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CYCLICALLY 5-EDGE CONNECTED NON-BICRITICAL CRITICAL SNARKS

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Abstract

Snarks are bridgeless cubic graphs with chromatic index $\chi' = 4$. A snark G is called critical if $\chi'(G - \{v, w\}) = 3$, for any two adjacent vertices v and w.

For any $k \geq 2$ we construct cyclically 5-edge connected critical snarks G having an independent set I of at least k vertices such that $\chi'(G - I) = 4$.

For $k = 2$ this solves a problem of Nedela and Škoviera [6].

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1. Introduction

A *snark* is a bridgeless cubic graph with chromatic index $\chi' = 4$. The study of the reduction of snarks is as old as the study of these graphs itself.

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For a detailed introduction to this topic we refer the reader to one of [3, 5, 6, 7, 10, 12, 13]. This note deals with a reduction of snarks introduced by Nedela and Škoviera in [6].

Let G be a snark and let $F \subset E(G)$ be a k-edge cut ($k \geq 0$) whose removal divides G into two components H_1 and H_2. If the chromatic index of one of the components is 4, say $\chi'(H_1) = 4$, then H_1 can be extended to a snark H with $|V(H)| \leq |V(G)|$ by adding edges and probably vertices. Graph H is called a *k-reduction* of G. If $|V(H)| < |V(G)|$, then H is called a *proper* k-reduction of G.

A snark is called *k-irreducible* if it has no proper m-reduction such that $m < k$, and it is called *irreducible* if it is k-irreducible for each $k \geq 1$.

A snark G is called *critical* if $\chi'(G - \{v, w\}) = 3$ for any two adjacent vertices $v, w \in V(G)$, it is called *cocritical* if $\chi'(G - \{v, w\}) = 3$ for any two non-adjacent vertices $v, w \in V(G)$, and it is called *bicritical* if it is critical and cocritical.

Nedela and Škoviera proved the following characterizations.

Theorem 1.1 [6]**.** *Let G be a snark. Then the following statements hold true.*

1. *If $5 \leq k \leq 6$, then G is k-irreducible if and only if it is critical.*
2. *If $k \geq 7$, then G is k-irreducible if and only if it is bicritical.*

Finally, it turns out that

Theorem 1.2 [6]**.** *A snark is irreducible if and only if it is bicritical.*

In [2] it is shown that there are cocritical snarks which are not critical, and that there are snarks which are neither critical nor cocritical. Further, each critical snark on less than 30 vertices is bicritical and henceforth it is irreducible. Nedela and Škoviera [6] state the following problem.

Problem 1.3 [6]**.** *Does there exist a snark that is critical but not bicritical? Equivalently, does there exist a 6-irreducible snark that is not irreducible?*

The answer is "yes". In [9] the latter author constructed infinite families of cyclically 4-edge connected critical snarks which are not bicritical. The smallest one has 32 vertices. M. Škoviera [8] found another infinite family of cyclically 4-edge connected snarks with these properties by using a different method.

In this note, we improve these results in two directions. We construct cyclically 5-edge connected critical snarks with the property that they have a large independent set of vertices whose removal does not yield an edge 3-colorable graph. Clearly, these graphs are not bicritical.

2. The Main Theorem

We will use the following lemma, due to Blanuša [1].

Lemma 2.1. (Parity Lemma) *Let M be a multigraph whose edges are colored with colors $1, \ldots, k$, and let a_i be the number of vertices v in M such that no edge incident to v is colored i. Then for all $i = 1, \ldots, k : a_i \equiv |V(M)| \pmod 2$.*

Proof. For $i = 1, \ldots k$ let E_i be the set of edges colored i. Then $a_i = |V(M)| - 2|E_i|$, and hence $a_i \equiv |V(M)| \pmod 2$. ■

Theorem 2.2. *For each $k \geq 1$ there is a cyclically 5-edge connected critical snark that has an independent set I of $2k+1$ vertices such that $\chi'(G-I) = 4$.*

Proof. The idea of the construction is as follows. Let $k \geq 1$ be fixed. We construct a multigraph M_k and specify three closed walks in this multigraph. We replace vertices of M_k by well specified graphs to obtain a cubic graph G_k. We show that G_k is cyclically 5-edge connected and that it is a snark. Furthermore, each of the walks in M_k can be extended to circuits in G_k to obtain a 2-factor of G_k with precisely two odd circuits. We then show that for each edge $e = vw$ in G_k there is a 2-factor of G_k with precisely two odd circuits which are connected by e. If this is true, then G_k has an edge 4-coloring with a color class consisting of precisely two edges, one of them incident to v and the other incident to w. Thus $G_k - \{v, w\}$ is edge 3-colorable. Hence G_k is critical.

Construction

Let $k \geq 1$ be fixed and $I = \{w_0, w_1, \ldots, w_{2k}\}$. Define M_k to be the multigraph with vertex set $\mathbb{Z}_{3(2k+1)} \cup I$, and for each $i \in \mathbb{Z}_{3(2k+1)}$ vertex i is joined to vertex $i+1$ by two parallel edges, $e(i, i+1)$ and $f(i, i+1)$, and by one edge with $w_m \in I$ if $i \equiv m \pmod{2k+1}$. We call the elements of $\mathbb{Z}_{3(2k+1)}$ the *outer* vertices of M_k. Multigraph M_1 is shown in Figure 1.

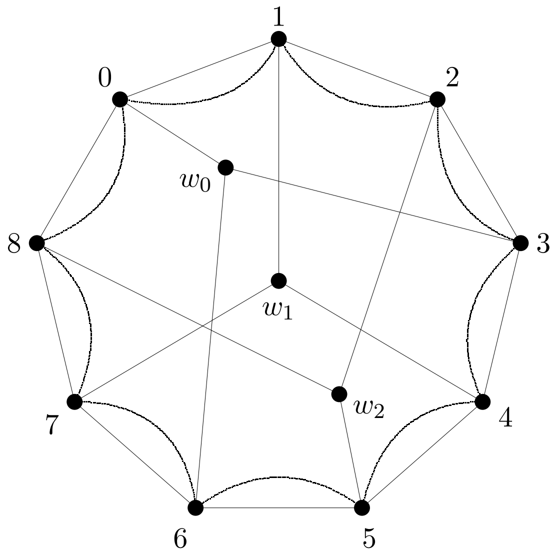

Figure 1. Multigraph M_1

For $k \geq 1$ the circuits P_e, P_f and the walk Q in M_k are defined as follows: Let $2k+1 = K$,

$P_e = e(0,1), e(1,w_1), e(w_1,K+1), e(K+1,K+2), e(K+2,w_2),\ e(w_2,2),$ $e(2,3), \ldots,\ e(2K-1,w_{K-1}), e(w_{K-1},K-1), e(K-1,K), e(K,w_0), e(w_0,0).$

$P_f = f(0,1), e(1,w_1), e(w_1,K+1), f(K+1,K+2), e(K+2,w_2),\ e(w_2,2),$ $f(2,3),\ \ldots,\ e(2K-1,w_{K-1}), e(w_{K-1},K-1), f(K-1,K), e(K,w_0), e(w_0,0).$

$Q = e(2K-1,2K), e(2K,2K+1),\ e(2K+1,2K+2), \ldots,\ e(3K-1,0),$ $f(0,3K-1),\ f(3K-1,3K-2), \ldots,\ f(2K,2K-1).$

The 7-block B^i ($i \in \mathbb{Z}_{3(2k+1)}$) is the graph obtained from a cycle C_6 on the vertices $v_0^i, v_1^i, \ldots, v_5^i$ by adding a vertex v_6^i and edges $v_1^i v_6^i$ and $v_4^i v_6^i$.
For each $k \geq 1$ construct the cubic graph G_k as follows:

For each $i \in \mathbb{Z}_{3(2k+1)}$ take block B^i and vertices $w_0, \ldots, w_{2k}$, add edges $v_0^i v_5^{i+1}$, $v_3^i v_2^{i+1}$ and for $0 \leq j \leq 2k$ add edges $w_j v_6^n$ if $n \equiv j \pmod{2k+1}$.

If we contract each subgraph B^i to a single vertex i, we obtain multigraph M_k. For the following we may assume that the edges $v_0^i v_5^{i+1}$ and $v_3^i v_2^{i+1}$ of G_k are denoted by $e(i,i+1)$ and $f(i,i+1)$ in the contracted graph, respectively. Vice versa G_k can be obtained from M_k by successively replacing the outer vertices by 7-blocks. A fact which we will use in the following.

Claim 2.2.1. *G_k is a cyclically 5-edge connected snark, that has an independent set I of $2k+1$ vertices such that $G_k - I$ is not edge 3-colorable.*

Proof. A 7-block B^i is obtained from the Petersen graph by removing three vertices of a path of length 2. Therefore it follows from Lemma 2.1 that for each 3-coloring of B^i the same colors are missing at v_0^i and v_3^i and two different colors are missing at v_2^i and v_5^i or vice versa. Thus $G_k - I$ cannot be edge 3-colorable.

Since between any two outer vertices of M_k there are five edge disjoint paths and since no 5-circuit of B^i can be separated by removing less than five edges from G_k, it follows that G_k is cyclically 5-edge connected. ■

Claim 2.2.2. *G_k is critical.*

Proof. We have to show that for each edge $vw = e \in E(G_k)$ there is a 2-factor $\mathcal{F}_e$ of G_k with precisely two odd circuits which are connected by e.

We reconstruct G_k from M_k by successively replacing outer vertices of M_k by the 7-blocks. We show that the circuits P_e, P_f and the walk Q can be extended to circuits of the new graphs. Eventually (after replacing all outer vertices of M_k), we obtain a spanning 2-factor of G_k with precisely two odd circuits.

In G_k we have basically the following types of edges:
$v_0^iv_1^i$, $v_1^iv_2^i$, $v_2^iv_3^i$, $v_3^iv_4^i$, $v_4^iv_5^i$, $v_5^iv_0^i$, $v_1^iv_6^i$, $v_4^iv_6^i$, $v_0^iv_5^{i+1}$, $v_3^iv_2^{i+1}$, and $v_6^iw_l$ (where $i \equiv l \pmod{2k+1}$).

We have $V(M_k) = \mathbb{Z}_{3(2k+1)} \cup \{w_0, \dots, w_{2k}\}$, and we define for $j = 1, \dots, 3(2k+1)-1$ the function $f_j : V(M_k) \to V(M_k)$ with $f_j(i) = i + j$ if $i \in \mathbb{Z}_{3(2k+1)}$ and $f_j(w_k) = w_{k+j}$, where the indices are added modulo $2k+1$. This function is an automorphism on M_k.

Thus the aforementioned construction can be applied on M_k where vertices v are labeled by $f_j(v)$, for each $j = 1, \dots, 3(2k+1)-1$. Hence it suffices to show that for each edge type there is a 2-factor of G_k containing precisely two odd circuits connected by at least one edge of that type.

It is easy to see that there are hamiltonian paths in B^i with terminal vertices v_0^i, v_6^i and v_5^i, v_6^i and v_2^i, v_6^i and v_3^i, v_6^i, respectively.

Then, if for $1 \le i \le 2(2k+1)-2$ vertex i of M_k is replaced by B^i circuits, P_e and P_f can be extended to circuits of the new graph, respectively.

Paths $v_2^i, v_1^i, v_6^i, v_4^i, v_3^i$ and v_0^i, v_5^i span B^i. Then, if for $0 \le l \le 2k$ vertex $2(2k+1)+l$ of M_k is replaced by $B^{2(2k+1)+l}$, the walk Q can be extended to a circuit of the new graph.

Let G_k^- be the graph obtained from M_k by replacing all degree 5 vertices but 0 and $2(2k+1)-1$. To show that G_k is critical we consider the following cases: Let $s = 2(2k+1) - 1$.

Case 1. We consider the extended circuit P_e and the extended walk Q in G_k^-.

Replace vertices 0 and s by B^0 and B^s, respectively. In B^0 extend P_e by the path v_0^0, v_1^0, v_6^0, and Q by the path $v_2^0, v_3^0, v_4^0, v_5^0$. In B^s extend P_e by the path v_5^s, v_4^s, v_6^s, and Q by the path $v_0^s, v_1^s, v_2^s, v_3^s$ to obtain two disjoint spanning circuits P^* and Q^* of odd length which form a 2-factor of G_k.

These two odd circuits are connected by edges $v_0^0v_5^0$, $v_1^0v_2^0$, $v_4^0v_6^0$, $v_1^sv_6^s$, $v_3^sv_4^s$, $v_3^{s-1}v_2^s$ and by edges $v_6^{s+1+l}w_l$ for $0 \leq l \leq 2k$

Case 2. We consider the extended circuit P_f and the extended walk Q in G_k^-.

Let vertices 0 and s be replaced by B^0 and B^s, respectively. In B^0 extend P_f by the path v_3^0, v_4^0, v_6^0 and Q by the path $v_2^0, v_1^0, v_0^0, v_5^0$. In B^s extend P_f by the path v_2^s, v_1^s, v_6^s, and Q by $v_0^s, v_5^s, v_4^s, v_3^s$. As in case 1 we obtain two disjoint spanning circuits P^* and Q^* of odd length which form a 2-factor of G_k. These two circuits are connected by $v_2^0v_3^0$, $v_4^0v_5^0$, $v_0^sv_1^s$ and $v_0^{s-1}v_5^s$. ■

Flower snark J_{2k+1} (cf. [5]) has vertex set $V(J_{2k+1}) = \{a_i, b_i, c_i, d_i | i = 1, 2, \dots, 2k+1\}$ and edge set $E(J_{2k+1}) = \{b_ia_i, b_ic_i, b_id_i; a_ia_{i+1}; c_id_{i+1}; d_ic_{i+1} | i = 1, 2, \dots, 2k+1\}$. The above construction can also be carried out by using copies of $J_7 - \{a_1, a_2, a_7\}$ instead of the B^i's. This yields a cyclically 5-edge connected snark H_k with girth 6. By the same argumentation as above, it follows that H_k is critical and that $\chi'(H_k - I) = 4$, for each $k \geq 1$. Because the proof is long and tedious we omit it here and state:

Theorem 2.3. *For any $k \geq 1$ there are cyclically 5-edge connected critical snarks with girth 6 having an independent set of $2k+1$ vertices whose removal does not yield an edge 3-colorable graph.*

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