 2k^2)/4 \rceil$ if $k \leq n/2$ and $e(G) \geq nk/2$ if $k > n/2$. Also, the lower bound for $\omega(G)$ is sharp for $k \leq n/2$ and each of the bounds for $e(G)$ is sharp for the appropriate range.*

Before completing the proof of Theorem 4, we will give a structure theorem for CP_4-free graphs that will be useful in the proof of Theorem 4.

Theorem 5. *If G is a connected CP_4-free graph, then the complement $\overline{G}$ is just a vertex disjoint union of complete bipartite graphs (possibly all trivial).*

Proof. We will first show that G has two disjoint cliques that span the vertices of G. This will be done by induction, and it is trivial for $n \leq 4$.

One characterization of a P_4-free graph G is that there is always a partition of any set of vertices of G into two sets A and B such that there are either no edges between A and B or all of the edges are in between A and B. This can be verified with a straightforward induction proof. Since G is connected, there must be such a partition, say $A\cup$B, of all of the vertices of G such that all of the edges between A and B are in.

Now, the graph spanned by A, which we will just denote by A, is P_4-free and the independence number $\alpha(A) \leq 2$, since G is C-free. This implies than $\overline{A}$ is P_4-free and also K_3-free. Thus, $\overline{A}$ is bipartite, and so A is spanned by two cliques.

Thus, we let $A = A_1 \cup A_2$, where both A_1 and A_2 are both cliques and disjoint. The same is true of B, so $B = B_1 \cup B_2$ such that both B_1 and B_2 are cliques. This gives that both $A_1 \cup B_1$ and $A_2 \cup B_2$ span cliques, and so, G has two disjoint cliques, say R and S, that span G, which was the claim. Consider the graph $\overline{G}$, which is a bipartite graph with only edges between the sets R and S. Since P_4 is self-complementary, $\overline{G}$ is also P_4-free. Thus, $\overline{G}$ must be a a vertex disjoint union of complete bipartite graphs to avoid an induced P_4. This completes the proof of Theorem 5. ■

***Proof of Theorem* 4.** From Theorem 5 we have the structure of G, which is $K_n - (B_1 \cup B_2 \cup \cdots \cup B_t)$ for some collection of vertex disjoint complete bipartite graphs B_i. Let $b_1, b_2, \cdots, b_t$ be the number of vertices in the bipartite graphs B_i respectively, and we can assume that $b_1 \leq b_2 \leq \cdots \leq b_t$.

Clearly $\omega(G) \geq \lceil n/2 \rceil$, since one of the cliques R or S will have at least that number of vertices. Determining the minimum number of edges in the graph $e(G)$ is equivalent to determining the maximum number of possible edges in the bipartite graphs $B_1 \cup B_2 \cup \cdots \cup B_t$. Observe that G being k-connected is equivalent to $b_t \leq n - k$.

If $k \leq n/2$, then under these conditions, to maximize the number of edges in $B_1 \cup B_2 \cup \cdots \cup B_t$, one should choose $t = 2$ with $b_1 = k$ and $b_2 = n - k$. Also, each of the bipartite graphs B_1 and B_2 should be as balanced as possible, and so $B_1 = K_{\lfloor k/2 \rfloor, \lceil k/2 \rceil}$ and $B_2 = K_{\lfloor (n-k)/2 \rfloor, \lceil (n-k)/2 \rceil}$. This implies that the graph $K_n - (B_1 \cup B_2)$ has at least $\binom{n}{2} - \lfloor k/2 \rfloor \lceil k/2 \rceil - \lfloor (n-k)/2 \rfloor \lceil (n-k)/2 \rceil = \lceil (n^2 + (2k-2)n - 2k^2)/4 \rceil$ edges. If $k \geq n/2$, then each vertex must have degree at least k, and so clearly $e(G) \geq nk/2$.

To see the sharpness of the result for $k \leq n/2$, consider the graph H of order n obtained from a C_4 by replacing the vertices around the cycle with cliques of orders $\lfloor k/2 \rfloor$, $\lceil (n-k)/2 \rceil$, $\lceil k/2 \rceil$, $\lfloor (n-k)/2 \rfloor$ respectively, and replacing each edge with the appropriate complete bipartite graph. The graph H is CP_4-free, $\omega(G) = \lceil n/2 \rceil$, and $e(G) = \lceil (n^2 + (2k-2)n - 2k^2)/4 \rceil$. If $k > n/2$, then let $H = K_n - (B_1 \cup B_2 \cup \cdots \cup B_t)$ where the B_i's are chosen such that each vertex of H has degree at least k but is as small as possible. Therefore, the number of vertices in each part of B_i will be at most $n-k-1$ and each B_i will be as balanced as possible. For values of k and n with appropriate divisibility properties this will give a regular graph of order k. This verifies the sharpness of the result and completes the proof of Theorem 4. ■

The extremal results for CZ_2-free graphs are very similar to those for CP_4-free graphs for sufficiently large order graphs, as the next result indicates. Of course, any CP_4-free graph is certainly CZ_2-free, so it is natural to expect some relationship between these extremal graphs.

Theorem 6. *Let G be a k-connected CZ_2-free graph with $k \geq 2$ and $\delta(G) \geq 3$. Then, for n sufficiently large, $e(G) \geq \lceil (n^2 + (2k-2)n - 2k^2)/4 \rceil$ and $\omega(G) \geq cn^{1/2}$ for some constant c. The lower bound on $e(G)$ is sharp, and the lower bound for $\omega(G)$ is at most $c(\log n)n^{2/3}$.*

Proof. The sharpness for the lower bounds on the number of edges in CZ_2-free graphs comes from the examples given in the proof of Theorem 4, since CP_4-free implies CZ_2-free. Recall that these examples came from a C_4 by replacing the vertices around the cycle with cliques of orders $\lfloor k/2 \rfloor$, $\lceil (n-k)/2 \rceil$, $\lceil k/2 \rceil$, $\lfloor (n-k)/2 \rfloor$ respectively, and replacing each edge with the appropriate complete bipartite graph. A bound on $\omega(G)$ comes from a result by Spencer which is stated in [5] and implies that the Ramsey number $r(\{C_3, C_4\}, K_n) > c(n/\log n)^{3/2}$ for some constant c. This implies that there is a graph H of order $c(n/\log n)^{3/2}$ with clique number $\omega(H) < n$. Also, since there is no C_3 in $\overline{H}$, H is C-free, and, since $\overline{H}$ does not contain a C_4, this implies that H is Z_2-free. Therefore for n sufficiently large there is a graph L of order n that is CZ_2-free for which $\omega(L) \leq c(\log n)n^{2/3}$.

To verify the lower bound for $e(G)$, consider a smallest minimal vertex cut S for the CZ_2-free graph G. Let A and B be the two components of $G-S$. (Note that since S is a minimal cut, each vertex of S has a adjacency in each component of $G-S$, and thus the C-free property implies there are just 2 components.) The C-freeness also implies that the neighborhood in A (or in B) of a vertex $x \in S$ induces a complete graph. We will first consider the case when $|A| \geq 2$ and $|B| \geq 3$. If a vertex x has at least two adjacencies in A, say a_1, a_2, then x must be adjacent to all of the vertices in B. Otherwise, there would be an Z_2 using a_1, a_2, x and a neighbor of x

in B along with an appropriate non-neighbor. This argument is symmetric with respect to A and B. So, if any vertex in $x \in S$ has two adjacencies in either A or B, then x will be adjacent to all of the vertices in $A \cup B$, and each of A and B is a complete graph. Under the assumption that there is such a vertex x, and if a vertex $y \in S$ has just one adjacency $b \in B$, then there will be an induced Z_2 using a triangle in B (recall that B is complete) containing b, y, and a vertex in A. Likewise, y adjacent to one vertex in A will result in a Z_2 with y on the triangle. Thus, we can conclude that each vertex in S is adjacent to each vertex in $A \cup B$, if at least one vertex of S has 2 adjacencies in A or B.

If each vertex of S has only one adjacency in each of A and B, then B could contain no triangle, for this would give immediately an induced Z_2. Also, this implies that each vertex in B has degree at most 2 relative to B, and so B is either a path or a cycle. In either case, this would give an induced C centered in B and using a vertex of S. From this contradiction we can conclude that each vertex of S is adjacent to each vertex of $A \cup B$.

Since G is k-connected, S has at least k vertices. Also, since G is C-free, $\overline{S}$ contains no triangles, and thus the number of edges in $\overline{S}$ is at most $\lceil |S|/2 \rceil \lfloor |S|/2 \rfloor$. Thus if $|A \cup B| \geq k$, the number of edges in $\overline{G}$ is at most $\lceil |S|/2 \rceil \lfloor |S|/2 \rfloor + \lceil |A \cup B|/2 \rceil \lfloor |A \cup B|/2 \rfloor$. This gives the required bound on $e(G)$, since the number of edges in G will be minimized by having $\lceil k/2 \rceil \lfloor k/2 \rfloor + \lceil (n-k)/2 \rceil \lfloor (n-k)/2 \rfloor$ edges in $\overline{G}$. If $|A \cup B| < k$, then select a smallest minimal cutset D of S. Thus, $S - D = S_1 \cup S_2$, where S_1 and S_2 are complete graphs. Note that $|S_1 \cup S_2| \geq k$, for otherwise each vertex in S would have degree at least $n - 2k$, which easily gives the required bound on $e(G)$. Now, the number of edges in $\overline{G}$ is at most $|S_1||S_2| + \lceil |A \cup B \cup D|/2 \rceil \lfloor |A \cup B \cup D|/2 \rfloor$. Just as before, this implies the required bound for $e(G)$, and so we have completed the proof of the case when $|A| \geq 2$ and $|B| \geq 3$.

Note that if $|A| = 2$ and $|B| = 2$, then $|S| = n - 4$, and this implies that $\delta(G) \geq n - 4$. Clearly in this case the required bound for $e(G)$ is satisfied since n is sufficiently large. Therefore we can assume that one of the components remaining after the deletion of a smallest minimal cutset has just one vertex.

Select a smallest minimal cutset S. By assumption, one of the components of $G - S$ will be a single vertex, which we will denote by v. Thus $d(v) = \delta(G) < (n + 2k)/2$. Let A be the other component. If some vertex $x \in S$ has only one adjacency $y \in A$, then a triangle in A will result in an induced Z_2. If there is no triangle in A, then each vertex in A has degree at most 2 relative to A, which implies a large independent set and so an induced C, since each of the vertices in A are adjacent to all but at most 2 of the vertices of S. Thus we can assume that each vertex of S has at least 2 adjacencies in A.

We next consider the case when some vertex $x \in S$ has neighborhood B in A, and that $|B| \leq |A| - 2$. If there is a vertex in $z \in A$ which is a distance 2 from B in A, say with path (z, y, b), then there will be an induced Z_2 unless y is adjacent to each vertex in B. Since G is CZ_2-free, the vertex y can have no other adjacencies in A except for $\{z\} \cup B$, and for the same reason no other vertex in $A - B$ can be adjacent to a vertex in B. Likewise, the vertex z can have no additional adjacencies in A, and so z is adjacent to all of the vertices in S except for x. Since G is C-free, there are no edges between B and $S - \{x\}$, which implies that $\{x, y\}$ is a minimal cutset in G with components with at least 2 and 3 vertices respectively. This case has already been considered, so, we can assume there are no vertices in A a distance 2 from B. Thus, each vertex in A is adjacent to some vertex in B.

If y and z are vertices of $A - B$, then their neighborhoods in B are disjoint, for otherwise there would be either a C or a Z_2. Also, for the same reason (existence of either an induced C or Z_2), there can be no edges in $A - B$ unless there are just 2 vertices in $A - B$ that dominate all of the vertices in B. Any vertex of S adjacent to all vertices of A except for possibly one would imply a C, so no such vertices exists. Therefore, each vertex in S must have a neighborhood with the same property as that of x. Thus, we can conclude that all of the vertices of S are adjacent to precisely the vertices B in A, since B is the only possible neighborhood with the required property. To avoid a C, the vertices in S must also have complete neighborhoods, so G has a clique with vertices $S \cup B$, and the remaining vertices have disjoint neighborhoods with at least 3 vertices in the clique. Hence the clique $S \cup B$ has at least $3n/4$ vertices, and this implies that $e(G) > \binom{3n/2}{2} \geq (9n^2)/32 - 3n/8$, gives the required bound for $e(G)$, since n is large.

We are left with the case in which each vertex of S is adjacent to all but possibly 1 vertex of A. In this case in $\overline{G}$ there are at most n edges in A, at most $2n$ edges between $S \cup \{v\}$ and A, and no more than $|S|^2/4$ edges in S. If $|S| > (n + 2k)/2$, then each vertex in G has degree at least $(n + 2k)/2$ and this gives the required bound on $e(G)$. If $|S| \leq (n + 2k)/2$, then $\overline{G}$ has at most $(n + 2k)^2/8 + 3n$ edges, and so again G has the required number of edges. This verifies the lower bound for $e(G)$.

The lower bound on $e(G)$ implies there is a vertex v of G of degree at least $n/2$. Let H be the graph spanned by the vertices in the neighborhood of v. Since G is C-free, there is no K_3 in $\overline{H}$, and so H must contain a clique with at least $\sqrt{n/2}$ vertices since $r(K_3, K_t) \leq t^2$ (see [9]). This completes the proof of Theorem 6. ■

3 FORBIDDEN SUBGRAPHS THAT IMPLY MODERATE DENSITY

For both paths P_m with $m \geq 5$ and Z_m for $m \geq 3$ the density implied by the property of graphs being CP_m-free or CZ_m-free graphs depends on m. We will not be able to give precise results in this case, but we will be able give some reasonable bounds, when the graph is of sufficiently large order. We start with the following result for paths.

Theorem 7. *Let $m \geq 5$ be an odd integer, and G be a CP_m-free connected graph of order n. If n is sufficiently large, then $\omega(G) \geq c_1 n^{2/(m-1)}$ and $e(G) \geq c_2 n^{1+1/(2^{(m-1)/2}-1)}$ for some constants c_1 and c_2 that do not depend on n. Also, the bounds for $\omega(G)$ and $e(G)$ are of the correct order of magnitude.*

Proof. For ease of calculation, we will let $m = 2t + 1$ with $t \geq 2$. We will first assume that G does not have $\omega(G) \geq c_1 n^{1/t}$, and show that this leads to a contradiction. Select an arbitrary vertex v of G, and consider the sets $N_1, N_2, \cdots, N_{t-1}$, where N_i denotes the vertices in G that are a distance i from v. Also, let $N_0 = \{v\}$. Observe that since G is C-free, $\alpha(N_1) \leq 2$. Therefore, if $|N_1| \geq (cn)^{2/t}$, then G will contain a $K_{(cn)^{1/t}}$, since $r(K_3, K_p) < p^2$. Also, note for $1 \leq i < t-1$ that the neighborhood in N_{i+1} of any vertex $x \in N_i$ is complete, for otherwise there would be a claw centered at x with 2 vertices in N_{i+1} and one in N_{i-1}. Thus, by assumption, each vertex in N_i has less than $(cn)^{1/t}$ adjacencies in N_{i+1}. This implies that $|N_{i+1}| \leq ((cn)^{1/t} - 1)|N_i|$, and so $|N_0 \cup N_1 \cup \cdots \cup N_i| \leq (cn)^{(i+1)/t}$. In particular, $|N_0 \cup N_1 \cup \cdots \cup N_{t-1}| \leq cn$. Thus, for appropriate c there is a longest distance path $P = (x_1, x_2, \cdots, x_p)$ with $p \geq t + 1$, and of course $p < m$ since P is an induced path.

Now select a series of sets $M_1, M_2, \cdots$ where M_i is the set of vertices that are a distance i from the set of vertices in P. Just as before, $M_1 < pcn^{2/t}$ and each vertex in M_i has less than $(cn)^{1/t}$ adjacencies in M_{i+1}. This implies that $|M_{i+1}| \leq ((cn)^{1/t} - 1)|M_i|$, and so $|M_0 \cup M_1 \cup \cdots \cup M_{t-1}| \leq cmn$. Thus there is for some r a longest distance path $P_1 = (y_1, y_2, \cdots, y_q = x_r)$ with $q \geq t + 1$. Therefore the graph H spanned by $P \cup P_1$ is induced except possibly for the additional edges $x_{r-1}y_{q-1}$ and $x_{r+1}y_{q-1}$. Note also that $q < m$, since P_1 is also an induced path. Starting with H form a third distance path P_2, which also will have $s \geq t+1$ vertices. If the last point on P_2 is some y_k on P_1, then by the choice of P_1, we must have $k \geq s \geq t+1$. This will give an induced path with at least $2(t+1)-1 \geq m$ vertices, since there will be at most a 2-chord spanned by P_2 and the subpath of P_1 preceding y_k. If P_2 terminates on P, then there will certainly be an induced P_m using the vertices of P_1, P_2 and possibly some vertices of P. This contradiction implies that $\omega(G) \geq cn^{1/t}$.

We will assume that $e(G) < c_2 n^{1+1/(2^t-1)}$, and show that this leads to a contradiction. Select a vertex v of smallest degree in G, which is a most $c'n^{1/(2^t-1)}$ for some constant c'. Just as one done earlier, consider the sets $N_1, N_2, \cdots, N_{t-1}$, where N_i denotes the vertices in G that are a distance i from v. Also, let $N_0 = \{v\}$, and let $n_i = |N_i|$ for $1 \leq i \leq t-1$. Observe that since G is C-free, the neighborhood in N_{i+1} of a vertex in N_i induces a complete graph for $i \geq 1$. Using this fact we will verify an upper bound for each n_i for $i \geq 2$. Note that the average degree in N_2 of the vertices in N_1 is at least n_2/n_1. In fact we can find disjoint subsets of the neighborhoods of vertices in N_1 that span N_2 and average at least n_2/n_1 vertices. Therefore by just counting the edges in the complete graph neighborhoods of the vertices in N_1 and using a convexity argument, there must be at least $n_1(n_2/n_1+1)(n_2/n_1)/2 \geq n_2^2/(2n_1)$ edges in N_2 and between N_1 and N_2. This gives the inequality $n_2^2/(2n_1) < e = e(G)$, which implies that $n_2 < \sqrt{2n_1 e}$. More generally, the same argument gives that $n_{i+1} < \sqrt{2n_i e}$ for $1 \leq i < t$. It follows that for each $2 \leq i \leq t$,

$$n_i < (2e)^{(2^{i-1}-1)/2^{i-1}} n_1^{1/2^{i-1}}.$$

More specifically, using the assumption that $e < c_2 n^{1+(1/(2^t-1))}$ and the fact that $n_1 \leq c'n^{1+(1/(2^t-1))}$, this gives that

$$n_t < (2e)^{(2^{t-1}-1)/2^{t-1}} n_1^{1/2^{t-1}} \leq cn^{1+(1/(2^t-1))-(1/2^{t-1})} = cn^{\beta},$$

where c depends on the constant c_2 and $\beta < 1$. It follows immediately that $|\{v\} \cup N_1 \cup N_2 \cdots N_t| \leq c^* n^{\beta}$ for some constant c^*. Therefore, there is some vertex in G at at distance at least $t+1$ from v.

Select some longest distance path P with at least $p \geq t+2$ vertices. Just as in the case of the proof for $\omega(G)$, start with the vertices P and let M_i denote the vertices of G at a distance i from P, and let $m_i = |M_i|$.

A repeat of the previous argument involving the sets N_i will give

$$m_t < p(2e)^{(2^{t-1}-1)/2^{t-1}} n_1^{1/2^{t-1}} \le cn^{1+(1/(2^t-1))-(1/2^{t-1})} = cn^{\beta},$$

and so there is a path of longest distance path P_1 from P with at least $q \ge t+1$ vertices. Just as before, the graph H spanned by these two paths is induced except possibly for two edges. A repeat of this procedure starting with H will give another distance path P_2 from H with at least $t+1$ vertices. A repeat of the previous argument used for $\omega(G)$ implies that G must have an induced path with at least $2t+1$ vertices. This contradiction implies that $e(G) \ge c_2 n^{1+1/(2^t-1)}$.

To show that the lower bound for $\omega(G)$ has the correct order of magnitude, consider the following "tree like" graph. Start with a $K_{c_1 n^{1/t}}$ and make each vertex of this complete adjacent to $c_2 n^{1/t}$ different vertices that form a complete graph as well. Do this until there are t levels. For each $1 \le i \le t$ there should be approximately $n^{i/t}$ vertices at that level. Appropriate choice of the constants c_i will yield a graph H_1 with n vertices. Also, the graph H_1 is C-free, the longest induced path has $2t = m-1$ vertices, and $\omega(H_1) = cn^{1/t} = cn^{2/(m-1)}$ for some constant c.

To show that the lower bound for $e(G)$ has the correct order of magnitude, consider a tree like graph with t levels just like the one considered for $\omega(G)$, except that the size of the complete graphs will vary depending on the level in the graph. For convenience let $\gamma = (1 + (1/(2^t-1))/2$. Then, for $1 \le i \le t$ the complete graphs at level i will have $c_i n^{\gamma/2^{i-1}}$ vertices. Therefore the order of magnitude of the number of vertices at level i will be $n^{((2^i-1)/2^{i-1})\gamma}$ and the order of magnitude of the number of edges at each level will be $n^{2\gamma}$. Thus, appropriate choice of the constants c_i will yield a CP_m-free graph H_2 of order n with $cn^{2\gamma} = cn^{1+(1/(2^t-1))}$ edges for some constant c. This completes the proof of Theorem 7. ∎

Since any P_m-free graph is clearly P_{m+1}-free, there is the immediate corollary to the proof and examples from Theorem 7.

Theorem 8. *Let $m \ge 6$ be an even integer, and G be a CP_m-free connected graph of order n. If n is sufficiently large, then $\omega(G) \ge c_1 n^{2/m}$ and $e(G) \ge c_2 n^{1+1/(2^{m/2}-1)}$ for some constants c_1 and c_2 that do not depend on n. Also, $\omega(G) \le c_1 n^{2/(m-2)}$ and $e(G) \le c_2 n^{1+1/(2^{(m-2)/2}-1)}$.*

Note that if G is a P_m-free graph, then G is also a Z_{m-2}-free graph, since P_m is an induced subgraph of Z_{m-2}. Again, a corollary of the examples and proof of Theorem 7 gives immediately the following result. The same proof techniques used for CP_m-free graphs will also work for Z_{m-2}-free graphs that have at least one vertex of degree at least 3, so we will not repeat them again.

Theorem 9. *Let $m \geq 3$, and let G be a CZ_m-free connected graph of order n for n sufficiently large with $\Delta(G) \geq 3$ (G is not C_n or P_n). If m is odd, then $\omega(G) \geq c_1 n^{2/(m+1)}$ and $e(G) \geq c_2 n^{1+1/(2^{(m+1)/2}-1)}$, and the bounds for $\omega(G)$ and $e(G)$ are of the correct order of magnitude for odd m. If m is even, then $\omega(G) \geq c_1 n^{2/(m+2)}$ and $e(G) \geq c_2 n^{1+1/(2^{(m+2)/2}-1)}$ for some constants c_1 and c_2 that do not depend on n. Also, $\omega(G) \leq c_1 n^{2/m}$ and $e(G) \leq c_2 n^{1+1/(2^{m/2}-1)}$.*

The connectivity, other than just being connected, does not play a role in any of the results of this section. Note that the examples given in the lower bounds can easily be modified to give an fixed connectivity k that does not depend on n. Thus, one could assume that all graphs considered were k-connected.

4 Forbidden Subgraphs and Sparse Graphs

Forbidding the pairs CW, CN, and CB, or more generally forbidding $B_{a,b}$ or $N_{a,b,c}$ graphs for integers $a, b, c \geq 1$ in a 2-connected C-free graphs G of order n does not imply that G has many edges or a large clique. The cycle C_n does not contain any of these graphs as induced subgraphs, and it clearly has a minimum number of edges and clique size for a hamiltonian graph. To avoid this trivial case, we will consider only graphs with minimum degree at least 3. The following Theorem 10 shows that the forced clique size is just 3, and the number of edges implied by the forbidden subgraph condition is linear in the number of vertices n. In fact, even if the connectivity $k = \kappa(G)$ is increased (but is fixed and is not a function of the order n of the graph), the clique size is still bounded by $k + 1$ and number of edges is still linear in n. This is indicated in Theorems 10, 11, and 12, which follow.

Theorem 10. *If G is a 2-connected (or in fact just 1-connected) graph of order n with $\delta(G) \geq 3$ that is $CB_{a,b}$-free, or $CN_{a,b,c}$-free for $a, b, c \geq 1$, then $\omega(G) \geq 3$ and $e(G) \geq \lceil 3n/2 \rceil$. Also, the lower bounds for $\omega(G)$ and $e(G)$ are sharp.*

Proof. The minimum degree condition implies that G must have at least $\lceil 3n/2 \rceil$ edges, and the claw-free condition along with $\delta(G) \geq 3$ implies that G must contain a K_3. To see that the bounds given are sharp, observe that the graph G_1 pictured in Figure 2 is a 2-connected $CB_{a,b}$-free, and $CN_{a,b,c}$-free graph for $a, b, c \geq 1$ with $\omega(G_1) = 3$ and $e(G_1) = 3n/2$ for n divisible by 4. Thus, the bounds cannot be improved, and this completes the proof of Theorem 10. ■

Since any $CN_{a,b,c}$-free graph is a C-free graph and the upper bound example in Theorem 2 was also $N_{a,b,c}$-free, then the following is a direct consequence of Theorem 2.

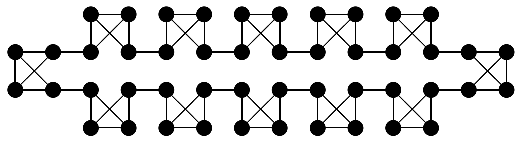

Figure 2

Theorem 11. *Let $k \geq 3$ and let G be a k-connected $CN_{a,b,c}$-free graph for $a, b, c \geq 1$ of order n. If n is sufficiently large, then $\omega(G) \geq \lceil (k+2)/2 \rceil$ and $e(G) \geq \lceil kn/2 \rceil$. Also, these lower bounds for $\omega(G)$ and $e(G)$ cannot be improved.*

The following result gives the corresponding extremal result for generalized Bulls that the previous result gave for generalized Nets.

Theorem 12. *Let $k \geq 3$ and let G be a k-connected $CB_{a,b}$-free graph of order n for $a, b \geq 1$. If n is sufficiently large, then $\omega(G) \geq 2\lceil k/2 \rceil$ and $e(G) \geq \lceil (3k-2)n/4 \rceil$. Also, these lower bounds for $\omega(G)$ and $e(G)$ cannot be improved for k even.*

Proof. For k even and $2n$ divisible by k, consider the graph G which has a vertex set which is partitioned into $2n/k$ sets $X_1, X_2, \cdots, X_{2n/k}$ each with $k/2$ vertices such that each set $X_i \cup X_{i+1}$ (with the indices taken modulo $2n/k$) induces a complete graph on k vertices. It is straightforward to check that G is CB-free (and hence $CB_{a,b}$-free for $a, b \geq 1$), $e(G) = (3k-2)n/4$, and $\omega(G) = k$.

Consider a k-connected graph G that is $CB_{a,b}$-free. We will show for n sufficiently large that $\omega(G) \geq 2\lceil k/2 \rceil$ and $e(G) \geq \lceil (3k-1)n/2 \rceil$. We will first show that if this is not true, then G will have a very large diameter. If there is a vertex u of G of degree at least $(k+1)^2$, then the neighborhood N of u has no $\overline{K}_3$ since G is C-free. Thus, N will contain a clique with at least $k+1$ vertices since $|N| > r(K_3, K_{k+1})$ (see [9]). In this case G has large diameter since $\Delta(G) \leq (k+1)^2$ implies this for n large. Now, consider the case when $e(G) < (3k-2)n/4$. Select a vertex v of smallest degree in G, which is a most $3k/2$. As has been done several times earlier, consider the sets $N_1, N_2, \cdots, N_j \cdots$, where N_j denotes the vertices in G that are a distance j from v. Also, let $N_0 = \{v\}$, and let $n_j = |N_j|$ for $1 \leq i$. Observe that since G is C-free, the neighborhood in N_{i+1} of a vertex in N_i induces a complete graph for $i \geq 1$. The average degree in N_2 of the vertices in N_1 is at least n_2/n_1. Therefore by counting the edges in the complete graph neighborhoods of the vertices in N_1 and using a convexity argument, there must be at least $n_1(n_2/n_1+1)(n_2/n_1)/2 \geq n_2^2/(2n_1)$ edges in N_2 and between N_1 and N_2. This gives the inequality $n_2^2/(2n_1) < e = e(G)$, which implies that $n_2 < \sqrt{2n_1 e} < 3k\sqrt{n}/2$. More generally, the same argument gives that $n_{i+1} < \sqrt{2n_i e}$ for $1 \leq i < t$. It follows that for each $2 \leq i \leq t$,

$$n_i < 3kn^{(2i-1)/2i}/2.$$

This impies that the diameter is a function of n, and so is large if n is sufficiently large.

Select a diameter path $P = (x_0, x_1, \cdots, x_d)$. Let x be a vertex not on P that is adjacent to a vertex on P. We will assume that $a \leq b$. The vertex x can be adjacent to a most 3 vertices, for otherwise the length of the path would be shortened. Also, if x is adjacent to x_i for $(0 < i < d)$, then to avoid a claw centered at x_i, x must be adjacent to either x_{i-1} or x_{i+1}. Also, if x is adjacent to just 2 vertices, say x_i and x_{i+1}, with $(a \leq i < d-a)$, then there will be an induced $B_{a,b}$ using x and vertices on the path P. Therefore, we can assume that each vertex x that is adjacent to a vertex x_i for $(a \leq i < d-a)$ must be adjacent to precisely 3 consecutive vertices on the path P. A corresponding adjacency pattern must be true at the end of the path P as well. Thus, we have a collection of sets $N_1, N_2, \cdots, N_{d-1}$ such that each vertex in N_i is adjacent to precisely x_{i-1}, x_i, x_{i+1} if $a \leq i < d-a$ and the remaining N_i's are adjacent to x_i, x_{i+1} and possibly x_{i-1}. Denote the vertices adjacent to precisely $\{x_0, x_1\}$ by N_0 and the vertices adjacent to just x_0 by N_0^-. There are the corresponding sets N_d and N_d^+. All of the

N_i's induce complete graphs, because there would be a claw otherwise. This includes N_0^- and N_d^+ as well.

For each i let $N_i^* = N_i \cup \{x_i\}$. If $a \le i < d-a$ and there are vertices $y_i \in N_i^*$ and $y_{i+1} \in N_{i+1}^*$, that are not adjacent, then there is an induced $B_{a,b}$ using y_i, y_{i+1} and vertices on the path P. Therefore all the edges between N_i^* and N_{i+1}^* are in G. A vertex adjacent to a vertex of N_i^* for $(a \le i < d-a)$ that is not on the path P and not in any N_j^* for $(0 < j < d)$ would imply a claw centered in N_i^*. Hence the graph spanned by $N_a^* \cup N_{a+1}^* \cup \cdots, N_{d-a-1}^*$ has no outside adjacencies except for $N_0^- \cup N_0^* \cup N_{a-1}^* \cup N_{d-a}^* \cup \cdots \cup N_d^+$. For $a \le i \le d-a-3$ the vertices $N_i^* \cup N_{i+2}^*$ form a cut set that separates N_{i+1}^* from the remainder of the graph, and so $|N_i^* \cup N_{i+2}^*| \ge k$. Therefore there is some (in fact many) j for which $|N_j^*| \ge \lceil k/2 \rceil$, and this implies that either $|N_{j-1}^* \cup N_j^*| \ge 2\lceil k/2 \rceil$ or $|N_j^* \cup N_{j+1}^*| \ge 2\lceil k/2 \rceil$, which gives the required clique of order at least $2\lceil k/2 \rceil$.

If the vertices $N_a^* \cup N_{a+1}^* \cup \cdots \cup N_{d-a}^*$ form a set of cut vertices of G, then this implies for $a < i \le d-a-1$, that the set N_i^* is a vertex cut of G, and so $|N_i^*| \ge k$. Thus, each of the vertices in $N_a^* \cup N_{a+1}^* \cup \cdots, N_{d-a}^*$ have degree at least $2k$, in fact at least $3k-1$, except for the vertices in N_a and N_{d-a}. This clearly implies that $e(G) > \lceil (3k-2)n/4 \rceil$. Therefore we can assume that there is a path Q from x_{a-1} to x_{d-a+1} that is disjoint from $N_a^* \cup N_{a+1}^* \cup \cdots \cup N_{d-a}^*$. Pick Q to be such a distance path, which must be of length at least $d-2a$.

The path Q will have the same properties as P, so there will be a family of sets that correspond to the sets N_i^*. In fact the path induced by P and Q from a vertex in the middle of P to the middle of Q will have the same property. The immediate consequence of this is that there is a cycle $C = (y_1, y_2, \cdots, y_p, y_1)$ in G with $d < p \le 2d$ with corresponding sets M_i^* associated with each of the vertices y_i, and the M_i^*'s have the same properties as the N_j^*'s. Thus, each vertex in G will be in some M_i^*. In particular $M_i^* \cup M_{i+2}^*$ is a cut set for G and so must have at least k vertices. This implies that the sum of the degrees of the vertices in G is at least $(3k/2-2)n$, and this which would occur if and only if each of the sets M_i^* had $k/2$ vertices. Therefore, $e(G) \ge (3k-2)n/4$, which completes the proof of Theorem 12. ■

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