

## DECOMPOSITION OF MULTIGRAPHS

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### Abstract

In this note, we consider the problem of existence of an edge-decomposition of a multigraph into isomorphic copies of 2-edge paths  $K_{1,2}$ . We find necessary and sufficient conditions for such a decomposition of a multigraph  $H$  to exist when

- (i) either  $H$  does not have incident multiple edges or
- (ii) multiplicities of the edges in  $H$  are not greater than two.

In particular, we answer a problem stated by Z. Skupień.

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Z. Skupień, at the conference in Zakopane (September 1994), stated a problem of decomposition of the edge set of a multigraph  $H$  into stars  $K_{1,2}$ , if it is assumed that multiplicities of the edges do not exceed 2. This property is denoted by  $K_{1,2}|H$ . It is known that if  $H$  is a simple graph, then  $K_{1,2}|H$  if and only if the size of every component of  $H$  is even. It is easy to verify that this condition is not sufficient to ensure the decomposition of a multigraph.

Let  $\mathcal{M}$  be the class of trees with a perfect matching. Denote by  $H^*$  the graph obtained from  $H$  by deleting all edges of multiplicity 1 and reducing the multiplicities of all the other edges to 1. Let  $\mathcal{M}(H)$  be the set of these components in  $H^*$  that belong to  $\mathcal{M}$ . Clearly, in case (i) members of  $\mathcal{M}(H)$

are simple edges and in case (ii) they can be isomorphic to any member of  $\mathcal{M}$ .

For  $\mathcal{A} \subset \mathcal{M}(H)$  denote by  $E_{\mathcal{A}}$  the set of edges in  $H$  with at least one end in a vertex of a component in  $\mathcal{A}$  (multiple edges are counted multiplicity many times). Let  $V(\mathcal{A})$  (resp.  $E(\mathcal{A})$ ) stand for the union of the vertex (resp. edge) sets of the components in  $\mathcal{A}$ . For a component  $t$  in  $\mathcal{M}(H)$  which corresponds to a multiple edge denote by  $m(t)$  the multiplicity of this edge in  $H$ . Finally, let  $O(H \setminus V(\mathcal{A}))$  be the number of components of  $H \setminus V(\mathcal{A})$  with an odd size which are not multiple edges.

Here are our main results.

**Theorem 1.** *Let  $H$  be a multigraph of an even size and with no incident multiple edges. Then  $K_{1,2}|H$  if and only if*

$$(1) \text{ for every set of edges } \mathcal{A} \subset \mathcal{M}(H), |E_{\mathcal{A}}| \geq 2 \sum_{t \in \mathcal{A}} m(t) + O(H \setminus V(\mathcal{A})).$$

Clearly, in the above theorem each member of  $\mathcal{M}(H)$  is an edge.

A *proper* simple cut-edge in a multigraph is a simple cut-edge whose deletion does not create a component consisting of one vertex.

**Corollary 1.** *Let  $H$  be a multigraph of even size, with no incident multiple edges and with no simple proper cut-edge. If each edge  $e$  of multiplicity  $m(t) > 1$  is incident to at least  $2m(t)$  edges of multiplicity 1 then  $K_{1,2}|H$ .*

**Theorem 2.** *Let  $H$  be a multigraph of an even size and let the multiplicities of the edges be not greater than 2. Then  $K_{1,2}|H$  if and only if*

$$(2) \quad \forall \mathcal{A} \subset \mathcal{M}(H), |E_{\mathcal{A}}| \geq 2|V(\mathcal{A})| + O(H \setminus V(\mathcal{A})).$$

Before proving these theorems let us make two remarks.

**Remark 1.** The problem of decomposing a multigraph  $H$  into  $K_{1,2}$  reduces to the case when  $H$  is connected.

**Remark 2.** Let  $H$  be a multigraph. If  $H$  contains a pair of incident multiple edges, then we can delete a copy of  $K_{1,2}$ ; we repeat this process for pairs of incident multiple edges until we obtain a multigraph  $H'$  (not unique) with no two incident multiple edges. If  $K_{1,2}|H'$  for some choice of  $H'$ , then by adding the deleted edges, we immediately get a decomposition of  $H$ .

## PROOFS OF RESULTS

For a multigraph  $H$  define a graph  $G(H)$  whose vertex set is  $E(H)$  and a pair of vertices in  $G(H)$  is an edge if the corresponding edges in  $H$  have exactly one common vertex.

**Lemma 1.**  $K_{1,2}|H$  if and only if  $G(H)$  has a perfect matching.

**Proof.** Suppose  $K_{1,2}|H$ . Since the vertices in  $G(H)$  correspond to edges in  $H$ , a decomposition of  $H$  into stars  $K_{1,2}$  defines a perfect matching in  $G(H)$ .

Conversely, suppose  $G(H)$  has a perfect matching. Each edge in this matching defines a copy of  $K_{1,2}$  in  $H$ . Since the matching covers all vertices in  $G(H)$ , the corresponding copies of  $K_{1,2}$  form an edge-decomposition of  $H$ . ■

By the result of Tutte [T],  $G(H)$  has a perfect matching if and only if

$$(3) \quad \forall S \subset V(G(H)), \quad O_V(G(H) \setminus S) \leq |S|,$$

where  $O_V(G(H) \setminus S)$  is the number of components of  $G(H) \setminus S$  with an odd number of vertices.

When writing this paper, we have been informed that J. Ivančo, M. Meszka and Z. Skupień [IMS] have made the same observation as in our Lemma 1. In particular, they concluded that deciding whether  $K_{1,2}|H$  for an instance multigraph  $H$  is a polynomial problem.

Assume now that  $H$  has no incident multiple edges. Call an edge  $e$  in  $H$  an  $m$ -bridge if it is a bridge in the component  $H_1$  of  $H$  containing  $e$  and if at least one of the components of  $H_1 - e$  is a multiple edge.

**Lemma 2.** Let  $H$  be a multigraph with no incident multiple edges. If  $e$  is not an  $m$ -bridge in  $H$  then

$$O_V(G(H) - e) \leq O_V(G(H)) + 1.$$

**Proof.** The lemma obviously holds when  $H$  is a multiple edge because then  $G(H)$  is an edgeless graph. Otherwise it follows from the observation that  $G(H_1) - e$  has exactly 2 components (where  $H_1$  is the component of  $H$  containing  $e$ ) and  $e$  is not an  $m$ -bridge. We leave routine details of this proof to the reader. ■

Let  $E(\mathcal{A}, H \setminus V(\mathcal{A}))$  (respectively  $E(\mathcal{A}, \mathcal{A})$ ) be the set of edges with one end-vertex in  $\mathcal{A} \subset \mathcal{M}(H)$  and the other one in  $V(H) \setminus V(\mathcal{A})$  (respectively with end-vertices in two different members of  $\mathcal{A}$ ).

**Proof of Theorem 1.** Since  $H$  has no incident multiple edges, the set  $\mathcal{M}(H)$  represents the set of multiple edges in  $H$ . The condition (1) is equivalent to

$$(4) \quad \begin{aligned} & \forall \mathcal{A} \subset \mathcal{M}(H), \\ & |E(\mathcal{A}, H \setminus V(\mathcal{A}))| + |E(\mathcal{A}, \mathcal{A})| \geq \sum_{t \in \mathcal{A}} m(t) + O(H \setminus V(\mathcal{A})). \end{aligned}$$

Suppose there is a decomposition  $\pi$  of  $H$  into stars  $K_{1,2}$ . For every component of  $H \setminus V(\mathcal{A})$  of odd size at least one copy of  $K_{1,2}$  in  $\pi$  has an edge in  $E(\mathcal{A}, H \setminus V(\mathcal{A}))$ . Moreover, for every multiple edge  $t \in \mathcal{A}$ ,  $m(t)$  copies of  $K_{1,2}$  in  $\pi$  have one edge in  $E(\mathcal{A}, H \setminus V(\mathcal{A})) \cup E(\mathcal{A}, \mathcal{A})$ . Hence

$$\forall \mathcal{A} \subset \mathcal{M}(H), \quad |E(\mathcal{A}, H \setminus V(\mathcal{A}))| + |E(\mathcal{A}, \mathcal{A})| \geq \sum_{t \in \mathcal{A}} m(t) + O(H \setminus V(\mathcal{A})),$$

which completes the proof of necessity.

To show sufficiency suppose that (1) is satisfied and  $H$  does not have a decomposition into stars  $K_{1,2}$ . By Lemma 1 and (3), we get

$$(5) \quad \exists S \subset V(G(H)), \quad O_V(G(H) \setminus S) > |S|.$$

Assume that  $S$  has the smallest cardinality among the sets satisfying the above inequality.

Suppose first that  $S = \emptyset$ . Then, at least one component of  $G(H)$  has an odd number of vertices. By the definition of  $G(H)$ , either one of the components of  $H$  which is not a multiple edge has an odd size or one of the components of  $H$  is a multiple edge. In the former case we get a contradiction to (4) because for  $\mathcal{A} = \emptyset$  we obtain  $O(H) = 0$ . To get a contradiction in the latter case, denote by  $e$  a multiple edge which is a component in  $H$ . The condition (4) yields a contradiction for  $\mathcal{A} = \{e\}$ . Hence  $S \neq \emptyset$ .

Suppose now that some  $e \in S \subset V(G(H)) = E(H)$  is not an  $m$ -bridge in  $H \setminus (S \setminus \{e\}) = (H \setminus S) \cup \{e\}$ . By minimality of  $S$ ,

$$O_V((G(H) \setminus S) \cup \{e\}) \leq |S \setminus \{e\}|.$$

The multigraph  $(H \setminus S) \cup \{e\}$  satisfies the assumptions of Lemma 2. Consequently,  $O_V(G(H) \setminus S) \leq O_V((G(H) \setminus S) \cup \{e\}) + 1$ , so

$$O_V(G(H) \setminus S) \leq |S \setminus \{e\}| + 1 = |S|,$$

a contradiction.

Thus all the edges  $e \in S \subset E(H)$  are  $m$ -bridges in  $(H \setminus S) \cup \{e\}$ . Let  $\mathcal{A}$  be the set of multiple edges which are components in  $H \setminus S$ . Then, clearly,  $S = E(\mathcal{A}, H \setminus V(\mathcal{A})) \cup E(\mathcal{A}, \mathcal{A})$ . By the definition of  $G(H)$  and (4)

$$O_V(G(H) \setminus S) = \sum_{t \in \mathcal{A}} m(t) + O(H \setminus V(\mathcal{A})) \leq |E(\mathcal{A}, H \setminus V(\mathcal{A}))| + |E(\mathcal{A}, \mathcal{A})| = |S|,$$

a contradiction to (5).  $\blacksquare$

**Proof of Corollary 1.** By the assumption of the corollary, for every set of multiple edges  $\mathcal{A}$ ,

$$(6) \quad 2|E(\mathcal{A}, \mathcal{A})| + |E(\mathcal{A}, H \setminus V(\mathcal{A}))| \geq \sum_{t \in \mathcal{A}} 2m(t).$$

Let  $\omega(H \setminus V(\mathcal{A}))$  be the number of components of  $H \setminus V(\mathcal{A})$  of order at least 2. Then, since no simple edge in  $H$  is a proper cut-edge,

$$(7) \quad |E(\mathcal{A}, H \setminus V(\mathcal{A}))| \geq 2\omega(H \setminus V(\mathcal{A})) \geq 2O(H \setminus V(\mathcal{A})).$$

By adding (6) and (7) we get (4) so by Theorem 1 the proof is complete.  $\blacksquare$

**Proof of Theorem 2.** To show necessity suppose the required decomposition exists. Let  $M$  be the perfect matching in the graph formed by the components in  $\mathcal{A}$ . Denote by  $B$  the set of edges obtained from  $E_{\mathcal{A}}$  by deletion of the edges of  $M$  and their doubles. Clearly,  $|B| = |E_{\mathcal{A}}| - 2e(M)$ . Note that  $O(H \setminus V(\mathcal{A})) = O(H \setminus B)$ , where  $H \setminus B$  stands for the multigraph obtained from  $H$  by removing the edges of  $B$ . By the existence of a  $K_{1,2}$ -decomposition of  $H$  at least  $O(H \setminus B) + 2e(M)$  different copies of  $K_{1,2}$  in the decomposition have one edge in  $B$ . Hence

$$|B| \geq O(H \setminus B) + 2e(M)$$

so

$$|E_{\mathcal{A}}| = |B| + 2e(M) \geq 4e(M) + O(H \setminus B) = 2|V(\mathcal{A})| + O(H \setminus V(\mathcal{A})).$$

Suppose sufficiency is false. Let  $H$  be a multigraph of an even size with the minimum number of doubled edges satisfying (2) and such that  $K_{1,2} \not\ll H$ . Assume first that  $H^*$  contains a component  $C$  of a positive size which is not a member of  $\mathcal{M}$ .

If the size of  $C$  is even, then  $K_{1,2} \ll C$ . Therefore, if we delete copies of every edge in  $C$  from  $H$ , then the resulting multigraph  $H'$  still has an even size, satisfies (2) and  $K_{1,2} \not\ll H'$  contradicting to the minimality of  $H$ .

Let the size of  $C$  be odd. Suppose first  $C$  contains a cycle and let  $e$  be one of its edges. It is routine to show that the multigraph  $C'$  obtained from  $C$  by doubling the edge  $e$  has a decomposition into stars  $K_{1,2}$ . Moreover, the multigraph  $H'$  obtained from  $H$  by deleting the edges of  $C'$  has an even size, satisfies (2) and  $K_{1,2} \nmid H'$  contradicting to the minimality of  $H$  again.

Let now  $C$  be a tree of an odd size. One can easily show that since  $C \notin \mathcal{M}$ ,  $C$  can be decomposed into graphs  $A$  and  $B$  such that  $K_{1,2} \mid A$  and  $B$  is isomorphic to  $K_{1,3}$ . If the size of  $A$  is positive then as before we can delete from  $H$  the edges of  $A$  and obtain a multigraph  $H'$  contradicting to the minimality of  $H$ .

Thus, we can assume that  $C$  is isomorphic to  $K_{1,3}$ . Let  $e$  and  $f$  be two of the edges of  $C$  and let  $e_1$  and  $e_2$  (respectively  $f_1$  and  $f_2$ ) be the parallel edges corresponding to  $e$  (resp.  $f$ ) in  $H$ . Subdivide the edges  $e_1$  and  $e_2$  by inserting two new vertices  $v_1$  and  $v_2$  into  $e_1$  and two new vertices  $u_1$  and  $u_2$  into  $e_2$ . Let  $e'_1$  (resp.  $e'_2$ ) denote the edge  $v_1v_2$  (resp.  $u_1u_2$ ). The resulting multigraph  $H'$  has an even size, satisfies (2) and, by the minimality of  $H$ ,  $H'$  admits a decomposition  $\pi'$  into stars  $K_{1,2}$ . Contract the copies of  $K_{1,2}$  in  $\pi'$  containing  $e'_1$  and  $e'_2$ . We get the multigraph  $H$  again. The decomposition  $\pi'$  of  $H'$  defines in  $H$  a decomposition  $\pi$  which (by  $K_{1,2} \nmid H$ ) is a decomposition into copies of  $K_{1,2}$  and the multigraph induced by the parallel edges  $e_1$  and  $e_2$ . In the latter case, consider the multigraph  $F$  induced by  $e_1$  and  $e_2$  and the edges of copies of  $K_{1,2}$  in  $\pi$  containing  $f_1$  and  $f_2$ . It is routine to show that  $K_{1,2} \mid F$ , so consequently  $K_{1,2} \mid H$ , a contradiction.

We have shown that all components of  $H^*$  are isomorphic to members of  $\mathcal{M}$ .

If all the components in  $\mathcal{M}(H)$  are single edges then by Theorem 1 the proof is complete. Suppose now that at least one of the components, say  $C$ , in  $\mathcal{M}(H)$  is a tree with a perfect matching different from a single edge. It is easy to notice that then there are edges  $e$  and  $f$  in  $C$  such that  $e$  is a pendant edge in  $C$  and  $f$  is the only edge in  $C$  incident to  $e$ . Denote by  $e_1, e_2$  (respectively  $f_1, f_2$ ) the parallel edges in  $H$  corresponding to  $e$  (respectively  $f$ ) in  $C$ . Subdivide  $f_1$  and  $f_2$  by inserting 2 new vertices  $x_1, x_2$  into  $f_1$  and  $y_1, y_2$  into  $f_2$ . Let  $f'_1$  (respectively  $f'_2$ ) denote the edge  $x_1x_2$  (respectively  $y_1y_2$ ).

Let us check the inequality (2) for  $H'$ . Note that  $\mathcal{M}(H') = (\mathcal{M}(H) \setminus \{C\}) \cup \{C_1, C_2\}$ , where  $C_1$  is the edge  $e$  and  $C_2 = C \setminus \{e, f\}$ .

Let  $\mathcal{A} \in \mathcal{M}(H')$ . The condition (2) is easy to verify when  $C_1, C_2 \in \mathcal{A}$  and when  $C_1, C_2 \notin \mathcal{A}$ . Thus suppose that  $C_2 \in \mathcal{A}$  and  $C_1 \notin \mathcal{A}$  (the case  $C_1 \in \mathcal{A}$  and  $C_2 \notin \mathcal{A}$  is analogous and we leave it to the reader).

Let  $\mathcal{A}' = \mathcal{A} \setminus \{C_2\}$ . Then by our assumption for  $H$

$$|E_{\mathcal{A}'}| \geq 2|V(\mathcal{A}')| + O(H \setminus V(\mathcal{A}')) = 2|V(\mathcal{A}')| + O(H' \setminus V(\mathcal{A}')).$$

Let  $k$  be the number of odd-sized components in  $H' \setminus V(\mathcal{A})$  which are not odd-sized components in  $H' \setminus V(\mathcal{A}')$ . Clearly each of them is joined to a vertex in  $C_2$  by at least one edge. Moreover, the component of  $H' \setminus V(\mathcal{A})$  containing  $C_1$  is joined to a vertex [4] of  $C_2$  by at least 2 edges. Hence

$$\begin{aligned} |E_{\mathcal{A}}| &\geq |E_{\mathcal{A}'}| + 2|E(C_2)| + k + 1 = |E_{\mathcal{A}'}| + 2|V(C_2)| - 2 + k + 1 \\ &\geq 2|V(\mathcal{A}')| + O(H' \setminus V(\mathcal{A}')) + 2|V(C_2)| + k - 1 \geq 2|V(\mathcal{A})| + O(H' \setminus V(\mathcal{A})) - 1. \end{aligned}$$

Note that  $|E_{\mathcal{A}}|$  and  $O(H' \setminus V(\mathcal{A}))$  have the same parity. Indeed,

$$\begin{aligned} 0 \equiv e(H') &= |E_{\mathcal{A}}| + \sum_{C \in EV} e(C) + \sum_{C \in OD} e(C) \equiv |E_{\mathcal{A}}| + |OD| \\ &= |E_{\mathcal{A}}| + O(H' \setminus V(\mathcal{A})) \pmod{2}, \end{aligned}$$

where  $EV$  (resp.  $OD$ ) stands for the set of even-sized (resp. odd-sized) components in  $H' \setminus V(\mathcal{A})$ . Consequently  $|E_{\mathcal{A}}| \geq 2|V(\mathcal{A})| + O(H' \setminus V(\mathcal{A}))$ . By the minimality of  $H$ ,  $H'$  admits a  $K_{1,2}$ -decomposition  $\pi'$ .

Contract the copies of  $K_{1,2}$  in  $\pi'$  containing  $f'_1$  and  $f'_2$ . We get again the multigraph  $H$ . The decomposition  $\pi'$  of  $H'$  defines in  $H$  a decomposition  $\pi$  which is either a  $K_{1,2}$ -decomposition (in this case the proof is complete) or a decomposition into copies of  $K_{1,2}$  and the multigraph induced by the parallel edges  $f_1, f_2$ . In the latter case consider the multigraph induced by  $f_1, f_2$  and the copies of  $K_{1,2}$  in  $\pi$  containing  $e_1$  and  $e_2$ . It is routine to show that this multigraph admits a  $K_{1,2}$ -decomposition. This contradiction completes our proof.  $\blacksquare$

**Remark 3.** One can easily deduce from Theorem 2 that a multigraph  $H$  with multiplicities of all edges equal to 2 is  $K_{1,2}$ -decomposable if and only if  $H^*$  is not a tree with a perfect matching. This result was earlier proved by Bondy [B].

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## REFERENCES

- [B] J.A. Bondy, *Perfect Path Double Covers of Graphs*, J. Graph Theory **14** (1990) 259–272.
- [IMS] J. Ivančo, M. Meszka and Z. Skupień Z, *Decomposition of multigraphs into isomorphic graphs with two edges*, to appear in Ars Combinatoria.
- [T] W.T. Tutte, *The factorisation of linear graphs*, J. London Math. Society **22** (1947) 107–111.

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