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A SUFFICIENT CONDITION FOR THE EXISTENCE OF k-KERNELS IN DIGRAPHS

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Abstract

In this paper, we prove the following sufficient condition for the existence of k-kernels in digraphs: Let D be a digraph whose asymmetrical part is strongly conneted and such that every directed triangle has at least two symmetrical arcs. If every directed cycle γ of D with $\ell(\gamma) \not\equiv 0(\text{mod } k)$, $k \geq 2$ satisfies at least one of the following properties: (a) γ has two symmetrical arcs, (b) γ has four short chords. Then D has a k-kernel.

This result generalizes some previous results on the existence of kernels and k-kernels in digraphs. In particular, it generalizes the following Theorem of M. Kwaśnik [5]: *Let D be a strongly connected digraph, if every directed cycle of D has length $\equiv 0(\text{mod}\, k)$, $k \geq 2$. Then D has a k-kernel.*

Keywords: digraph, kernel, k-kernel.

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1. Introduction

The concept of k-kernel of a digraph was introduced by Kwaśnik in [5, 6] who also obtained an interesting theorem about the existence of k-kernels in a strongly connected digraph, which is a generalization of Richardson's Theorem: *Let D be a strongly connected digraph, if every directed cycle of*

D has length $\equiv 0 (mod\, k)$, $k \geq 2$. Then D has a k-kernel. In this paper, we present a generalization of the result of Kwaśnik. For general concepts we refer the reader to [1]. Let D be a digraph. $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively. An arc $(u_1, u_2) \in A(D)$ is called *asymmetrical* (resp. *symmetrical*) if $(u_2, u_1) \notin A(D)$ (resp. $(u_2, u_1) \in A(D)$). The asymmetrical part of D (resp. symmetrical part of D) which is denoted $Asym(D)$ (resp. $sym(D)$) is the spanning subdigraph of D whose arcs are the asymmetrical (resp. symmetrical) arcs of D.

A directed walk of D is a sequence of vertices of D $T = [z_0, z_1, \ldots, z_n]$ such that $(z_i, z_{i+1}) \in A(D)$, for $0 \leq i \leq n-1$. A directed path T of D from z_0 to z_n is a sequence of distinct vertices $T = [z_0, z_1, \ldots, z_n]$ belonging to $V(D)$ such that $(z_i, z_{i+1}) \in A(D)$ for $i = 1, 2, \ldots, n-1$. The length of T we shall denote by $\ell(T)$ is n. For convenience we shall mean T as a subdigraph of D. For $z_i, z_j \in V(T)$ we denote by $[z_i, T, z_j]$ the directed walk from u to v contained in T. A chord of the directed walk T is an arc of D of the form (z_i, z_j) where $j \neq i+1$ and $\{z_i, z_j\} \subseteq \{z_0, z_1, \ldots, z_n\}$, and a short chord of T is an arc of the form $(i, i+2)$ with $0 \leq i \leq n-2$. By the directed distance $d_D(x, y)$ from the vertex x to vertex y in a digraph D we mean the length of the shortest directed path from x to y in D. A directed cycle of D is a sequence of vertices belonging to $V(D)$, $\mathcal{C} = [z_0, z_1, \ldots, z_n, z_0]$ such that $z_i \neq z_j$, for $i \neq j$ and $(z_i, z_{i+1}) \in A(D)$, for $0 \leq i \leq n$ (notation modulo n). A chord of the directed cycle $\mathcal{C}$ is an arc of D of the form (z_i, z_j) with $j \neq i+1$ (modulo n). The chord is short when $j = i+2$ (modulo n).

The union of two digraphs D and H is denoted $D \cup H$ and defined as follows: $V(D \cup H) = V(D) \cup V(H)$ and $A(D \cup H) = A(D) \cup A(H)$. Finally, we will write $D_1 \subseteq D_2$ when $V(D_1) \subseteq V(D_2)$ and $A(D_1) \subseteq A(D_2)$.

Definition 1.1 [5]**.** Let k be a natural number with $k \geq 2$. A set $J \subseteq V(D)$ will be called a *k-kernel* of the digraph D iff:

1) For each $x, x' \in J$, $x \neq x'$ we have $d_D(x, x') \geq k$ and
2) For each $y \in V(D) - J$, there exists $x \in J$ such that $d_D(y, x) \leq k-1$.

Notice that for $k = 2$ we have a kernel in the sense of Berge [1].

2. A Sufficient Condition for the Existence of k-Kernels in Digraphs

The main result of this paper is Theorem 2.1, to prove it we need the following Lemma.

Lemma 2.1. *Let D be a digraph, $u, v, w \in V(D)$, T_1 a directed path from u to v, T_2 a vw-directed path of length at most 1 (possible $v = w$), T_3 a wu-directed path, and denote by $\gamma = T_1 \cup T_2 \cup T_3$. If $\ell(\gamma) \not\equiv 0 (mod\, k)$, $k \geq 2$, then there exists a directed cycle $\mathcal{C}$ contained in γ with $\ell(\mathcal{C}) \not\equiv 0 (mod\, k)$ and vertices $u', v', w' \in V(\mathcal{C})$ such that $[u', \mathcal{C}, v']$ is a subpath of T_1, $[v', \mathcal{C}, w']$ is a subpath of T_2 and $[w', \mathcal{C}, u']$ is a subpath of T_3. (possibly $\ell(T_2) = 0$ and possibly $\ell[v', \mathcal{C}, w'] = 0$).*

Proof. We proceed by induction on $\ell(\gamma)$.

If $\ell(\gamma) = 2$, clearly γ is a directed cycle with the required properties. Suppose the result is valid for γ' with the properties of Lemma 2.1 such that $\ell(\gamma') < n$ and let $\gamma = T_1 \cup T_2 \cup T_3$ with $\ell(\gamma) = n$.

If $V(T_1) \cap V(T_3) = \{u\}$, then $v \neq w$, γ is a directed cycle and the result follows.

If $V(T_1) \cap V(T_3) \neq \{u\}$, we take z the first vertex of T_1 different of u which is in T_3.

Since $\ell(\gamma) \not\equiv 0 (\mathrm{mod}\, k)$ we have that at least one of the following assertions holds:

(a) $\ell\left([u, T_1, z] \cup [z, T_3, u]\right) \not\equiv 0 (\mathrm{mod}\, k)$

(b) $\ell\left([z, T_1, v] \cup T_2 \cup [w, T_3, z]\right) \not\equiv 0 (\mathrm{mod}\, k)$.

If (a) holds, we take $\gamma' = [u, T_1, z] \cup [z, T_3, u]$, $u' = u$, $v' = w' = z$, clearly $\ell(\gamma') < n$; and by the inductive hypothesis on γ' we have that there exists a directed cycle $\mathcal{C}$ contained in γ' and hence in γ with the required properties.

When (b) holds, we take $\gamma' = [z, T_1, v] \cup T_2 \cup [w, T_3, z]$, $u' = z$, $v' = v$, $w' = w$; clearly $\ell(\gamma') < n$; and by the inductive hypothesis of γ' we have that there exists a directed cycle $\mathcal{C}$ contained in γ' and hence in γ with the required properties. ■

Theorem 2.1. *Let D be a digraph such that $Asym(D)$ is strongly connected and each directed cycle of length 3 has at least two symmetrical arcs. If for every directed cycle γ of D with $\ell(\gamma) \not\equiv 0 (mod\, k)$ either (a) or (b) is satisfied where:*

(a) *γ has two symmetrical arcs,*

(b) *γ has four short chords,*

then D has a k-kernel ($k \geq 2$).

Proof. Let $m_0 \in V(D)$ be any vertex, and for each $0 \leq i < k$ let $N_i \subseteq V(D)$ be defined as follows:

$N_i = \{z \in V(D) \mid$ the shortest directed path from m_0 to z contained in $Asym(D)$ has length $\equiv i (\mathrm{mod}\, k)\}$.

(1) Clearly $N_i \cap N_j = \emptyset$ for $i \neq j$, $0 \leq i, j < k$ and

(2) $\bigcup_{i=0}^{k-1} N_i = V(D)$.

This follows directly from the fact $Asym(D)$ is strongly connected. Moreover, we shall prove that:

(3) Every arc of D with initial endpoint in N_i has terminal endpoint in N_{i+1} (notation modulo k).

Let (x, y) be an arc with initial endpoint in N_i, and take: a shortest directed path T_x from m_0 to x contained in $Asym(D)$, a shortest directed path T_y from m_0 to y contained in $Asym(D)$ and a shortest directed path T from y to m_0 contained in $Asym(D)$: It should be noted that such paths exist because $Asym(D)$ is strongly connected.

(3.1) $\ell(T_x) \equiv i (\mathrm{mod}\, k)$.

This follows from the definition of N_i and the fact that $x \in N_i$.

(3.2) T_x has no short chord in D.

Since T_x is the shortest directed path from m_0 to x contained in $Asym(D)$, we have that T_x has no short chord contained in $Asym(D)$. Let $T_x = [m_0 = z_0, z_1, \ldots, z_n = x]$. If (z_i, z_{i+2}), $0 \leq i \leq n-2$ is a symmetrical short chord of T_x, we have that $[z_i, z_{i+1}, z_{i+2}, z_i]$ is a directed triangle with at most one symmetrical arc (because $\{(z_i, z_{i+1}), (z_{i+1}, z_{i+2})\} \subseteq A(T_x) \subseteq A(Asym(D))$, contradicting the assumption of Theorem 2.1. We conclude that T_x has no short chord in D. Similarly it can be proved the following two assertions:

(3.3) T_y has no short chord in D.

(3.4) T has no short chord in D.

Now we will analyze the two possible subcases:

Case 1. $y \in T_x$.

Here we will analyse the several possible subcases:

Case 1.a $\ell([m_0, T_x, y] \cup T) \not\equiv 0 (\mathrm{mod}\, k)$.

In this case it follows from Lemma 2.1 (taking $u = m_0$, $v = w = y = z_i$, $T_1 = [m_0, T_x, y]$ $T_2 = [v = w = y = z_i]$ and $T_3 = T$), that there exists a directed cycle $\mathcal{C}$ contained in $[m_0, T_x, y] \cup T$ with $\ell(\mathcal{C}) \not\equiv 0 (\mathrm{mod}\, k)$ and vertices u', v', w' such that $[u', \mathcal{C}, v']$ is a subpath of $[m_o, T_X, y]$, $v' = w'$ because $u = v$, and $(v' = w', \mathcal{C}, u')$ is a subpath of T. And we have:

(1.a.1) $\mathcal{C} \subseteq Asym(D)$.

This follows from the facts $\mathcal{C} \subseteq [m_0, T_x, y] \cup T \subseteq T_x \cup T \subseteq Asym(D)$.

(1.a.2) $[u', \mathcal{C}, v']$ has no short chord.

It is a consequence of (3.2) and the fact that $[u', \mathcal{C}, v']$ is a subpath of $[m_0, T_x, y]$ which is a subpath of T_x.

Similarly:

(1.a.3) $[v', \mathcal{C}, u']$ has no short chord.

Since $\ell(\mathcal{C}) \not\equiv 0 (\mathrm{mod}\, k)$, it follows from (1.a.1) and the assumption of Theorem 2.1 that $\mathcal{C}$ has four short chords. Let

$$\mathcal{C} = \left[u' = z_0, z_1, \ldots, z_{i-1}, z_i = v', z_{i+1}, \ldots, z_n, z_0\right];$$

we have from (1.a.2) and (1.a.3) that the only possible short chords of $\mathcal{C}$ are (z_{i-1}, z_{i+1}) and (z_n, z_1), contradicting the hypothesis of Theorem 2.1.

Case 1.b $\ell\left([y, T_x, x] \cup [x, y]\right) \not\equiv 0 (\mathrm{mod}\, k)$.

In this case we have the directed cycle

$$\mathcal{C} = [y, T_x, x] \cup [x, y] = [y = w_0, w_1, \ldots, w_n = x, w_0]$$

with $\ell(\mathcal{C}) \not\equiv 0 (\mathrm{mod}\, k)$. Since $[y, T_x, x]$ is a subpath of T_x and $T_x \subseteq Asym(D)$, we have that the only possible symmetrical arc of $\mathcal{C}$ is (x, y). Hence it follows from the assumption of Theorem 2.1 that $\mathcal{C}$ has four short chords. But it follows from (3.2) and the fact that $[y, T_x, x]$ is a subpath of T_x that the only possible short chords of $\mathcal{C}$ are: (w_{n-1}, w_0) and (w_n, w_1), contradicting the assumption of Theorem 2.1. So the only possible case is:

Case 1.c $\ell\left([m_0, T_x, y] \cup T\right) \equiv 0 (\mathrm{mod}\, k)$ and $\ell\left([y, T_x, x] \cup [x, y]\right) \equiv 0 (\mathrm{mod}\, k)$. In this case we have that:

$$\ell\left([m_0, T_x, y] \cup T\right) + \ell\left([y, T_x, x] \cup [x, y]\right) \equiv 0 (\mathrm{mod}\, k)$$

$$\text{(i.e.)} \quad \ell\left(T_x \cup [x, y] \cup T\right) \equiv 0 (\mathrm{mod}\, k).$$

Hence $\ell\left(T_x \cup [x, y] \cup T\right) \equiv \ell\left([m_0, T_x, y] \cup T\right) (\mathrm{mod}\, k)$ and it follows that $\ell\left(T_x \cup [x, y]\right) \equiv \left([m_0, T_x, y]\right) (\mathrm{mod}\, k)$. Then

$$\ell\left([m_0, T_x, y]\right) \equiv \ell(T_x) + 1 \ (\mathrm{mod}\, k)$$

and we have from (3.1) that $\ell\left([m_0, T_x, y]\right) \equiv i+1 \bmod k$. Finally, notice that since T_x is the shortest directed path from m_0 to x contained in $Asym(D)$ and $[m_0, T_x, y]$ is a subpath of T_x we have that $[m_0, T_x, y]$ is a shortest directed path from m_0 to y contained in $Asym(D)$. We conclude from the definition of N_{i+1} that $y \in N_{i+1}$.

Case 2. $y \notin T_x$.
In this case we will prove that $\ell(T_y) \equiv i + 1(\bmod k)$. Again we will analyze several possible cases:

Case 2.a $\ell(T_y \cup T) \not\equiv 0(\bmod k)$.
It this case it follows from Lemma 2.1 (Taking $u = m_0, v = w = y$, $T_1 = T_y$, $T_2 = (v = w = y)$ and $T_3 = T$) that there exists a directed cycle $\mathcal{C}$ of length $\ell(\mathcal{C}) \not\equiv 0 \ (\bmod k)$, u', $v' = w' \in V(\mathcal{C})$ such that $[u', \mathcal{C}, v']$ is a subpath of T_y and $[v', \mathcal{C}, u']$ is a subpath of T. Since $T_y \subseteq Asym(D)$ and $T \subseteq Asym(D)$, we have that $\mathcal{C} \subseteq Asym(D)$. Hence it follows from the assumption of Theorem 2.1 that $\mathcal{C}$ has four short chords. But it follows from (3.3), (3.4) and by the facts:

$[u', \mathcal{C}, v']$ is a subpath of T_y and $[v', \mathcal{C}, u']$ is a subpath of T that if $\mathcal{C} = [u' = z_0, z_1, \ldots, z_{i-1}, z_i = v', z_{i+1}, \ldots, z_n, z_0]$, then the only possible short chords of $\mathcal{C}$ are (z_{i-1}, z_{i+1}) and (z_n, z_1), contradicting the hypothesis of Theorem 2.1.

Case 2.b $\ell\left(T_x \cup [x, y] \cup T\right) \not\equiv 0 \ (\bmod k)$.
It follows from Lemma 2.1 (Taking $u = m_0, v = x, w = y$, $T_1 = T_x$, $T_2 = (x, y)$ and $T_3 = T$) that there exists a directed cycle $\mathcal{C}$ of length $\ell(\mathcal{C}) \not\equiv 0(\bmod k)$ $u', v', w' \in V(\mathcal{C})$ such that $[u', \mathcal{C}, v']$ is a subpath of T_x, $[v', \mathcal{C}, w']$ is a subpath of $[x, y]$ (possibly $v' = w'$) and $[w', \mathcal{C}, u']$ is a subpath of T_3. Since $T_x \subseteq Asym(D)$ and $T \subseteq Asym(D)$, we have that the only possible symmetrical arc of $\mathcal{C}$ is (x, y). Hence it follows from the assumption of Theorem 2.1 that $\mathcal{C}$ has four short chords. However; if $\mathcal{C} = [u' = z_0, z_1, \ldots, z_{i-1}, z_i = v', z_{i+1} = w', z_{i+2}, \ldots, z_n, z_0]$, then it follows from (3.2), (3.4) and by the facts: $[u', \mathcal{C}, v']$ is a subpath of T_x, $[w', \mathcal{C}, u']$ is a subpath of T, that the only possible short chords of $\mathcal{C}$ are $(z_{i-1}, z_{i+1}), (z_i, z_{i+2})$ and (z_n, z_1), contradicting the assumption of Theorem 2.1.

We conclude from cases 2.a and 2.b that:

Case 2.c $\ell(T_y \cup T) \equiv 0(\bmod k)$ and $\ell\left(T_x \cup [x, y] \cup T\right) \equiv 0(\bmod k)$.
Hence
$$\ell(T_y \cup T) \equiv \ell(T_x \cup [x, y] \cup T) \ (\bmod k)$$

so

$$\ell(T_y) \equiv \ell\left(T_x \cup [x,y]\right) \equiv \ell(T_x) + 1 \ (\mathrm{mod}\, k)$$

and since $\ell(T_x) \equiv i(\mathrm{mod}\, k)$ we have $\ell(T_y) \equiv i+1(\mathrm{mod}\, k)$ and we conclude $y \in N_{i+1}$. Clearly it follows from (1), (2) and (3) that each N_i $(0 \leq i \leq k-1)$ is a k-kernel of D, and Theorem 2.1 is proved. ■

Remark 2.1. The assumption *Each directed triangle has at least two symmetrical arcs* is not needed for $k \neq 3$ (For $k \neq 3$, we have $3 \not\equiv 0(\mathrm{mod}\, k)$ and it follows from the other assumption that any directed cycle of length 3 has at least two symmetrical arcs). So we can state the following

Theorem 2.2. *Let D be a digraph such that $Asym(D)$ is strongly connected. If every directed cycle γ of D with $\ell(\gamma) \not\equiv 0(mod\, k)$, $k \geq 2$, $k \neq 3$ either (a) or (b) is satisfied where:*

(a) *γ has two symmetrical arcs,*

(b) *γ has four short chords,*

then D has a k-kernel $(k \geq 2, k \neq 3)$.

Remark 2.2. For $n = 2$ P. Duchet [2] has proved that if every directed cycle of odd length has at least two symmetrical arcs, then D has a kernel (2-kernel). Here the assumption that $Asym(D)$ is strongly connected is not necessary but for $k \geq 3$ we need the hypothesis $Asym(D)$ is strongly connected, as we can see in the following remark.

Remark 2.3 [4]**.** The hypothesis $Asym(D)$ is strongly connected in Theorem 2.1 and Theorem 2.2 cannot be changed by $Asym(D)$ is connected (for $k \geq 3$). For $k \geq 3$ consider the digraph H_k defined in [4] as follows:

$$\begin{aligned} V(H_k) &= \{0,1,2,\ldots,k^2+k+1\}, \\ A(H_k) &= \{(i,i+1) \quad i \in \{0,1,\ldots,k^2+k\} \cup (k^2+k+1,0)\} \\ &\qquad \cup \{(ik+2, ik+1), \quad i \in \{1,2,\ldots,k\}\,. \end{aligned}$$

And D_k is also defined in [4] as follows: For each $i \in V(H_k)$, let T_i^k an iz-directed path of length k such that $T_i^k \cap T_j^k = \{z\}$ $T_i^k \cap H_k = \{i\}$ and let $D_k = H_k \cup \bigcup_{i=0}^{k^2+k+1} T_i^k$. It is easy to see that: D_k does not have a k-kernel, $Asym(D)$ is a connected digraph and each directed cycle of length $\not\equiv 0(\mathrm{mod}\, k)$ has at least two symmetrical arcs.

Remark 2.4. For $k = 2$ P. Duchet [3] has proved the following result: *Let D be a digraph, if each directed triangle is symmetrical and each directed cycle of odd length has two short chords, then D has a kernel* (2-*kernel*). *He also conjectured: If each odd directed cycle has two short chords, then D has a kernel* (2-*kernel*). This question can be generalized as follows:

Question 2.1. *If each directed cycle of length $\not\equiv 0(mod\, k)$ has two short chords, then D has a k-kernel.*

Finally, we show some consequences of Theorem 2.1.

Corollary 2.1 [P. Duchet [2]]**.** *Let D be a digraph. If every odd directed cycle in D has at least two symmetrical arcs, then D has a kernel.*

Corollary 2.2 [H. Galeana [4]]**.** *Let D be a digraph such that $Asym(D)$ is strongly connected. If every directed cycle of length $\not\equiv 0(mod\, k)$ has at least two symmetrical arcs, then D has a k-kernel.*

Corollary 2.3 [M. Kwaśnik [5]]**.** *Let D be a strongly connected digraph. If every directed cycle of D has length $\equiv 0(mod\, k)$, $k \geq 2$, then D has a k-kernel.*

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