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2-HALVABLE COMPLETE 4-PARTITE GRAPHS

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Abstract

A complete 4-partite graph K_{m_1,m_2,m_3,m_4} is called d-halvable if it can be decomposed into two isomorphic factors of diameter d. In the class of graphs K_{m_1,m_2,m_3,m_4} with at most one odd part all d-halvable graphs are known. In the class of biregular graphs K_{m_1,m_2,m_3,m_4} with four odd parts (i.e., the graphs $K_{m,m,m,n}$ and $K_{m,m,n,n}$) all d-halvable graphs are known as well, except for the graphs $K_{m,m,n,n}$ when $d = 2$ and $n \neq m$. We prove that such graphs are 2-halvable iff $n, m \geq 3$. We also determine a new class of non-halvable graphs K_{m_1,m_2,m_3,m_4} with three or four different odd parts.

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1. Introductory Notes and Definitions

A *factor* F of a graph $G = G(V, E)$ is a subgraph of G having the same vertex set V. A *decomposition* of a graph $G(V, E)$ into two factors $F_1(V, E_1)$ and $F_2(V, E_2)$ is a pair of factors such that $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = E$. A decomposition of G is *isomorphic* if $F_1 \cong F_2$. An isomorphic decomposition is also called a *halving* of a graph G. An isomorphism $\phi : F_1 \to F_2$ is then also called a *halving isomorphism*, and the factors F_1 and F_2 the *halves* of G. The *diameter* diam G of a connected graph G is the maximum of the set of distances $\mathrm{dist}_G(x, y)$ among all pairs of vertices of G. If G is disconnected, then we define $\mathrm{diam}\, G = \infty$. A graph, having a decomposition into two halves (i.e., isomorphic factors) of diameter d is called *d-halvable*

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or simply *halvable*. The *order* of a graph G (or of a partite set of a complete multipartite graph G) is the number of vertices of G (or of the partite set of G). For terms not defined here, see [1].

Since 1968, when the well-known article by Bosák, Rosa and Znám [2] on decompositions of complete graphs into factors with given diameters was published, different aspects of such decompositions of various classes of graphs have been studied. See, e.g., A. Kotzig and A. Rosa [9], P. Tomasta [11], D. Palumbíny [10], P. Híc and D. Palumbíny [8] for decompositions of complete graphs into isomorphic factors with a given diameter. E. Tomová [12] studied decompositions of complete bipartite graphs into two factors with given diameters, T. Gangopadhyay [7] dealt with decompositions of complete r-partite graphs ($r \geq 3$) into two factors with given diameters.

Halvability of complete bipartite and tripartite graphs into connected factors was studied by the author [3]; it was shown that a complete bipartite or tripartite graph is d-halvable (for a finite d) only if $d = 3, 4, 5, 6$ or $d = 2, 3, 4, 5$, respectively. For each of the diameters all respective d-halvable graphs were determined. Also halvability of complete r-partite graphs for $r > 4$ [4] into factors with finite diameters and complete r-partite graphs for $r \geq 2$ into disconnected factors [5] was studied by the author. Halvability of complete 4-partite graphs into factors with finite diameters was studied in a common article by the author and J. Širáň [6]. For graphs with at most one odd part it was shown that they are d-halvable for a finite diameter d only if $d = 2, 3, 4$ or 5 and for each of the diameters all such d-halvable graphs were determined. As the complete 4-partite graphs with two or three odd parts have an odd number of edges and therefore are not halvable, the only remaining class containing halvable graphs is the class of graphs with all odd parts. In this class, however, only partial results were proved. There remain two gaps to be filled. The first one concerns halvability of graphs with parts of three or four different orders, the other 2-halvability of graphs $K_{n,n,m,m}$. The purpose of this article is to narrow the first gap and to fill the other.

2. Graphs With Three or Four Different Odd Order Parts

In this section, we present certain class of non-halvable complete 4-partite graphs with parts of three or four different odd orders. Before we do that, we briefly summarize relevant results proved in [6]. The first one restricts the range of possible finite diameters, which in the general case includes diameters 2, 3, 4 and 5.

Theorem A. *The graph K_{m_1,m_2,m_3,m_4} has no 5-halving for any odd numbers m_1, m_2, m_3, m_4.*

One of the two existing classes of biregular complete 4-partite graphs with all odd parts, namely the class of graphs $K_{r,r,r,s}$, is shown not to be halvable by the following result.

Theorem B. *Let r, s be odd numbers, $r \neq s$. Then the graph $K_{r,r,r,s}$ has no halving.*

We use below a modification of the methods used in [6] in the proof of Theorem B to generalize the result for a certain class of complete 4-partite graphs with all odd parts. Theorem B then follows as an immediate corollary of the more general result.

The proof of Theorem B is based on the concept of halvability of a degree sequence of a graph. Recall that the *degree sequence* of a graph G with vertices $v_1, v_2, \ldots, v_n$ is the non-increasing sequence $D = (d_1, d_2, \ldots, d_n)$ where $d_i = \deg v_{\phi(i)}$ for a suitable permutation ϕ of the set $\{1, 2, \ldots, n\}$.

In general, two sequences $B = (b_1, b_2, \ldots, b_n)$ and $C = (c_1, c_2, \ldots, c_n)$ of non-negative integers will be called *matchable* if there exists a permutation ψ of the set $\{1, 2, \ldots, n\}$, called a *matching permutation*, such that $b_i = c_{\psi(i)}$. A sequence $A = (a_1, a_2, \ldots, a_n)$ is said to have a *halving* if there exist matchable sequences $B = (b_1, b_2, \ldots, b_n)$ and $C = (c_1, c_2, \ldots, c_n)$ such that $a_i = b_i + c_i$ for each $i \in \{1, 2, \ldots, n\}$.

To prove the above mentioned result, we use a slightly more general concept. A sequence $A^m = (a_1^m, a_2^m, \ldots, a_n^m)$ is called an *m-modular sequence* of a sequence $A = (a_1, a_2, \ldots, a_n)$ if $a_i^m \in \{0, 1, \ldots, m-1\}$ and $a_i \equiv a_i^m (\bmod\ m)$ for $i = 1, 2, \ldots, n$. An m-modular sequence $A^m = (a_1^m, a_2^m, \ldots, a_n^m)$ is similarly said to have a *halving* if there exist matchable sequences $B^m = (b_1^m, b_2^m, \ldots, b_n^m)$ and $C^m = (c_1^m, c_2^m, \ldots, c_n^m)$ of integers of the set $\{0, 1, 2, \ldots, m-1\}$ such that $b_i^m + c_i^m \equiv a_i^m (\bmod\ m)$ for each $i \in \{1, 2, \ldots, n\}$.

Obviously, if a graph G has a halving, then the degree sequence of G has a halving. It is also easy to see that if two sequences B and C are matchable and B^m and C^m are their respective m-modular sequences for an arbitrary m, then B^m and C^m must be matchable. Furthermore, if a sequence A is halvable into sequences B and C, and A^m, B^m, C^m are their respective

m-modular sequences for an arbitrary m, then the m-modular sequence A^m is halvable into B^m and C^m.

Therefore, to prove that a graph G is not halvable it suffices to show that there exists a number m such that the m-modular sequence of the degree sequence of G is not halvable. We use the idea now to prove the main result of this section.

Theorem 2.1. *Let K_{m_1,m_2,m_3,m_4} be a complete 4-partite graph with all parts odd and $m = 2^r$ be a number such that $m_1 \equiv m_2 \equiv m_3 \not\equiv m_4 (\bmod\ m)$. Then the graph K_{m_1,m_2,m_3,m_4} is not halvable.*

Proof. According to what we have seen above, to prove the assertion it is enough to show that the m-modular sequence of the graph is not halvable. Let $P_i, i = 1, \ldots, 4$ be the partite set of order m_i. If we suppose that $m_1 \equiv m_2 \equiv m_3 \equiv u (\bmod\ m)$, $m_4 \equiv v (\bmod\ m)$ and $u \not\equiv v (\bmod\ m)$, we can see that the degree of every vertex $x_1 \in P_1$ is $m_2 + m_3 + m_4 \equiv 2u + v \equiv s (\bmod\ m)$. Analogically, the degree of every vertex $x_2 \in P_2$ is $m_1 + m_3 + m_4 \equiv 2u + v \equiv s (\bmod\ m)$ and also the degree of every vertex $x_3 \in P_3$ is $m_1 + m_2 + m_4 \equiv 2u + v \equiv s (\bmod\ m)$. On the other hand, the degree of every vertex $x_4 \in P_4$ is $m_1 + m_2 + m_3 \equiv 3u \equiv t (\bmod\ m)$. We suppose w.l.o.g. that $s, t \in \{0, 1, \ldots, m-1\}$. As $u \not\equiv v (\bmod\ m)$, it follows that $2u + v \not\equiv 3v (\bmod\ m)$ or, which is equivalent, $s \neq t$. Thus the m-modular sequence of the degree sequence of K_{m_1,m_2,m_3,m_4} contains an odd number $p = m_1 + m_2 + m_3$ of entries equal to s and $q = m_4$ (also an odd number) entries equal to t. We now denote the m-modular sequence of the graph K_{m_1,m_2,m_3,m_4} as $A = (a_1, a_2, \ldots, a_p, a'_1, a'_2, \ldots, a'_q)$, where $a_1 = a_2 = \ldots = a_p = s$, $a'_1 = a'_2 = \ldots = a'_q = t$ and we suppose that it is halvable into matchable m-modular sequences $B = (b_1, b_2, \ldots, b_p, b'_1, b'_2, \ldots, b'_q)$ and $C = (c_1, c_2, \ldots, c_p, c'_1, c'_2, \ldots, c'_q)$ such that $b_i + c_i \equiv a_i (\bmod\ m)$ and $b'_i + c'_i \equiv a'_i (\bmod\ m)$ for every i.

Above we have defined a matching permutation ψ. For convenience we now define a *halving permutation* θ of the sequence A as a permutation induced by the matching permutation ψ as follows: $\theta(a_i) = a_{\psi(i)}$. This means that $\theta(a_i) = a_j$ iff $b_i = c_j$, where $j = \psi(i)$. (Here the symbols a, b, c stand also for a', b', c', respectively.) Let γ be a cycle of a halving permutation θ. We want to show that γ always contains an even number of entries of the subsequence $(a_1, a_2, \ldots, a_p)$ as well as of the subsequence $(a'_1, a'_2, \ldots, a'_q)$. Let $\theta(a'_1) = a_1$, $\theta(a_i) = a_{i+1}$ for $i = 1, 2, \ldots, k-1$ and $\theta(a_k) = a'_2$. As $b_i + c_i \equiv a_i$ and $b_i = c_{i+1}$ for $i = 1, 2, \ldots, k-1$, it is easy to see that $c_1 = b_2 = c_3 = b_4 = \ldots = b_{2j} = c_{2j+1} = \ldots$ and $b'_1 = c_1 = b_k = c'_2$ if k is even and $b'_1 = c_1 = c_k$ if k is odd. In the latter case it holds then that $b_1 = b_k$ and hence $c'_2 = b_1$. We can now remove all entries $a_1, a_2, \ldots, a_k$ (if k is even) or the entries $a_2, \ldots, a_k$ (if k is odd) to "shorten" the cycle γ without changing parity of appearance of the entries of the subsequence $(a_1, a_2, \ldots, a_p)$ in the new cycle. Therefore, to simplify our considerations, we can define (with a slight abuse of the definition of a cycle) a *shortened cycle* γ' of the cycle γ as a cycle in which no two consecutive entries belong both to the subsequence $(a_1, a_2, \ldots, a_p)$ or to $(a'_1, a'_2, \ldots, a'_q)$. In fact, the shortened cycle $\gamma' = (a_{i_1}, a'_{i_2}, a_{i_3}, a'_{i_4}, \ldots, a_{i_{n-1}}, a'_n)$ of length n is just a subsequence of the entries forming the cycle γ with the property that $b_{i_j} = c'_{i_{j+1}}$ and $b'_{i_{j+1}} = c_{i_{j+2}}$ for $j = 1, 2, \ldots, n-2$ and $b'_{i_n} = c'_{i_1}$. Notice that n must be even, otherwise there are two consecutive entries $a_{i_j}, a_{i_{j+1}}$ or $a'_{i_j}, a'_{i_{j+1}}$.

It is clear that at least one shortened cycle of the permutation θ must be non-empty, as shortening removes even numbers of entries from both subsequences that contain an odd number of entries each. Now we want to show that every non-empty shortened cycle contains an even number of entries of each of the subsequences to arrive at a contradiction. Let $(a_1, a'_1, a_2, a'_2, \ldots, a_l, a'_l)$ be a shortened cycle. Then $b_1 = c'_1$ and $b'_1 \equiv a'_1 - c'_1 \equiv t - b_1 (\mathrm{mod}\ m)$. Because $c_2 = b'_1$ and $a_2 \equiv s \equiv b_2 + c_2 (\mathrm{mod}\ m)$, we get $b_2 \equiv s - c_2 \equiv s - t + b_1 (\mathrm{mod}\ m)$. Now b_2 can be equal to b_1 only if $s = t$, which is not the case. Hence a_2 is not identical with a_1 and the cycle γ' continues by a'_2. Thus after another "forwards" step we get $c'_2 = b_2 \equiv s - t + b_1$ and similarly as above $b'_2 \equiv a'_2 - c'_2 \equiv t - (s - t + b_1) \equiv 2t - s - b_1 (\mathrm{mod}\ m)$. Now we have to take a "backwards" step and we get $c_3 \equiv 2t - s - b_1 (\mathrm{mod}\ m)$ and $b_3 \equiv a_3 - c_3 \equiv 2(s-t) + b_1 (\mathrm{mod}\ m)$. In general, after j pairs of steps we get $b_j \equiv j(s-t) + b_1 (\mathrm{mod}\ m)$. To close the cycle, we need $b_j \equiv b_1 (\mathrm{mod}\ m)$, which can occur only if $j(s-t) \equiv 0 (\mathrm{mod}\ m)$. But this is possible only if $j \equiv 0 (\mathrm{mod}\ m)$ since we have supposed that $s \neq t$. This yields $j | m$. Since by our assumption $m = 2^r$, it follows that $j = 2^{r'}$ for some $r' \leq r$ and hence j must be even. So we have shown that every shortened cycle (and consequently every cycle) contains an even number of entries from each of the subsequences $(a_1, a_2, \ldots, a_p)$ and $(a'_1, a'_2, \ldots, a'_q)$. This is the desired contradiction, since both p and q are odd and there remains at least one entry in each subsequence not belonging to any cycle, which is absurd. Hence the m-modular sequence of the degree sequence of K_{m_1, m_2, m_3, m_4} is not halvable and thus neither is the graph itself. ■

If we set $m_1 = m_2 = m_3 \neq m_4$, Theorem B follows as an immediate corollary. To see this, we just choose r such that $m = 2^r > \max\{m_1, m_4\}$.

Corollary 2.2. (Theorem B) *Let r, s be odd numbers with $r \neq s$. Then the graph $K_{r,r,r,s}$ has no halving.*

The existence of another class of non-halvable graphs follows from the following simple observation.

Theorem 2.3. *Let K_{m_1,m_2,m_3,m_4} be a complete 4-partite graph with all parts odd such that $m_1 \leq m_2 \leq m_3$ and $m_4 > m_1+m_2+2m_3$. Then K_{m_1,m_2,m_3,m_4} is not halvable.*

Proof. Let A be the degree sequence of K_{m_1,m_2,m_3,m_4} and B and C the degree sequences of factors F_1 and F_2 of K_{m_1,m_2,m_3,m_4}, respectively. Our aim is to show that the first $m_1 + m_2 + m_3$ entries of A are more than two times greater than the remaining entries and therefore always one entry of the pair b_i, c_i $(i \leq m_1+m_2+m_3)$ is greater than any $a_j,\ j > m_1+m_2+m_3$. Because $m_1 + m_2 + m_3$ is an odd number, at least one of the first $m_1 + m_2 + m_3$ entries of B (or C) has no match among the entries $c_1, c_2, \ldots, c_{m_1+m_2+m_3}$ (or $b_1, b_2, \ldots, b_{m_1+m_2+m_3}$). The largest of such entries is indeed greater than any of $a_{m_1+m_2+m_3+1}, a_{m_1+m_2+m_3+2}, \ldots, a_{m_1+m_2+m_3+m_4}$ and therefore cannot be matched by any of $b_{m_1+m_2+m_3+1}, b_{m_1+m_2+m_3+2}, \ldots, b_{m_1+m_2+m_3+m_4}$ (or $c_{m_1+m_2+m_3+1}, c_{m_1+m_2+m_3+2}, \ldots, c_{m_1+m_2+m_3+m_4}$) either.

Obviously, $a_1 = a_2 = \ldots = a_{m_1} = m_2 + m_3 + m_4$,
$a_{m_1+1} = a_{m_1+2} = \ldots = a_{m_1+m_2} = m_1 + m_3 + m_4$,
$a_{m_1+m_2+1} = a_{m_1+m_2+2} = \ldots = a_{m_1+m_2+m_3} = m_1 + m_2 + m_4$ and
$a_{m_1+m_2+m_3+1} = a_{m_1+m_2+m_3+2} = \ldots = a_{m_1+m_2+m_3+m_4} = m_1 + m_2 + m_3$.

Of course, $a_1 \geq a_2 \geq \ldots \geq a_{m_1+m_2+m_3} > a_{m_1+m_2+m_3+1} = \ldots = a_{m_1+m_2+m_3+m_4}$.

According to our assumption, it moreover holds that
$a_{m_1+m_2+m_3} = m_1 + m_2 + m_4 \geq 2m_1 + 2m_2 + 2m_3 + 1 = 2a_{m_1+m_2+m_3+1} + 1$.
If we put $m_1 + m_2 + m_3 = 2k + 1$ (which is possible, as m_1, m_2, m_3 are all odd), we can see that one of the subsequences $(b_1, b_2, \ldots, b_{m_1+m_2+m_3})$, $(c_1, c_2, \ldots, c_{m_1+m_2+m_3})$, say the former, contains at least $k + 1$ entries $b_{j_1}, b_{j_2}, \ldots, b_{j_{k+1}}$ that are greater than or equal to $m_1 + m_2 + m_3 + 1 = a_{m_1+m_2+m_3+1} + 1$, while the latter contains at most k such entries. If we want F_1 to be isomorphic to F_2, the sequences B and C must be matchable and therefore at least one of the entries

$b_{j_1}, b_{j_2}, \ldots, b_{j_{k+1}}$ must be matched by an entry belonging to the subsequence $(c_{m_1+m_2+m_3+1}, c_{m_1+m_2+m_3+2}, \ldots, c_{m_1+m_2+m_3+m_4})$. This is impossible, since $c_i \leq a_i$ for any i and $b_{j_l} > m_1+m_2+m_3 = a_{m_1+m_2+m_3+1} = a_{m_1+m_2+m_3+2} = \ldots = a_{m_1+m_2+m_3+m_4}$ for $l = 1, 2, \ldots, k+1$. Thus B and C are not matchable and F_1 cannot be isomorphic to F_2. ■

A slightly weaker but simpler version of Theorem 2.3 can be formulated as follows.

Corollary 2.4. *Let K_{m_1,m_2,m_3,m_4} be a complete 4-partite graph with all parts odd such that $m_1 \leq m_2 \leq m_3$ and $m_4 > 4m_3$. Then K_{m_1,m_2,m_3,m_4} is not halvable.*

The following is an easy but interesting consequence of Theorem 2.3.

Corollary 2.5. *Let m_1, m_2, m_3 be odd numbers. Then there are at most finitely many halvable graphs K_{m_1,m_2,m_3,m_4}.*

3. 2-Halvable Graphs $K_{2n+1,2n+1,2m+1,2m+1}$

The following results were proved for the class of graphs $K_{2n+1,2n+1,2m+1,2m+1}$ in [6].

Theorem C. *A complete 4-partite graph $K_{1,1,2m+1,2m+1}$ has no 2-halving for any $m \geq 1$.*

The assumption $m \geq 1$ is actually redundant, as for $m = 0$ we get a graph K_4 which has only one halving, namely the 3-halving into paths P_4.

Theorem D. *Let r, s be odd integers. A complete 4-partite graph $K_{r,r,s,s}$ has a d-halving for a finite diameter d if*

(a) *$d = 4$ and $\max\{r, s\} \geq 3$, or*
(b) *$d = 3$ and $r, s \geq 1$, or*
(c) *$d = 2$ and $r = s \geq 3$.*

To obtain complete results on d-halvability of complete biregular 4-partite graphs for a finite d, we have to solve the problem of 2-halvability of graphs $K_{r,r,s,s}$ with r, s odd, $3 \leq r < s$. The following construction shows that all such graphs are 2-halvable.

Construction 3.1. We first present a 2-halving of the smallest graph of the class, $K_{3,3,5,5}$ and then we show how to extend the factors into isomorphic factors of diameter 2 of any graph of the class $K_{r,r,s,s}$ with r, s odd. The assumption $r < s$ is not necessary for $r \geq 5$, as the construction actually also covers the case $5 \leq r = s$.

The main idea is the following. We take a complete bipartite graph $K_{8,8}$ with the parts $V_1 = \{v_{11}, v_{12}, \ldots, v_{18}\}$ and $V_2 = \{v_{21}, v_{22}, \ldots, v_{28}\}$ and find its 3-halving into factors F_1 and F_2 with the property that the complementing isomorphism $\phi : F_1 \to F_2$ takes V_1 onto V_2 and vice versa. Moreover, we construct the factors such that in F_1 every vertex v_{1i} has at least one neighbour in the subset $V_{21} = \{v_{21}, v_{22}, v_{23}\}$ and another one in the subset $V_{22} = \{v_{24}, v_{25}, \ldots, v_{28}\}$. One can see that since $\mathrm{diam} F_1 = 3$, then $\mathrm{dist}_{F_1}(v_{1i}, v_{1j}) = \mathrm{dist}_{F_1}(v_{2i}, v_{2j}) = 2$ for every $i \neq j$, $i, j \in \{1, 2, \ldots, 8\}$. Obviously the vertices at distance 3 apart belong to different parts. If we now add to the factor F_1 of $K_{8,8}$ all edges of the complete bipartite graph $K_{3,5}$ with the parts induced by the sets V_{21} and V_{22}, we get a factor F_1' of the graph $K_{3,3,5,5}$ with partite sets V_{11}, V_{12} (which we determine instantly), V_{21} and V_{22}. As every vertex v_{1i} has one neighbour in V_{21} and one in V_{22} and every vertex of V_{21} is in F_1' adjacent to all vertices of V_{22}, we can see that $\mathrm{dist}_{F_1'}(v_{1i}, v_{2j}) \leq 2$ for every $i, j \in \{1, 2, \ldots, 8\}$. If we now choose V_{11} as $\phi(V_{21})$ and V_{12} as $\phi(V_{22})$ and add the edges of the complete bipartite graph $K_{3,5}$ with the parts V_{11} and V_{12} to the factor F_2, we get the other factor F_2' of the graph $K_{3,3,5,5}$.

One can check that the factor F_1' with the adjacency matrix A shown in Figure 1 and the complementing isomorphism ϕ defined by the cycles $(v_{11}, v_{23}, v_{13}, v_{21}), (v_{12}, v_{22}, v_{14}, v_{24}), (v_{15}, v_{27}, v_{17}, v_{25}), (v_{16}, v_{26}, v_{18}, v_{28})$ possess the required properties and together with its complement with respect to $K_{3,3,5,5}$, F_2', forms a 2-halving of $K_{3,3,5,5}$. The partite sets are $V_{11} = \{v_{11}, v_{13}, v_{14}\}$, $V_{12} = \{v_{12}, v_{15}, \ldots, v_{18}\}$, $V_{21} = \{v_{21}, v_{22}, v_{23}\}$ and $V_{22} = \{v_{24}, v_{25}, \ldots, v_{28}\}$.

To construct factors of graphs $K_{2k+3,2k+3,5,5}$, we extend the parts V_{11} and V_{21} of order 3 into parts of any order $2k + 3$ just by "blowing up" every vertex of the cycle $(v_{11}, v_{23}, v_{13}, v_{21})$ by adding k "copies" of the respective vertex in both factors. To be more precise, we add k new vertices $v_{11}^1, v_{11}^2, \ldots, v_{11}^k$ into V_{11} and join them in both factors exactly to the neighbours of v_{11}. Similarly we add k new vertices $v_{21}^1, v_{21}^2, \ldots, v_{21}^k$ into V_{21} and join them in both factors to the neighbours of v_{21}. We do the same for v_{13} and v_{23}. To obtain a 2-halving of any graph $K_{2k+3,2k+3,2l+5,2l+5}$, we extend the parts V_{12} and V_{22} in the same manner—we choose vertices of one of the "pure" cycles $(v_{15}, v_{27}, v_{17}, v_{25})$ or $(v_{16}, v_{26}, v_{18}, v_{28})$. ■

$$A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Figure 1

Now we can state the complete version of Theorem D. The proof follows immediately from Theorem D and Construction 3.1.

Theorem 3.2. (Theorem D+) *Let r, s be odd integers. A complete 4-partite graph $K_{r,r,s,s}$ has a d-halving for a finite diameter d if and only if*

(a) $d = 4$ *and* $\max\{r, s\} \geq 3$*, or*
(b) $d = 3$ *and* $r, s \geq 1$*, or*
(c) $d = 2$ *and* $\min\{r, s\} \geq 3$.

References

[1] M. Behzad, G. Chartrand and L. Lesniak-Foster, Graphs and Digraphs (Prindle, Weber & Schmidt, Boston, 1979).

[2] J. Bosák, A. Rosa and Š. Znám, *On decompositions of complete graphs into factors with given diameters*, in: Theory of Graphs, Proc. Coll. Tihany 1966 (Akadémiai Kiadó, Budapest, 1968) 37–56.

[3] D. Fronček, *Decompositions of complete bipartite and tripartite graphs into selfcomplementary factors with finite diameters*, Graphs. Combin. **12** (1996) 305–320.

[4] D. Fronček, *Decompositions of complete multipartite graphs into selfcomplementary factors with finite diameters*, Australas. J. Combin. **13** (1996) 61–74.

[5] D. Fronček, *Decompositions of complete multipartite graphs into disconnected selfcomplementary factors*, Utilitas Mathematica **53** (1998) 201–216.

[6] D. Fronček and J. Širáň, *Halving complete* 4*-partite graphs*, Ars Combinatoria (to appear).

[7] T. Gangopadhyay, *Range of diameters in a graph and its r-partite complement*, Ars Combinatoria **18** (1983) 61–80.

[8] P. Híc and D. Palumbíny, *Isomorphic factorizations of complete graphs into factors with a given diameter*, Math. Slovaca **37** (1987) 247–254.

[9] A. Kotzig and A. Rosa, *Decomposition of complete graphs into isomorphic factors with a given diameter*, Bull. London Math. Soc. **7** (1975) 51–57.

[10] D. Palumbíny, *Factorizations of complete graphs into isomorphic factors with a given diameter*, Zborník Pedagogickej Fakulty v Nitre, Matematika **2** (1982) 21–32.

[11] P. Tomasta, *Decompositions of graphs and hypergraphs into isomorphic factors with a given diameter*, Czechoslovak Math. J. **27** (1977) 598–608.

[12] E. Tomová, *Decomposition of complete bipartite graphs into factors with given diameters*, Math. Slovaca **27** (1977) 113–128.

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