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ON VARIETIES OF GRAPHS

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Abstract

In this paper, we introduce the notion of a variety of graphs closed under isomorphic images, subgraph identifications and induced subgraphs (induced connected subgraphs) firstly and next closed under isomorphic images, subgraph identifications, circuits and cliques. The structure of the corresponding lattices is investigated.

Keywords and phrases: graph, subgraph identification, variety.

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1. Introduction

In the theory of algebraic structures a lot of attention is paid to investigations of lattices of varieties of algebras. *A variety of algebras* is a nonempty class of algebras of the same type which is closed under homomorphic images, subalgebras and direct products (see [3, p. 61]). The notion of a variety may have different meanings depending on the contexts it is used in. We usually have a family $\mathbb{F}$ of algebraic structures of the same type and a finite set of closure operators $O_1, \ldots, O_m$ defined on $\mathbb{F}$. Then a variety $\mathbb{V} \subseteq \mathbb{F}$ of algebraic structures is a subfamily closed under all operators $O_1, \ldots, O_m$. For instance in [5] a variety of posets was defined as a class of posets closed under retracts and nonvoid direct products. Varieties (sets) of graphs closed

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under isomorphic images and induced subgraphs were investiged in [9] and varieties of graphs closed under isomorphic images and generalized hereditary operators were investigated in [2].

A set of all varieties (of given type) with set inclusion as the partial ordering is a complete lattice. The complete lattice of all varieties of graphs closed under isomorphic images and aditive and hereditary operators was investigated in [8]. In this context, the term *a variety of graphs* is replaced by the term *a property of graphs.* An interesting survey paper on additive and hereditary properties of graphs with open problems may be found in [1].

Another definition of a variety of graphs can be obtained by combining the "super-graph" and the "vertex-spliting" closure operators. Then a variety of graphs contains with a given graph $\mathcal{G}$ all graphs containing $\mathcal{G}$ as a minor. It follows from the famous theorem of Robertson and Seymour [10, Th. 5.1] that every such variety is finitely generated. For instance, the variety $V(\mathcal{K}_5, \mathcal{K}_{3,3})$ generated by the complete graph $\mathcal{K}_5$ and the complete bipartite graph $\mathcal{K}_{3,3}$ coincides with the family of all non-planar graphs, what is nothing else than a reformulation of the Kuratovski theorem [4, p. 163].

Of course, the question how to choose the closure operators in the definition of varieties of graphs (in order to have a chance to obtain some relevant results) is extremely important. One can believe that there is no general suggestion how to do it. On the other hand, it turns out that the following criteria should indicate that the choice of closure operators is "good" :

(a) the varieties can be characterized in terms of some invariants of graphs
(b) the lattice of varieties is "nice".

The aim of the paper is to present a new approach to the definition of a variety of graphs. Namely, varieties of graphs closed under isomorphic images, induced subgraphs (induced connected subgraphs) and identifications of graphs in connected induced subgraphs are investigated in Section 3, while in Section 4 varieties closed under isomorphic images, identifications of graphs in connected induced subgraphs and closed under the operator which ensures to contain with any graph all its circuits and cliques, are considered. Particular attention is paid to the study of structure of the corresponding lattices.

For the undefined terminology we refer the reader to [4].

2. Preliminary Results

By a graph $\mathcal{G} = (V, E)$ we mean an undirected finite graph without loops and multiple edges with a vertex set V and an edge set E. In the whole paper it is assumed that the vertex set V of every graph under consideration is a subset of a fixed countable infinite set W containing the set of natural numbers and containing $[a,b]$ if $a,b \in W$.

The following operation of *a subgraph identification* of two graphs in a connected induced subgraph generalizes the operation of the union of graphs [7].

Let $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ be disjoint graphs. Let $\mathcal{G}_1' = (V_1', E_1')$ and $\mathcal{G}_2' = (V_2', E_2')$ be connected induced subgraphs of $\mathcal{G}_1, \mathcal{G}_2$, respectively and $f : \mathcal{G}_1' \to \mathcal{G}_2'$ be an isomorphism. The *subgraph identification* of $\mathcal{G}_1$ with $\mathcal{G}_2$ under f is the graph $\mathcal{G} = \mathcal{G}_1 \cup^f \mathcal{G}_2 = (V, E)$, where

$$\begin{aligned}&V = V_1 \cup (V_2 - V_2')\\&E = \{\{a,b\}\,;\, a,b \in V \quad \text{and} \quad \{a,b\} \in E_1 \cup E_2, \quad \text{or} \quad \{f(a), b\} \in E_2\}.\end{aligned}$$

Obviously, $\mathcal{G}_1 \cup^f \mathcal{G}_2 \cong \mathcal{G}_2 \cup^{f^{-1}} \mathcal{G}_1$. The subgraph identification of two graphs can be extended to graphs with a non-empty intersection as follows.

Let $\mathcal{G}_i = (V_i, E_i),\ i = 1,2$ be two graphs (not necessarily disjoint) and let $f : \mathcal{G}_1' \to \mathcal{G}_2'$ be an isomorphism of a connected induced subgraph $\mathcal{G}_1'$ of $\mathcal{G}_1$ onto a connected induced subgraph $\mathcal{G}_2'$ of $\mathcal{G}_2$. Put

$$W = \{\,[v,0];\, v \in V_1 \cap V_2\} \quad \text{and} \quad V_3 = W \cup (V_2 - V_1).$$

Define a bijection $g : V_2 \to V_3$ as follows:

$$g(x) = \begin{cases} x, & \text{if } x \in V_2 - V_1, \\ [x,0], & \text{otherwise.} \end{cases}$$

If we put $E_3 = g(E_2) = \{\{g(x), g(y)\};\ \{x,y\} \in E_2\}$, then $g : \mathcal{G}_2 \to \mathcal{G}_3$, $\mathcal{G}_3 = (V_3, E_3)$, is a graph isomorphism. Note that the graphs $\mathcal{G}_1$ and $\mathcal{G}_3$ are disjoint. The subgraph identification of $\mathcal{G}_1$ with $\mathcal{G}_3$ under $g \circ f$ (i.e. the graph $\mathcal{G} = \mathcal{G}_1 \cup^{g \circ f} \mathcal{G}_3$) will be called the subgraph identification of $\mathcal{G}_1$ with $\mathcal{G}_2$ under f. We denote it again $\mathcal{G}_1 \cup^f \mathcal{G}_2$.

The fact that $f : \mathcal{G}_1' \to \mathcal{G}_2'$ is an isomorphism of the connected induced subgraph $\mathcal{G}_1' \subseteq \mathcal{G}_1$, onto the connected induced subgraph $\mathcal{G}_2' \subseteq \mathcal{G}_2$ will be denoted by $f : \mathcal{G}_1 \mapsto \mathcal{G}_2$.

Clearly, the subgraph identification of connected graphs is again a connected graph. It is easy to see that if f is an automorphism of $\mathcal{G}_1$, then $\mathcal{G}_1 \cup^f \mathcal{G}_1 = \mathcal{G}_1$.

Let $\mathbb{K}$ be a family of graphs. Denote

$$\gamma(\mathbb{K}) = \{\mathcal{G}_1 \cup^f \mathcal{G}_2;\, \mathcal{G}_1, \mathcal{G}_2 \in \mathbb{K}, f : \mathcal{G}_1 \mapsto \mathcal{G}_2\}$$

and
$$\Gamma(\mathbb{K}) = \gamma(\mathbb{K}) \cup \gamma^2(\mathbb{K}) \cup \cdots = \bigcup_{n=1}^{\infty} \gamma^n(\mathbb{K}),$$
where $\gamma^k(\mathbb{K}) = \gamma(\gamma^{k-1}(\mathbb{K}))$, for every $k \ > 1$.
Because $\mathcal{G} \cup^{id} \mathcal{G} = \mathcal{G}$, where id is the identity on the graph $\mathcal{G}$, we have
$$\mathbb{K} \subseteq \gamma(\mathbb{K}) \subseteq \gamma^2(\mathbb{K}) \subseteq \gamma^3(\mathbb{K}) \subseteq \dots \ .$$
Further, let $S(\mathbb{K})$, $I(\mathbb{K})$ denote the set of all induced subgraphs of graphs in $\mathbb{K}$ and the set of all isomorphic images of graphs in $\mathbb{K}$, respectively. If $\mathbb{K}$ is a set of connected graphs, we will denote by $S_c(\mathbb{K})$ the set of all connected induced subgraphs of graphs in $\mathbb{K}$.

A set $\mathbb{K}$ of graphs will be called *a variety closed under isomorphic images, induced subgraphs and subgraph identifications* if
$$I(\mathbb{K}) \subseteq \mathbb{K},\ S(\mathbb{K}) \subseteq \mathbb{K} \quad \text{and} \quad \Gamma(\mathbb{K}) \subseteq \mathbb{K}.$$
A set $\mathbb{K}$ of connected graphs will be called *a variety closed under isomorphic images, induced connected subgraphs and subgraph identifications* if
$$I(\mathbb{K}) \subseteq \mathbb{K}, S_c(\mathbb{K}) \subseteq \mathbb{K} \quad \text{and} \quad \Gamma(\mathbb{K}) \subseteq \mathbb{K}.$$
Let $\mathbb{K}$, $\mathbb{K}_1 \subseteq \mathbb{K}_2$ be a sets of graphs and O be one of the operators I, S, Γ. Then $\mathbb{K} \subseteq O(\mathbb{K})$, $O(O(\mathbb{K})) = O(\mathbb{K})$ and $O(\mathbb{K}_1) \subseteq O(\mathbb{K}_2)$. The same statement holds, provided $\mathbb{K}$, $\mathbb{K}_1 \subseteq \mathbb{K}_2$ are sets of connected graphs and $O \in \{I, S_c, \Gamma\}$. Thus I, Γ and $S (I, \Gamma$ and $S_c)$ are, respectively, closure operators on the system of all sets of graphs (of connected graphs). By [3, Theorem 5.2] the following proposition holds.

Proposition 2.1. *The set of all varieties of graphs* (*of connected graphs*) *closed under isomorphic images, subgraph identifications and induced subgraphs* (*induced connected subgraphs*) *with set inclusion as the partial ordering is a complete lattice.*

Let $\mathcal{G}$ be a graph. By *a circuit* $\mathcal{C} \subseteq \mathcal{G}$ in $\mathcal{G}$ we mean a connected subgraph of $\mathcal{G}$ in which every vertex has degree two. A complete graph $\mathcal{K}$ with n-vertices is called (in this paper) *a clique* of $\mathcal{G}$ if $\mathcal{K}$ is a subgraph of $\mathcal{G}$ and

– $\mathcal{K}$ is not a proper subgraph of a complete subgraph of $\mathcal{G}$, if $n \geq 3$,
– $\mathcal{K}$ is a bridge of $\mathcal{G}$, if $n = 2$,
– $\mathcal{K}$ is a one-vertex subgraph of $\mathcal{G}$.

The following lemma is basically included in the proof of Brooks' theorem [4, p. 223–225].

Lemma 2.2. *Let $\mathcal{G} = (V, E)$ be a connected graph, which is neither a complete graph nor a circuit. Then there are two nonadjacent vertices u, v in $\mathcal{G}$ such that $\mathcal{G} - \{u, v\}$ is a connected graph.*

Theorem 2.3. *If $\mathcal{G}$ is a connected graph which is neither a circuit nor a complete graph, then $\mathcal{G}$ contains proper connected induced subgraphs $\mathcal{G}_1$, $\mathcal{G}_2$ such that $\mathcal{G} = \mathcal{G}_1 \cup^f \mathcal{G}_2$, where $f : \mathcal{G}_1 \mapsto \mathcal{G}_2$ is an isomorphism.*

Proof. By Lemma 2.2, there are two nonadjacent vertices u, v in $\mathcal{G}$ such that $\mathcal{G} - \{u, v\}$ is connected. Let f be the identity on $\mathcal{G} - \{u, v\}$. The graphs $\mathcal{G}_1 = \mathcal{G} - \{u\}$ and $\mathcal{G}_2 = \mathcal{G} - \{v\}$ are proper connected induced subgraphs of $\mathcal{G}$. Since u, v are nonadjacent vertices $\mathcal{G} = \mathcal{G}_1 \cup^f \mathcal{G}_2$ holds. ■

Corollary 2.4. *Let $\mathcal{G}$ be a connected graph. There exist the family $\mathbb{C}l$ of cliques of $\mathcal{G}$ and a family $\mathbb{C}$ of circuits of $\mathcal{G}$ such that $\mathcal{G} \in \Gamma(\mathbb{C}l \cup \mathbb{C})$.*

Proof. The statement follows from Theorem 2.3 (repeating the process of decomposition of $\mathcal{G}$ into the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$). ■

Let $\mathbb{K}$ be a family of graphs. Denote $F(\mathbb{K}) = \mathbb{K} \cup cl(\mathbb{K}) \cup C(\mathbb{K})$, where $cl(\mathbb{K})$ is the set of all cliques of graphs in $\mathbb{K}$ and $C(\mathbb{K})$ is the set of all circuits of graphs in $\mathbb{K}$.

A set $\mathbb{K}$ of graphs will be called *a variety closed under isomorphic images, cliques, circuits and subgraph identifications* if

$$I(\mathbb{K}) \subseteq \mathbb{K}, F(\mathbb{K}) \subseteq \mathbb{K} \quad \textit{and} \quad \Gamma(\mathbb{K}) \subseteq \mathbb{K}.$$

Analogously as above, the following statement can be proved.

Proposition 2.5. *The set of all varieties of graphs closed under isomorphic images, cliques, circuits and subgraph identifications with set inclusion as the partial ordering is a complete lattice.*

3. The Lattice of Varieties Closed under the Operators **I**, **S**, Γ

The proof of the following statement is straightforward.

Lemma 3.1. *For every family $\mathbb{K}$ of graphs holds*

$$II(\mathbb{K}) = I(\mathbb{K}),\ SI(\mathbb{K}) = IS(\mathbb{K}),\ SS(\mathbb{K}) = S(\mathbb{K}),$$

$$\Gamma\Gamma(\mathbb{K}) = \Gamma(\mathbb{K}),\ \Gamma I(\mathbb{K}) = I\Gamma(\mathbb{K})$$

and for every family $\mathbb{K}$ *of connected graphs holds*

$$S_cI(\mathbb{K}) = IS_c(\mathbb{K}),\ S_cS_c(\mathbb{K}) = S_c(\mathbb{K}).$$

Lemma 3.2. *For every family* $\mathbb{K}$ *of graphs* $\Gamma S(\mathbb{K}) \subseteq IS\Gamma(\mathbb{K})$ *and for every family* $\mathbb{K}$ *of connected graphs* $\Gamma S_c(\mathbb{K}) \subseteq IS_c\Gamma(\mathbb{K})$.

Proof. It follows immediately that

$$\gamma I(\mathbb{K}) \subseteq I\gamma(\mathbb{K}), \tag{3.1}$$

for any family $\mathbb{K}$ of graphs.

Now we will show that

$$\gamma S(\mathbb{K}) \subseteq IS\gamma(\mathbb{K}), \tag{3.2}$$

for any family of graphs.

Let $\mathcal{G} \in \gamma S(\mathbb{K})$, i.e. there are graphs $\mathcal{G}_1, \mathcal{G}_2 \in S(\mathbb{K})$ and $f : \mathcal{G}_1 \mapsto \mathcal{G}_2$ such that $\mathcal{G} = \mathcal{G}_1 \cup^f \mathcal{G}_2$. Since $\mathcal{G}_1, \mathcal{G}_2 \in S(\mathbb{K})$, there exist $\mathcal{H}_1, \mathcal{H}_2 \in \mathbb{K}$ such that $\mathcal{G}_i$ is an induced subgraph of $\mathcal{H}_i$ for $i = 1, 2$. Clearly, the graph $\mathcal{H}_1 \cup^f \mathcal{H}_2$ contains an induced subgraph which is isomorphic to the graph $\mathcal{G}$. Hence $\mathcal{G} \in IS\gamma(\mathbb{K})$.

Now suppose $\mathcal{G} \in \Gamma S(\mathbb{K})$. Then there is $n > 0$ such that $\mathcal{G} \in \gamma^n S(\mathbb{K})$. Combining (3.1) with (3.2) and using $II(\mathbb{K}) = I(\mathbb{K})$ we obtain

$$\mathcal{G} \in \gamma^n S(\mathbb{K}) \subseteq \gamma^{n-1} IS\gamma(\mathbb{K}) \subseteq \gamma^{n-2} IS\gamma^2(\mathbb{K}) \subseteq \ldots \subseteq IS\gamma^n(\mathbb{K}) \subseteq IS\Gamma(\mathbb{K}).$$

The same argument can be used for the operators Γ and S_c. ■

Let $\mathbb{K}$ be a family of graphs (of connected graphs). The intersection of all varieties of graphs (of connected graphs) containing $\mathbb{K}$ is a variety $V(\mathbb{K})$ ($V_c(\mathbb{K})$) which is called *the variety generated by* $\mathbb{K}$ (the variety of connected graphs generated by $\mathbb{K}$). If $\mathbb{K} = \{\mathcal{G}_1, \ldots, \mathcal{G}_n\}$, then the variety $V(\mathbb{K})$ ($V_c(\mathbb{K})$) will be denoted by $V(\mathcal{G}_1, \ldots, \mathcal{G}_n)$ (by $V_c(\mathcal{G}_1, \ldots, \mathcal{G}_n)$).

Theorem 3.3. *For every set* $\mathbb{K}$ *of graphs*

$$V(\mathbb{K}) = IS\Gamma(\mathbb{K}).$$

For every set $\mathbb{K}$ *od connected graphs*

$$V_c(\mathbb{K}) = IS_c\Gamma(\mathbb{K}).$$

Proof. Obviously, $IS\Gamma(\mathbb{K}) \subseteq V(\mathbb{K})$. In order to prove the reverse inclusion it is sufficient to prove that $IS\Gamma(\mathbb{K})$ is a variety containing $\mathbb{K}$, i.e. it is sufficient to realize that

$$I(IS\Gamma(\mathbb{K})) \subseteq IS\Gamma(\mathbb{K}),\ S(IS\Gamma(\mathbb{K})) \subseteq IS\Gamma(\mathbb{K}) \text{ and } \Gamma(IS\Gamma(\mathbb{K})) \subseteq IS\Gamma(\mathbb{K}).$$

However, this is an easy consequence of Lemmas 3.1 and 3.2. In the same manner we can verify that $V_c(\mathbb{K}) = IS_c\Gamma(\mathbb{K})$. ∎

In order to be precise, in the rest of the paper the symbol $\mathcal{K}_n$ will denote the complete graph with the vertex set $V = \{1, 2, \ldots, n\}$. Similarly, the symbol $\mathcal{C}_n$ will be used for the circuit with the vertex set $V = \{1, 2, \ldots, n\}$ and with the edge set $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n, 1\}\}$.

It is obvious that $V(\mathcal{K}_1) \subseteq \mathbb{V}$ for every non-empty variety $\mathbb{V}$.

By $L_c(I, S_c, \Gamma)$ we will denote the lattice of all non-empty varieties of connected graphs closed under isomorphic images, induced connected subgraphs and subgraph identifications and by $L(I, S, \Gamma)$ we will denote the lattice of all non-empty varieties of graphs closed under the operators I, S, Γ.

The least element of the lattice $L_c(I, S_c, \Gamma)$ of varieties of connected graphs is the variety $V_c(\mathcal{K}_1) = V(\mathcal{K}_1)$ of single-vertex graphs. This variety will be denoted by $\mathbf{0}$. Clearly, the variety $\mathbf{0}$ is the least element of the lattice $L(I, S, \Gamma)$, too.

Firstly, we turn our attention to the lattice $L_c(I, S_c, \Gamma)$.

Lemma 3.4. *The only atom of the lattice $L_c(I, S_c, \Gamma)$ is the variety of all trees $V_c(\mathcal{K}_2)$.*

Proof. It is easy to check that if $\mathbb{V} \neq \mathbf{0}$ is a variety of the lattice $L_c(I, S_c, \Gamma)$, then $\mathcal{K}_2 \in \mathbb{V}$. Hence $V_c(\mathcal{K}_2) \subseteq \mathbb{V}$. On the other hand, no graph $\mathcal{G}$ in $V_c(\mathcal{K}_2)$ contains a circuit. Using the induction on the number of vertices it is easy to see that every tree belongs to $V_c(\mathcal{K}_2)$. ∎

Theorem 3.5. *The only variety covering the variety of trees in $L_c(I, S_c, \Gamma)$ is the variety $V_c(\mathcal{C}_4)$. Moreover, $V_c(\mathcal{C}_4)$ is the variety of all connected bipartite graphs.*

Proof. Let $\mathbb{V}$ be a variety of connected graphs such that $\mathbb{V} \supseteq V_c(\mathcal{K}_2)$ and $\mathbb{V} \neq V_c(\mathcal{K}_2)$. Then there is a graph $\mathcal{G} \in \mathbb{V}$ containing an induced circuit with n-vertices for some $n \geq 3$. Therefore, $\mathcal{C}_n \in \mathbb{V}$. If $n \geq 4$, then gluing two copies of $\mathcal{C}_n$ in a path of length $n-2$, we obtain a graph which contains a graph isomorphic with a circuit $\mathcal{C}_4$ as an induced subgraph (see Figure 1).

Hence $\mathcal{C}_4 \in \mathbb{V}$ and $V_c(\mathcal{C}_4) \subseteq \mathbb{V}$. If $n = 3$, then observe that the graph $\mathcal{H}$ depicted on Figure 2 is in $\mathbb{V}$. Since $\mathcal{C}_5$ is isomorhic with an induced subgraph of $\mathcal{H}$, we have $\mathcal{C}_5 \in \mathbb{V}$. Then gluing two copies of $\mathcal{C}_5$ in a path of length 3, we obtain a graph containing an induced subgraph isomorphic with $\mathcal{C}_4$. Again $V_c(\mathcal{C}_4) \subseteq \mathbb{V}$. Hence, the only variety covering the variety of trees is the variety $V_c(\mathcal{C}_4)$.

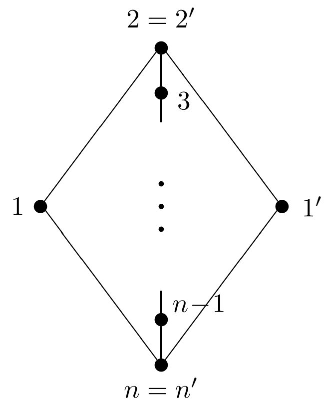

Figure 1

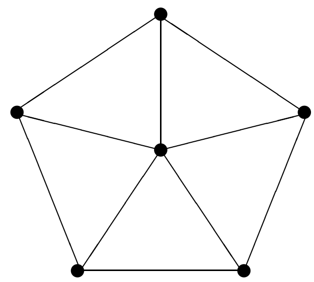

Figure 2

It remains to prove that $V_c(\mathcal{C}_4)$ coincides with the family $\mathbb{B}$ of all connected bipartite graphs. In order to prove $V_c(\mathcal{C}_4) \subseteq \mathbb{B}$, it is sufficient to realize that $\mathcal{C}_4$ is a bipartite graph and that the operators I, S_c and Γ preserve the biparticity. It follows directly from the definitions I and S_c that they preserve the biparticity. Let $\mathcal{G}_1$, $\mathcal{G}_2$ be connected bipartite graphs and $\mathcal{G} = \mathcal{G}_1 \cup^f \mathcal{G}_2$, where $f : \mathcal{G}_1 \mapsto \mathcal{G}_2$ is a subgraph isomorphism. It is known that the isomorphism f either preserves the colour of every vertex or it reverses the colour of every vertex. In the first case, the 2-coloring of $\mathcal{G}_1$, $\mathcal{G}_2$ induces 2-coloring of $\mathcal{G}_1 \cup^f \mathcal{G}_2$. In the other case, we may change colours in $\mathcal{G}_2$ thus obtaining the first case. Hence $\mathcal{G} = \mathcal{G}_1 \cup^f \mathcal{G}_2$ is bipartite in both cases.

Now we are going to prove $\mathbb{B} \subseteq V_c(\mathcal{C}_4)$. Using four copies of $\mathcal{C}_4$ we obtain the graph isomorphic to $\mathcal{H}$ in Figure 3. Since $\mathcal{C}_8$ is isomorphic to an induced subgraph of $\mathcal{H}$, we have $\mathcal{C}_8 \in V_c(\mathcal{C}_4)$. Gluing two copies of $\mathcal{C}_8$ in a path of length five we obtain a graph containing as an induced subgraph a circuit isomorphic with $\mathcal{C}_6$. Hence $\mathcal{C}_4, \mathcal{C}_6, \mathcal{C}_8 \in V_c(\mathcal{C}_4)$.

Assume $\mathcal{C}_{2k} \in V_c(\mathcal{C}_4)$, $k > 2$. Let $f : \mathcal{C}_{2k} \mapsto \mathcal{C}_6$ be a subgraph isomorphism mapping a path of lenght two of $\mathcal{C}_{2k}$ onto a path of lenght two of $\mathcal{C}_6$. Then $\mathcal{C}_{2k} \cup^f \mathcal{C}_6$ contains as an induced subgraph a circuit isomorphic with $\mathcal{C}_{2k+2}$. Thus every circuit of an even lenght belongs to $V_c(\mathcal{C}_4)$. However, it follows from Corollary 2.4 that every bipartite graph belongs to $V_c(\{\mathcal{K}_2\} \cup \mathbb{C})$, where $\mathbb{C}$ is the family of all even circuits. Thus $\mathbb{B} \subseteq V_c(\mathcal{C}_4)$ and the result follows. ■

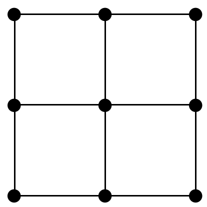

Figure 3

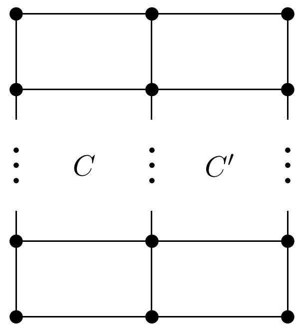

Figure 4

Lemma 3.6. *Let $\mathbb{V}$ be a variety of connected graphs. Let $\mathbb{V} \supseteq V_c(\mathcal{C}_4)$ and $\mathbb{V} \neq V_c(\mathcal{C}_4)$. Then every circuit $\mathcal{C}_n$, $n \geq 4$, belongs to $\mathbb{V}$.*

Proof. Since $\mathbb{V} \supseteq V_c(\mathcal{C}_4)$ and $\mathbb{V} \neq V_c(\mathcal{C}_4)$ there exists a non-bipartite connected graph $\mathcal{G}$ in $\mathbb{V}$. If a circuit isomorphic with $\mathcal{C}_3$ is an induced subgraph of $\mathcal{G}$, then $\mathcal{C}_5 \in \mathbb{V}$ (see Figure 2). Otherwise the shortest circuit of odd lenght in $\mathcal{G}$ is an induced subgraph of $\mathcal{G}$. Hence there is a circuit $\mathcal{C}_m$, $m \geq 5$, m odd, in $\mathbb{V}$. Since $\mathbb{V} \supseteq V_c(\mathcal{C}_4)$, by Theorem 3.5 every circuit of an even lenght belongs to $\mathbb{V}$. Gluing $\mathcal{C}_m$ and $\mathcal{C}_{m+1}$, $m \geq 5$, in a path of length $m-2$ we obtain a graph containing a circuit isomorphic with $\mathcal{C}_5$ as an induced subgraph. Thus in both cases $\mathcal{C}_5 \in \mathbb{V}$. Let $f : \mathcal{C}_5 \mapsto \mathcal{C}_{2k}$, $k > 2$, be a subgraph isomorphism mapping a path of length two onto a path of length two. Then $\mathcal{C}_{2k+1} \simeq \mathcal{C} \subseteq \mathcal{C}_5 \cup^f \mathcal{C}_{2k}$, where $\mathcal{C}$ is an induced subgraph of $\mathcal{C}_5 \cup^f \mathcal{C}_{2k}$, hence $\mathcal{C}_{2k+1} \in \mathbb{V}$ for every $k \geq 2$. ■

Corollary 3.7. *If $\mathcal{C}_3 \in \mathbb{V}$, then every circuit belongs to $\mathbb{V}$.*

Lemma 3.8. *Let $\mathbb{V}$ be a variety of connected graphs. If $\mathbb{V}$ is not the variety of all connected graphs and $V_c(\mathcal{C}_5) \subseteq \mathbb{V}$, $V_c(\mathcal{C}_5) \neq \mathbb{V}$, then there exists $n \geq 3$ such that $\mathbb{V} = V_c(\mathcal{K}_n)$.*

Proof. By Corollary 2.4 every graph $\mathcal{G}$ in $\mathbb{V}$ can be obtained by successive subgraph identifications of a family of its complete subgraphs (cliques) and some its circuits.

Let $cl(\mathbb{V})$ be a family of all pairwise nonisomorhic complete subgraphs in $\mathbb{V}$. Assume that $cl(\mathbb{V})$ is infinite. Then it contains complete graphs of arbitrary large order. By Corollary 3.7 and 2.4 $\mathbb{V}$ is the variety of all graphs, a contradiction. Hence we may assume that there is a maximal complete graph $\mathcal{K}_n$ of order n in $cl(\mathbb{V})$. Then $V_c(\mathcal{K}_n) \subseteq \mathbb{V}$ and $\mathcal{K}_m \in V_c(\mathcal{K}_n)$ for every $m \leq n$ (as an induced subgraph of $\mathcal{K}_n$). By Corollary 3.7 every circuit $\mathcal{C}_n$ belongs into $V_c(\mathcal{K}_n)$. Using Corollary 2.4 we get $\mathbb{V} = V_c(\mathcal{K}_n)$. ∎

Corollary 3.9. $\mathbb{V}$ *is the variety of all connected graphs if and only if* $\mathbb{V}$ *contains a complete graph of arbitrary large order.*

Let $\mathcal{G}$ be a connected graph. We denote by $cq(\mathcal{G})$ *the clique number* of $\mathcal{G}$, i.e. $cq(\mathcal{G})$ is the order of a maximal clique of $\mathcal{G}$. The results of this section are summarized in the following two theorems.

Theorem 3.10. *Let* $\mathcal{G}$ *be a connected graph. Then*

(i) $V_c(\mathcal{G}) = V_c(\mathcal{K}_2)$, *if* $\mathcal{G}$ *is a tree,*
(ii) $V_c(\mathcal{G}) = V_c(\mathcal{C}_4)$, *if* $\mathcal{G}$ *is a bipartite graph with a circuit,*
(iii) $V_c(\mathcal{G}) = V_c(\mathcal{C}_5)$, *if there is a circuit of an odd lenght in* $\mathcal{G}$ *and* $cq(\mathcal{G}) = 2$,
(iv) $V_c(\mathcal{G}) = V_c(\mathcal{K}_n)$, *if the clique number* $cq(\mathcal{G}) = n \geq 3$.

Denote by $\mathbb{G}_c$ the variety of all connected graphs.

Theorem 3.11. *The lattice* $L_c(I, S_c, \Gamma)$ *is the chain*

$$V_c(\mathcal{K}_1) < V_c(\mathcal{K}_2) < V_c(\mathcal{C}_4) < V_c(\mathcal{C}_5) < V_c(\mathcal{K}_3)$$
$$< V_c(\mathcal{K}_4) < \ldots < V_c(\mathcal{K}_n) < \ldots < \mathbb{G}_c$$

isomorphic with the ordinal number $\omega + 1$.

Denote by $\mathbb{G}$ the set of all graphs and by the disconnected graph with two vertices 1 and 2.

A family $\mathbb{K}$ of graphs is called hereditary (by [9]) if it is closed under induced subgraphs. Thus, every variety closed under the operators I, S, Γ is hereditary.

A family $\mathbb{K}$ of graphs is called aditive ([8]) if it is closed under disjoint union of graphs. It is easy to check that a variety $\mathbb{V}$ of graphs closed under the operators I, S, Γ is aditive if $\mathbb{K}$ contains a graph with at least two vertices. Indeed, the disjoint union of two graphs $\mathcal{G}$ and $\mathcal{H}$ belonging to $\mathbb{V}$ can be constructed as follows. First, observe that $\mathbb{V}$ contains necessarily a graph isomorphic to $\tilde{\mathcal{K}}_2$. Such a graph is always a subgraph of a graph which we get by gluing a nontrivial graph in $\mathbb{V}$ with itself in a one-vertex graph. Further, applying Γ operator on $\mathcal{G}$ and $\tilde{\mathcal{K}}_2$ we form a new graph $\mathcal{G}' \in \mathbb{V}$ having two connectivity components, one being isomorphic to $\mathcal{G}$ and the other containg only one vertex. Finally, gluing $\mathcal{G}'$ and $\mathcal{H}$ in the one-vertex component of $\mathcal{G}'$ we obtain a graph isomorphic to the disjoint union of $\mathcal{G}$ and $\mathcal{H}$.

Taking into account that every aditive variety of graphs is determined uniquely by the set of all connected graphs which belong to it, we see that Theorem 11 has the following corollary.

Corollary 3.12. *The lattice $L(I, S, \Gamma)$ is the chain*

$$V(\mathcal{K}_1) < V(\tilde{\mathcal{K}}_2) < V(\mathcal{K}_2) < V(\mathcal{C}_4) < V(\mathcal{C}_5)$$
$$< V(\mathcal{K}_3) < V(\mathcal{K}_4) < V(\mathcal{K}_5) < \ldots < \mathbb{G}.$$

Let $\mathbb{V}$ be a variety of graphs. An integer $c(\mathbb{V})$ such that $\mathbb{V}$ contains a complete graph on $c(\mathbb{V}) + 1$ vertices but $\mathbb{V}$ does not contain a complete graph on $c(\mathbb{V}) + 2$ vertices is called *a completness* of $\mathbb{V}$ (see [6]). By Theorem 3.10 we have that $\mathbb{V} = V(\mathcal{G}) > V(\mathcal{C}_5)$ implies $cq(\mathcal{G}) = c(\mathbb{V}) + 1$.

Remark. It follows from Theorem 3.10 and Corollary 2.4 that every n-colourable graph ($n \geq 3$) belongs into $V(\mathcal{K}_n)$. However, the converse statement is not true. The graphs in Figure 5a are 3-colourable but their identification in Figure 5b is not 3-colourable.

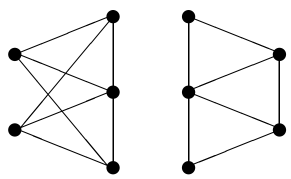

Figure 5a

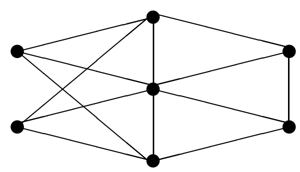

Figure 5b

4. The Lattice of Varieties Closed under the Operators I, F, Γ

In this section, by a variety we mean a variety of graphs closed under isomorphic images, cliques, circuits and subgraph identifications.

Theorem 4.1. *For any set* $\mathbb{K}$ *of graphs*

$$V(\mathbb{K}) = I\Gamma F\Gamma(\mathbb{K}).$$

Proof. It is obvious that $II = I$, $\Gamma I = I\Gamma$ and $\Gamma\Gamma = \Gamma$. Therefore

$$I(I\Gamma F\Gamma(\mathbb{K})) = I\Gamma F\Gamma(\mathbb{K}) \quad \text{and} \quad \Gamma(I\Gamma F\Gamma(\mathbb{K})) = I\Gamma F\Gamma(\mathbb{K}).$$

Now, it suffices to show that

$$F(I\Gamma F\Gamma(\mathbb{K})) \subseteq I\Gamma F\Gamma(\mathbb{K}).$$

A clique $\mathcal{K}_n$ of a subgraph identification $\mathcal{G}_1 \cup^f \mathcal{G}_2$ (of a graph $\mathcal{G}_1$ with $\mathcal{G}_2$ under f) is a clique of $\mathcal{G}_1$, or $\mathcal{K}_n$ is isomorphic with a clique of $\mathcal{G}_2$. Hence

$$\mathcal{K}_n \in F(I\Gamma F\Gamma(\mathbb{K})) \Longrightarrow \mathcal{K}_n \in I\Gamma F\Gamma(\mathbb{K})$$

for any complete graph $\mathcal{K}_n$, $n \geq 1$.

From Theorems 3.10 and 3.11 it follows that

1. $IS_c\Gamma(\mathbb{K})$ contains all circuits and hence also $IF\Gamma(\mathbb{K})$ contains all circuits if there exists a graph $\mathcal{G} \in \mathbb{K}$ containing a circuit of an odd length.
2. $IS_c\Gamma(\mathbb{K})$ and hence also $IF\Gamma(\mathbb{K})$ contains all circuits of even length but no circuit of odd length if there exists a graph $\mathcal{G} \in \mathbb{K}$ containing a circuit of even length but no graph from $\mathbb{K}$ contains a circuit of odd length.
3. $IS_c\Gamma(\mathbb{K})$ and hence also $IF\Gamma(\mathbb{K})$ contains no circuit if any graph from $\mathbb{K}$ contains no circuit.

This gives

$$\mathcal{C}_n \in F(I\Gamma F\Gamma(\mathbb{K})) \Longrightarrow \mathcal{C}_n \in I\Gamma F\Gamma(\mathbb{K})$$

for any circuit $\mathcal{C}_n$ of length $n \geq 3$. ∎

Corollary 4.2. *If a set* $\mathbb{K}$ *of graphs contains no disconnected graph, then the variety* $V(\mathbb{K})$ *contains no disconnected graph, too.*

It is obvious that $V(\mathcal{K}_1) \subseteq \mathbb{V}$ for every non-empty variety $\mathbb{V}$.

By $L_c(I, F, \Gamma)$ we will denote the lattice of all non-empty varieties of connected graphs closed under the operators I, F, Γ and by $L(I, F, \Gamma)$ we will denote the lattice of all non-empty varieties of graphs closed under the operators I, F, Γ.

Theorem 4.3. *The lattice $L_c(I, F, \Gamma)$ is isomorphic with the direct product of 2-element lattice and the lattice $2 \oplus B_\omega$, where $\oplus$ denotes the ordinal sum and B_ω is a Boolean lattice isomorphic with the lattice of all subset of the set N (of all natural numbers).*

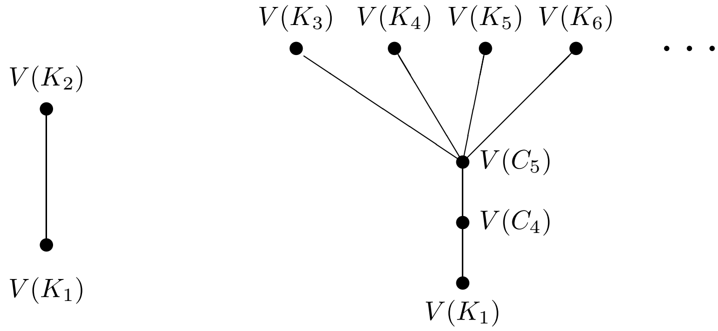

Figure 6

Proof. One can check that $cl(\mathbb{K}) = cl(V(\mathbb{K}))$ for any set $\mathbb{K}$ of graphs. If any complete graph $\mathcal{K}_n$, $n > 2$, belongs to a class $\mathbb{K}$ (of graphs), then $V(\mathbb{K})$ contains an odd circuit and therefore $V(\mathbb{K})$ contains all circuits. Each variety of graphs is (by Corollary 2.4) uniquely determined by its system of circuits and complete graphs. Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be any set of complete graphs of order at least 3. It is easy to see that

$$(a)\ \ V(\mathbb{K}_1) \vee V(\mathbb{K}_2) = V(\mathbb{K}_1 \cup \mathbb{K}_2) \quad \text{and} \quad V(\mathbb{K}_1) \wedge V(\mathbb{K}_2) = V(\mathbb{K}_1 \cap \mathbb{K}_2),$$

where $V(\emptyset) = V(\mathcal{C}_5)$ is the least variety containing all circuits.

Let $\mathcal{P}(M)$ be the Boolean lattice of all subsets of the set $M = \{3, 4, 5, \dots\}$ all integers $n \geq 3$. For each subset $A \subseteq M$ we denote $\mathbb{K}_A$ the set of complete graphs given by

$$\mathcal{K}_n \in \mathbb{K}_A \Longleftrightarrow n \in A.$$

The mapping $f : P(M) \to L_c(I, F, \Gamma)$ given by $f(A) = V(\mathbb{K}_A)$ is the monomorphism (the embeding) of the Boolean lattice $\mathcal{P}(M)$ to $L_c(I, F, \Gamma)$ (it is easy that $A \neq B$ implies $V(\mathbb{K}_A) \neq V(\mathbb{K}_B)$). Denote $f(P(M)) = B_\omega$.

The variety $V(\mathcal{C}_4)$ is an atom of $L_c(I, F, \Gamma)$ and the variety $V(\mathcal{C}_5)$ covers the variety $V(\mathcal{C}_4)$ (similarly as in the lattice $L_c(I, S, \Gamma)$). The atom $V(\mathcal{K}_2)$ is noncomparable with every variety from $B_\omega \cup \{V(\mathcal{C}_4), V(\mathcal{C}_5)\}$. Moreover, (by Corollary 2.4) we get that $V(\mathcal{K}_2) \vee W$ covers W for each variety from $B_\omega \cup \{V(\mathcal{C}_4), V(\mathcal{C}_5)\}$. If W_1 and W_2 are different varieties noncomparable with $V(\mathcal{K}_2)$, then $W_1 \vee V(\mathcal{K}_2)$ and $W_2 \vee V(\mathcal{K}_2)$ are different too. The mapping φ given by

$$[\mathbf{0}, \mathbb{V}] \mapsto \mathbb{V} \quad \text{and} \quad [\mathbf{1}, \mathbb{V}] \mapsto \mathbf{V}(\mathcal{K}_\mathbf{2}) \vee \mathbb{V}$$

is the isomorphism of the direct product of the 2-element lattice with universe $\{\mathbf{0}, \mathbf{1}\}$ and the lattice $2 \oplus B_\omega$ onto the lattice $L_c(I, F, \Gamma)$ The proof is complete. ■

Corollary 4.4. *The lattice* $L_c(I, F, \Gamma)$ *is distributive.*

By Theorem 4.1 and Corollary 4.2 the lattice $L_c(I, F, \Gamma)$ is a sublattice of the lattice $L(I, F, \Gamma)$ of all varieties of graphs closed under the operators I, F, Γ. However, the lattice $L(I, F, \Gamma)$ is not distributive. Denote by $\tilde{\mathcal{K}}_n$ the n-vertex graph without edges complementary to $\mathcal{K}_n$. The varieties

$$\mathbb{V}_1 = V(\tilde{\mathcal{K}}_4), \quad \mathbb{V}_2 = V(\tilde{\mathcal{K}}_5) \quad \text{and} \quad \mathbb{V}_3 = V(\tilde{\mathcal{K}}_6)$$

generate a non-distributive sublattice of the lattice $L(I, F, \Gamma)$. The graph $\tilde{\mathcal{K}}_9$ belongs to the varieties $\mathbb{V}_1 \vee \mathbb{V}_2$ and $\mathbb{V}_1 \vee \mathbb{V}_3$ (we can get it by gluing $\tilde{\mathcal{K}}_5$ with $\tilde{\mathcal{K}}_5$ and $\tilde{\mathcal{K}}_4$ with $\tilde{\mathcal{K}}_6$, respectively) but it does not belong to $\mathbb{V}_1 \vee (\mathbb{V}_2 \wedge \mathbb{V}_3)$, since the least non-trivial element in $\mathbb{V}_2 \wedge \mathbb{V}_3$ is $\tilde{\mathcal{K}}_{21}$.

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