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GRAPHS MAXIMAL WITH RESPECT TO ABSENCE OF HAMILTONIAN PATHS

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Abstract

Two classes of graphs which are maximal with respect to the absence of Hamiltonian paths are presented. Block graphs with this property are characterized.

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In [3], page 43, a conjecture of I. Broere and M. Frick [2] is quoted. This conjecture concerns $\mathcal{W}_k$-maximal graphs on $k+2$ vertices, i.e. graphs maximal with respect to absence of Hamiltonian paths. We shall present some results which disprove this conjecture.

All graphs under consideration will be finite undirected graphs without loops and multiple edges. If G is a graph and M is a subset of its vertex set, then $G\langle M\rangle$ denotes the subgraph of G induced by M. A Hamiltonian path in G is a path in G which contains all vertices of G. A block graph is a graph, all of whose blocks are complete graphs. Two blocks of a graph G are called neighbouring, if they have a common vertex (an articulation of G). A block subgraph of G is an induced subgraph G_0 of G such that each block of G_0 is a block of G. A block graph G is called linear, if its blocks form a finite sequence $B_1, \ldots, B_m$ in such a way that two blocks are neighbouring if and only if one of them follows immediately after the other in this sequence. The following lemma is easy to prove.

Lemma. *A block graph G has a Hamiltonian path, if and only if it is linear.*

Now let k be a positive integer. The symbol $\mathcal{W}_k$ denotes the class of all graphs in which the longest path has length $k+1$. A graph G is called

$\mathcal{W}_k$- maximal, if G is in $\mathcal{W}_k$, while each graph obtained from G by joining a pair of its non-adjacent vertices by an edge is not in $\mathcal{W}_k$. In particular, if G has $k+2$ vertices and is $\mathcal{W}_k$-maximal, then it is maximal with respect to absence of Hamiltonian paths; it has no Hamiltonian path, but each graph obtained from it by joining a pair of non-adjacent vertices by an edge has a Hamiltonian path. In [3] all disconnected $\mathcal{W}_k$-maximal graphs with $k+2$ vertices are characterized; they are graphs consisting of two connected components, both being complete graphs. The mentioned conjecture of I. Broere and M. Frick says that connected $\mathcal{W}_k$-maximal graphs are block graphs with at least three blocks each.

We shall show two types (classes) of $\mathcal{W}_k$-maximal graphs with $k+2$ vertices, among which there are graphs with one block (2-connected graphs). At the end, we shall characterize block graphs which are $\mathcal{W}_k$-maximal graphs with $k+2$ vertices.

We describe two constructions of classes of graphs.

Construction I. Let p be a non-negative integer, let $a_1, \dots, a_{p+2}$ be positive integers. Let $U_0, U_1, \dots, U_{p+2}$ be pairwise disjoint sets of vertices such that $|U_0| = p$ and $|U_i| = a_i$ for $i = 1, \dots, p+2$. The vertex set of the graph G will be $V(G) = \bigcup_{i=0}^{p+2} U_i$. The edge set of G is such that the induced subgraphs $G\langle U_0 \cup U_i\rangle$ for $i = 1, \dots, p+2$ are complete graphs and the graph G is their union. Any graph G obtained by this construction will be called a graph of type I.

Construction II. Let $p, q, r, a_1, \dots, a_p$, $b_1, \dots, b_q$, $c_1, \dots, c_r$ be positive integers, let s be a non-negative integer. Let $U_0, U_1, \dots, U_p$, $V_0, V_1, \dots, V_q$, W_0, $W_1, \dots, W_r, X$ be pairwise disjoint sets of vertices such that $|U_0| = p$, $|U_i| = a_i$ for $i = 1, \dots, p$, $|V_0| = q$, $|V_i| = b_i$ for $i = 1, \dots, q$, $|W_0| = r$, $|W_i| = c_i$ for $i = 1, \dots, r$, $|X| = s$. The vertex set of the graph G is $V(G) = (\bigcup_{i=0}^{p} U_i) \cup (\bigcup_{i=0}^{q} V_i) \cup (\bigcup_{i=0}^{r} W_i) \cup X$. The edge set of G is such that the induced subgraphs $G\langle U_0 \cup U_i\rangle$ for $i = 1, \dots, p$, $G\langle V_0 \cup V_i\rangle$ for $i = 1, \dots, q, G\langle W_0 \cup W_i\rangle$ for $i = 1, \dots, r$ and $G\langle U_0 \cup V_0 \cup W_0 \cup X\rangle$ are complete graphs and the graph G is their union. Any graph G obtained by this construction will be called a graph of type II.

Now we state a theorem concerning these graphs.

Theorem 1. *Let G be a graph with $k+2$ vertices and of type* I *or of type* II. *Then G is $\mathcal{W}_k$-maximal.*

Proof. We shall do the proof only for graphs of type I; the proof for graphs of type II would be more complicated, but quite analogous. Thus let G be a graph of type I with $k+2$ vertices. Suppose that G has a Hamiltonian path P. By deleting U_0 from G the graph with $p+2$ connected components $G\langle U_i\rangle$ for $i = 1, \ldots, p+2$ is obtained. But, as $|U_0| = p$ and P is a path, from P by deleting U_0 a graph with at most $p+1$ connected components may be obtained and these components must contain vertices of all sets U_i for $i = 1, \ldots, p+2$, which is a contradiction. Therefore G has no Hamiltonian path.

Let G' be a graph obtained from G by adding an edge e; without loss of generality we may suppose that e joins a vertex $u \in U_1$ with a vertex $v \in U_2$. We shall construct a Hamiltonian path in G'. We start by taking a Hamiltonian path of the (complete) graph $G\langle U_1\rangle$ which ends in u, then edge $e = uv$, then a Hamiltonian path in $G\langle U_2\rangle$ starting in v. We continue in such a way that for $i = 2, \ldots, p+1$ we join the end vertex of the constructed Hamiltonian path in $G\langle U_i\rangle$ with a vertex of U_0 which was still not used, then we join this vertex with a vertex of U_{i+1} and construct a Hamiltonian path in $G\langle U_{i+1}\rangle$ starting in that vertex. After doing this for all $i = 2, \ldots, p+1$ we have a Hamiltonian path in G'. This proves the assertion. ■

Note that the graphs of type I with $p = 0$ are exactly all graphs which were proved to be exactly all disconnected $\mathcal{W}_k$-maximal graphs with $k+2$ vertices in the above mentioned result from [3].

We express a conjecture.

Conjecture. Every $\mathcal{W}_k$-maximal graph with $k+2$ vertices is either of type I, or of type II. ■

At the end we prove a theorem concerning block graphs.

Theorem 2. *A block graph G with $k+2$ vertices, where k is a positive integer, is $\mathcal{W}_k$-maximal if and only if one of the following conditions is satisfied:*

(i) *G has exactly three pairwise neighbouring blocks;*

(ii) *G has exactly four blocks, three of which are pairwise non-neighbouring and the fourth is neighbouring to all of them.*

Proof. If the condition (i) is satisfied, then G is of type I for $p = 1$. If the condition (ii) is satisfied, then G is of type II for $p = q = r = 1$.

Now let G be a block graph with $k+2$ vertices. If G has one or two blocks, then it is linear and has a Hamiltonian path. If G has three blocks,

then either it satisfies the condition (i), or it is linear. If G has four blocks, then either it satisfies the condition (ii), or it is linear, or it contains a block graph satisfying (i) as its proper block subgraph. Thus let G contain at least five blocks. If there is an articulation in G common to at least three blocks, then G contains a proper block subgraph satisfying (i). If each articulation is common to exactly two blocks and there exists a block containing at least three articulations of G, then G contains a proper block subgraph satisfying (ii). In the remaining case G is linear.

Now suppose that G contains a proper block subgraph G_0 satisfying (i) or (ii). Then there exist neighbouring blocks B_1, B_2, of G such that B_1 is a block of G_0 and B_2 is not. Let u be a vertex of B_1 and v a vertex of B_2, none of which is the common vertex of B_1 and B_2. Let G' be the graph obtained from G by adding the edge $e = uv$. Further let G'' be the graph obtained from G by joining each vertex of B_1 with each vertex of B_2 by an edge. The graph G'' is a block graph and contains G' as a spanning subgraph. The graph G'' contains a block subgraph satisfying (i) or (ii), therefore it has no Hamiltonian path and neither has G'. We have proved that G is not $\mathcal{W}_k$-maximal.

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