Discussiones Mathematicae
Graph Theory 18 (1998) 171–181

ON INDEPENDENT SETS AND NON-AUGMENTABLE PATHS IN DIRECTED GRAPHS

H. Galeana-Sánchez

Instituto de Matemáticas, UNAM
Circuito Exterior, Ciudad Universitaria
04510 México, D.F., México

Abstract

We investigate sufficient conditions, and in case that D be an asymmetrical digraph a necessary and sufficient condition for a digraph to have the following property: "In any induced subdigraph H of D, every maximal independent set meets every non-augmentable path". Also we obtain a necessary and sufficient condition for any orientation of a graph G results a digraph with the above property. The property studied in this paper is an instance of the property of a conjecture of J.M. Laborde, Ch. Payan and N.H. Huang: "Every digraph contains an independent set which meets every longest directed path" (1982).

Keywords: digraph, independent set, directed path, non-augmentable path.

1991 Mathematics Subject Classification: 05C20.

1. Introduction

For general concepts we refer the reader to [1]. Let D be a digraph; $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D respectively. If D_0 is a subdigraph (resp. induced subdigraph) of D, we write $D_0 \subset D$ (resp. $D_0 \subset^* D$). If $S_1, S_2 \subset V(D)$ the arc (u_1, u_2) of D will be called an S_1S_2-arc whenever $u_1 \in S_1$ and $u_2 \in S_2$; $D[S_1]$ will denote the subdigraph induced by S_1. The set $I \subset V(D)$ is independent if $A(D[I]) = \emptyset$.

An arc $(u_1, u_2) \in A(D)$ is called asymmetrical (resp. symmetrical) if $(u_2, u_1) \notin A(D)$ (resp. $(u_2, u_1) \in A(D)$). The asymmetrical part of D which is denoted by Asym (D) is the spanning subdigraph of D whose arcs are the asymmetrical arcs of D; D is called an asymmetrical digraph if Asym $(D) = D$.

A path $\mathcal{M} = (x_0, x_1, \ldots, x_k)$ will be always a directed elementary path (i.e. $\mathcal{M} = (x_0, x_1, \ldots, x_k)$ is a sequence of vertices of D, $x_i \neq x_j$ for any $i \neq j$, and $(x_i, x_{i+1}) \in A(D)$ for each i, $0 \leq i \leq k-1$). It is a longest path if k is maximum. For $H \subset^* D$ a path $\mathcal{M} \subset H$ will be called non-augmentable in H if for every vertex $a \in V(H)$, none of the sequences: $(a, x_0, x_1, \ldots, x_k)$, $(x_0, x_1, \ldots, x_i, a, x_{i+1}, \ldots, x_k)$ or $(x_0, x_1, \ldots, x_k, a)$ are paths. When $H = D$ we simply say that $\mathcal{M}$ is a non-augmentable path.

Let $G = (V(G), E(G))$ be a graph; an orientation $\vec{G}$ of G is a digraph obtained from G by orientation of each edge of G in at least one of the two possible directions.

The problem considered in this paper is: for which digraphs do we have $\mathcal{M} \cap S \neq \emptyset$ for any maximal independent set and for every non-augmentable path $\mathcal{M}$? This problem is an instance of a conjecture of J.M. Laborde, Ch. Payan and N.H. Huang [4] "Every digraph contains an independent set which meets every longest directed path" (1982).

It is not true that in any digraph every maximal indepedent set meets every non-augmentable path. Consider for example the digraph with a vertex set $\{a, b, c, d\}$ and arc set $\{(a, b), (c, b), (c, d)\}$.

When the vertices of D are elements of a poset and the arcs of D represents the partial order, we have a result due to Grillet [3], who proved that if every induced subdigraph isomorphic to $P = (V(P), A(P))$, $V(P) = \{a, b, c, d\}$, $A(P) = \{(a, b), (c, b), (c, d)\}$ is contained in an induced subdigraph isomorphic to $Q = (V(Q), A(Q))$, $V(Q) = \{a, b, c, d, e\}$, $A(Q) = \{(a, b), (c, b), (c, d), (c, e), (e, b)\}$ then every maximal independent set meets every non-augmentable path.

When D is an asymmetrical digraph we have the following result due to H. Galeana-Sánchez and H.A. Rincón-Mejía [2], they proved that if D is an asymmetrical digraph with no subdigraph isomorphic to $P = (V(P), A(P))$, $V(P) = \{a, b, c, d\}$, $A(P) = \{(a, b), (c, b), (c, d)\}$, and no subgraph isomorphic to $Q = (V(Q), A(Q))$, $V(Q) = \{a, b, c, d\}$, $A(Q) = \{(a, b), (c, b), (c, d), (b, d)\}$. Then any maximal independent set meets every non-augmentable path.

2. Independent Sets and Non-Augmentable Paths

In this section, sufficient conditions for any maximal independent set to meet every non-augmentable path are studied.

Definition 1. For each $m \in \mathbb{N}$ let $X_m = \{x_0, x_1, \ldots, x_m\}$ and $Y = \{y_0, y_1\}$ be two disjoint sets of cardinality $m+1$ and 2, respectively. We will denote by D_m the digraph defined as follows:

$$V(D_m) = X_m \cup Y,$$

$A(D_m) = \{(x_i, x_{i+1}) \mid 0 \le i \le m-1\} \cup \{(x_i, y_0) \mid 0 \le i \le m-1\} \cup \{(y_1, x_i) \mid 1 \le i \le m\}$. See Figure 1.

Theorem 1. *Let D be a digraph such that for each i $(1 \le i \le m)$, $D_i \subset D$ implies*

$$A(D) \cap (\{(y_0, x_j) \mid 1 \le j \le i\} \cup \{(x_j, y_1) \mid 0 \le j \le i\}$$

$$\cup \{(y_1, x_0), (x_i, y_0), (y_0, y_1), (y_1, y_0)\}) \neq \emptyset.$$

Then any maximal independent set of D meets every non-augmentable path of length at most m.

Proof. We proceed by contradiction. Suppose that D satisfies the hypothesis but there exists a maximal independent set S and a non-augmentable path $T = (x_0, x_1, \ldots, x_n)$ of length n, $n \le m$ such that $S \cap T = \emptyset$. Since S is a maximal independent set, T non-augmentable and $S \cap T = \emptyset$, we have that there exists $y \in S$ such that $(x_0, y) \in A(D)$ and $(y, x_0) \notin A(D)$.

Notice that since T is non-augmentable and $S \cap T = \emptyset$; there is no $\{x_n\}S$-arc in D. So we can define: $p = \min\{t \in \{0, 1, \ldots, n\} \mid$ there is no $\{x_t\}S$-arc in $D\}$. Observe that the above observation implies $p \ge 1$. Moreover, since S is a maximal independent set, the definition of p implies that: There exists $y_1 \in S$ such that $(y_1, x_p) \in$ Asym (D).

We will get a contradiction from the following assertion:

(I) For each j $(0 \le j \le p)$, $(y_1, x_j) \in A(D)$.
(In particular, $(y_1, x_0) \in A(D)$ contradicting that T is non-augmentable).

In order to prove (I) we proceed again by contradiction; suppose that there exists t, $(0 \le t \le p)$ such that $(y_1, x_t) \notin A(D)$ and let $k = \max\{t \in \{0, 1, \ldots, p\} \mid (y_1, x_t) \notin A(D)\}$. Clearly, $k < p$ (because $(y_1, x_p) \in$ Asym (D)).

Since $k < p$ the definition of p implies that there exists $y_0 \in S$ such that $(x_k, y_0) \in A(D)$.

(I.1) For any j, $(k \le j \le p-1)$, $(x_j, y_0) \in A(D)$.
To prove proposition (I.1) we proceed by contradiction. Suppose that there exists t $(k \le t \le p-1)$ suth that $(x_t, y_0) \notin A(D)$ and let $\ell = \min\{t \in \{k, k+1, \ldots, p-1\} \mid (x_t, y_0) \notin A(D)\}$ be.

(I.1.a) For each j, $(k \le j \le \ell - 1)$; $(x_j, y_0) \in A(D)$.
It is a direct consequence of the definition of ℓ.

(I.1.b) For each j, $(k+1 \le j \le \ell)$; $(y_1, x_j) \in A(D)$.
It follows directly from the definition of k.

(I.1.c) $D_{\ell-k} \subset D[\{x_k, x_{k+1}, \ldots, x_\ell\} \cup \{y_0, y_1\}] \subset^* D$.
It is a consequence of Definition 1, (I.1.a) and (I.1.b).

The hypothesis of Theorem 1 and (I.1.c) imply

$$\begin{aligned} A(D) \quad \cap \quad & (\{(y_0, x_j) \mid k+1 \le j \le \ell\} \cup \{(x_j, y_1) \mid k \le j \le \ell\} \\ & \cup \{(y_1, x_k), (x_\ell, y_0), (y_0, y_1), (y_1, y_0)\}) \ne \emptyset. \end{aligned}$$

If $A(D) \cap \{(y_0, x_j) \mid k+1 \le j \le \ell\} \ne \emptyset$ we take t, $k+1 \le t \le \ell$ such that $(y_0, x_t) \in A(D)$. Then $k \le t-1 \le \ell - 1$ and (I.1.a) implies $(x_{t-1}, y_0) \in A(D)$. So we have $\{(x_{t-1}, y_0), (y_0, x_t)\} \subseteq A(D)$ and hence the succession $(x_0, x_1, \ldots, x_{t-1}, y_0, x_t, \ldots, x_n)$ is a path. A contradiction (because T is non-augmentable).

If $A(D) \cap \{(x_j, y_1) \mid k \le j \le \ell\} \ne \emptyset$ then, consider t, $k \le t \le \ell$ such that $(x_t, y_1) \in A(D)$; we have $k+1 \le t+1 \le \ell+1 \le p$ and the definition of k implies $(y_1, x_{t+1}) \in A(D)$. So we have $\{(x_t, y_1), (y_1, x_{t+1})\} \subseteq A(D)$ and hence $(x_0, \ldots, x_t, y_1, x_{t+1}, \ldots, x_n)$ is a path. A contradiction.

If $A(D) \cap \{(y_1, x_k), (x_\ell, y_0), (y_0, y_1), (y_1, y_0)\} \ne \emptyset$, then $A(D) \cap \{(y_1, x_k), (x_\ell, y_0)\} \ne \emptyset$ (because $\{y_0, y_1\} \subseteq S$ and S is an independent set). Now, notice that the definition of ℓ implies $(x_\ell, y_0) \notin A(D)$; and the definition of k implies $(y_1, x_k) \notin A(D)$. So Proposition (I.1) is proved.

(I.2) For each j, $(k+1 \le j \le p)$, $(y_1, x_j) \in A(D)$.
It follows directly from the definition of y_1 and the definition of k.

(I.3) $D_{p-k} \subset D[\{x_k, x_{k+1}, \ldots, x_p\} \cup \{y_0, y_1\}] \subset^* D$.
It is a direct consequence of (I.1) and (I.2).

Now (I.3) and the hypothesis of Theorem 1 imply

$$\begin{aligned} A(D) \quad \cap \quad & (\{(y_0, x_j) \mid k+1 \le j \le p\} \cup \{(x_j, y_1) \mid k \le j \le p\} \\ & \cup \{(y_1, x_k), (x_p, y_0), (y_0, y_1), (y_1, y_0)\}) \ne \emptyset. \end{aligned}$$

If $A(D) \cap \{(y_0, x_j) \mid k+1 \le j \le p\} \ne \emptyset$, then we take t, $k+1 \le t \le p$ such that $(y_0, x_t) \in A(D)$ and we have $k \le t-1 \le p-1$. Proposition (I.1) implies $(x_{t-1}, y_0) \in A(D)$, so $\{(x_{t-1}, y_0), (y_0, x_t)\} \subseteq A(D)$ and the succession $(x_0, x_1, \ldots, x_{t-1}, y_0, x_t, \ldots, x_n)$ is a path. A contradiction (because T is non-augmentable).

If $A(D) \cap \{(x_j, y_1) \mid k \leq j \leq p\} \neq \emptyset$ then, taking t, $k \leq t \leq p$ such that $(x_t, y_1) \in A(D)$ we have that $t \leq p-1$, (recall the definition of p) hence $k+1 \leq t+1 \leq p$ and (I.2) implies $(y_1, x_{t+1}) \in A(D)$. We conclude $\{(x_t, y_1), (y_1, x_{t+1})\} \subseteq A(D)$ and the succession $(x_0, \ldots, x_t, y_1, x_{t+1}, \ldots, x_n)$ is a path. A contradiction.

If $A(D) \cap \{(y_1, x_k), (x_p, y_0), (y_0, y_1), (y_1, y_0)\} \neq \emptyset$ then $A(D) \cap \{(y_1, x_k), (x_p, y_0)\} \neq \emptyset$ because $\{y_0, y_1\} \subseteq S$ and S is an independent set. Notice that the definition of k implies $(y_1, x_k) \notin A(D)$ and the definition of y_0 and p imply $(x_p, y_0) \notin A(D)$. ■

Corollary 1. *Let D be a digraph such that for each i, $(1 \leq i \leq m)$ $D_i \subset D$ implies*

$$A(D) \cap (\{(y_0, x_j) \mid 1 \leq j \leq i\} \cup \{(x_j, y_1) \mid 0 \leq j \leq i\}$$

$$\cup \{(y_1, x_0), (x_i, y_0), (y_0, y_1), (y_1, y_0)\}) \neq \emptyset,$$

and H an induced subdigraph of D. Then any maximal independent set of H meets every non-augmentable in H path of H whose length is at most m.

Corollary 2. *Let D be a digraph such that for each natural number i, $D_i \subset D$ implies*

$$A(D) \cap (\{(y_0, x_j) \mid 1 \leq j \leq i\} \cup \{(x_j, y_1) \mid 0 \leq j \leq i\}$$

$$\cup \{(y_1, x_0), (x_i, y_0), (y_0, y_1), (y_1, y_0)\}) \neq \emptyset$$

Then for any induced subdigraph H of D, every maximal independent set of H meets any non-augmentable in H path of H.

Theorem 2. *Let D be a digraph such that for each i $(1 \leq i \leq m)$ $D_i \subset D$ implies*

$$A(D) \cap (\{(y_0, x_j) \mid 0 \leq j \leq i\} \cup \{(x_j, y_1) \mid 0 \leq j \leq i-1\}$$
$$\cup \{(y_1, x_0), (x_i, y_0), (y_0, y_1), (y_1, y_0)\}) \neq \emptyset.$$

Then any maximal independent set meets every non-augmentable path of length at most m.

Proof. We proceed by contradiction. Suppose that D satisfies the hypothesis but there exists a maximal independent set S and a non-augmentable path $T = (x_0, x_1, \ldots, x_n)$ of length n, $n \leq m$ such that $S \cap T = \emptyset$. Since S

is a maximal independent set, T non-augmentable and $S \cap T = \emptyset$ we have that there exists $y \in S$ such that $(y, x_n) \in A(D)$ and $(x_n, y) \notin A(D)$.

Notice that, since G is non-augmentable and $S \cap T = \emptyset$; there is no $S\{x_0\}$-arc in D. So we can define $p = \max\{i \in \{0, 1, \ldots, n\} \mid$ there is no $S\{x_i\}$-arc in $D\}$, the observation of above implies $p \leq n - 1$. Moreover, since S is a maximal independent set, the definition of x_p implies that there exists $y_o \in S$ such that $(x_p, y_0) \in$ Asym (D).

We will get a contradiction from the following assertion:

(I) For each j $(p \leq j \leq n)$, $(x_j, y_0) \in A(D)$.
(In particular, $(x_n, y_0) \in A(D)$ and then the succession $(x_0, x_1, \ldots, x_n, y_0)$ is path contradicting that T is non-augmentable).

In order to prove (I) we proceed again by contradiction; suppose that there exists j, $(p \leq j \leq n)$ such that $(x_j, y_0) \notin A(D)$ and let $k = \min\{j \in \{p, \ldots, n\} \mid (x_j, y_0) \notin A(D)\}$. Clearly, $k > p$ because $(x_p, y_0) \in$ Asym (D). Since $k > p$ the definition of p implies that there exists $y_1 \in S$ such that $(y_1, x_k) \in A(D)$.

(I.1) For each j, $(p + 1 \leq j \leq k)$, $(y_1, x_j) \in A(D)$.
We proceed by contradiction to prove Proposition (I.1). Suppose that there exists t $(p + 1 \leq t \leq k)$ suth that $(y_1, x_t) \notin A(D)$ and let $\ell = \max\{t \in \{p + 1, \ldots, k\} \mid (y_1, x_t) \notin A(D)\}$ be.

(I.1.a) For each j, $(\ell + 1 \leq j \leq k)$, $(y_1, x_j) \in A(D)$.
It is a direct consequence of the definition of ℓ.

(I.1.b) For each j, $(\ell \leq j \leq k - 1)$, $(x_j, y_0) \in A(D)$.
It follows directly from the definition of k.

(I.1.c) $D_{k-\ell} \subset D[\{x_\ell, \ldots, x_k\} \cup \{y_0, y_1\}] \subset^* D$.
It is a consequence of (I.1.a) and (I.1.b). The hypothesis of Theorem 2 and (I.1.c) imply

$$\begin{aligned} A(D) \quad \cap \quad & (\{(y_0, x_j) \mid \ell \leq j \leq k\} \cup \{(x_j, y_1) \mid \ell \leq j \leq k - 1\} \\ & \cup \{(y_1, x_\ell), (x_k, y_0), (y_0, y_1), (y_1, y_0)\}) \neq \emptyset. \end{aligned}$$

If $A(D) \cap \{(y_0, x_j) \mid \ell \leq j \leq k\} \neq \emptyset$, we take t, $\ell \leq t \leq k$ such that $(y_0, x_t) \in A(D)$. The definition of ℓ implies $\ell - 1 \geq p$, hence $p \leq \ell - 1 \leq t - 1 \leq k - 1$, and the definition of k implies $(x_{t-1}, y_0) \in A(D)$. So, we have $\{(x_{t-1}, y_0), (y_0, x_t)\} \subseteq A(D)$ and then the succession $(x_0, x_1, \ldots, x_{t-1}, y_0, x_t, \ldots, x_n)$ is a path. A contradiction (because T is non-augmentable).

If $A(D) \cap \{(x_j, y_1) \mid \ell \leq j \leq k-1\} \neq \emptyset$, let t, $\ell \leq t \leq k-1$ be such that $(x_t, y_1) \in A(D)$. Then $\ell + 1 \leq t + 1 \leq k$ and the definition of ℓ implies $(y_1, x_{t+1}) \in A(D)$. Hence we have $\{(x_t, y_1), (y_1, x_{t+1})\} \subseteq A(D)$ and the succession $(x_0, \ldots, x_t, y_1, x_{t+1}, \ldots, x_n)$ is a path. A contradiction.

If $A(D) \cap \{(y_1, x_\ell), (x_k, y_0), (y_0, y_1), (y_1, y_0)\} \neq \emptyset$, then $A(D) \cap \{(y_1, x_\ell), (x_k, y_0)\} \neq \emptyset$ because $\{y_0, y_1\} \subseteq S$ and S is an independent set. But the definition of ℓ implies $(y_1, x_\ell) \notin A(D)$ and the definition of k implies $(x_k, y_0) \notin A(D)$. So Proposition (I.1) is proved.

(I.2) For each j, $(p \leq j \leq k-1)$, $(x_j, y_0) \in A(D)$.
It follows directly from the definition of k.

(I.3) $D_{k-p} \subset D[\{x_p, x_{p+1}, \ldots, x_k\} \cup \{y_0, y_1\}] \subset^* D$.
It is a direct consequence of (I.1) and (I.2).

Now (I.3) and the hypothesis of Theorem 2 imply

$$\begin{aligned} A(D) \quad \cap \quad & (\{(y_0, x_j) \mid p \leq j \leq k\} \cup \{(x_j, y_1) \mid p \leq j \leq k-1\} \\ & \cup \{(y_1, x_p), (x_k, y_0), (y_0, y_1), (y_1, y_0)\}) \neq \emptyset. \end{aligned}$$

If $A(D) \cap \{(y_0, x_j) \mid p \leq j \leq k\} \neq \emptyset$, then we take t, $p \leq t \leq k$ such that $(y_0, x_t) \in A(D)$. The definition of p, and the fact $y_0 \in S$ imply $t \neq p$, so $p + 1 \leq t \leq k$ and $p \leq t - 1 \leq k - 1$. Now it follows from (I.2) that $(x_{t-1}, y_0) \in A(D)$. Hence $\{(x_{t-1}, y_0), (y_0, x_t)\} \subseteq A(D)$ and so the succession $(x_0, x_1, \ldots, x_{t-1}, y_0, x_t, \ldots, x_n)$ is a path. A contradiction.

If $A(D) \cap \{(x_j, y_1) \mid p \leq j \leq k-1\} \neq \emptyset$, then there exists t, $p \leq t \leq k-1$ such that $(x_t, y_1) \in A(D)$. Since $p + 1 \leq t + 1 \leq k$, it follows from (I.1) that $(y_1, x_{t+1}) \in A(D)$. Hence $\{(x_t, y_1), (y_1, x_{t+1})\} \subseteq A(D)$ and the succession $(x_0, x_1, \ldots, x_t, y_1, x_{t+1}, \ldots, x_n)$ is a path. A contradiction.

If $A(D) \cap \{(y_1, x_p), (x_k, y_0), (y_0, y_1), (y_1, y_0)\} \neq \emptyset$ then $A(D) \cap \{(y_1, x_p), (x_k, y_0)\} \neq \emptyset$ because $\{y_0, y_1\} \subseteq S$ and S is an independent set. But the definition of p (and the fact $y_1 \in S$) implies $(y_1, x_p) \notin A(D)$ and the definition of k implies that $(x_k, y_0) \notin A(D)$. So Proposition (I) is proved. ■

Corollary 3. *Let D be a digraph such that for each i, $(1, \leq i \leq m)$, $D_i \subset D$ implies*

$$\begin{aligned} A(D) \quad \cap \quad & (\{(y_0, x_j) \mid 0 \leq j \leq i\} \cup \{(x_j, y_1) \mid 0 \leq j \leq i-1\} \\ & \cup \{(y_1, x_0), (x_i, y_0), (y_0, y_1), (y_1, y_0)\}) \neq \emptyset, \end{aligned}$$

and H an induced subdigraph of D. Then any maximal independent set of H meets every non-augmentable in H path of H whose length is at most m.

Corollary 4. *Let D be a digraph such that for each natural number i, $D_i \subset D$ implies*

$$A(D) \quad \cap \quad (\{(y_0, x_j) \mid 0 \leq j \leq i\} \cup \{(x_j, y_1) \mid 0 \leq j \leq i-1\} \cup \{(y_1, x_0), (x_i, y_0), (y_0, y_1), (y_1, y_0)\}) \neq \emptyset.$$

Then for any induced subdigraph H of D, every maximal independent set of H meets every non-augmentable in H path of H.

Theorem 3. *Let D be an asymmetrical digraph. The two following statements are equivalent:*

(i) *For each i $(1 \leq i \leq m)$ $D_i \subset D$ implies*

$$A(D) \cap (\{(x_0, y_1), (y_1, x_0), (y_0, x_i), (x_i, y_0), (y_0, y_1), (y_1, y_0)\}) \neq \emptyset\,.$$

(ii) *For any induced subdigraph $H \subseteq^* D$ it holds that every maximal independent set of H meets each non-augmentable in H path of length at most m.*

Proof. It follows directly from Theorem 1 that (i) implies (ii). Now suppose (ii) holds and let $i \in \{1, \ldots, m\}$ such that $D_i \subset D$; denote $H = D[V(D_i)]$. Suppose by contradiction that $A(D) \cap \{(x_0, y_1), (y_1, x_0), (y_0, x_i), (x_i, y_0), (y_0, y_1), (y_1, y_0)\} \neq \emptyset$. Then $A(H) \cap \{(x_0, y_1), (y_1, x_0), (y_0, x_i), (x_i, y_0), (y_0, y_1), (y_1, y_0)\} \neq \emptyset$. So $T = (x_0, x_1, \ldots, x_i)$ is a non-augmentable in H path of length $i \leq m$, and $S = \{y_0, y_1\}$ is a maximal independent set in H such that $S \cap T = \emptyset$ contradicting our assumption (ii). ■

If β is a class of graphs, a graph G is said to be a β-free graph whenever G has no induced subgraph isomorphic to a member of β. In what follows, we will denote by $\mathcal{F}$ the set $\mathcal{F} = \{F_1, F_2\}$ where F_1, F_2 are the graphs of Figure 2.

Theorem 4. *Let G be a graph. The following statements are equivalent:*

(i) *G is an $\mathcal{F}$-free graph.*

(ii) *For any orientation $\vec{G}$ of G and any induced subdigraph $H \subseteq^* \vec{G}$ of $\vec{G}$; if T_H is a non-augmentable in H path and I_H is a maximal independent set of H, then $T_H \cap I_H \neq \emptyset$.*

Proof. First let G be an $\mathcal{F}$-free graph and $\vec{G}$ any orientation of G. We will prove the following assertion:

(a) If $D_i \subset D$, then $A(D) \cap \{(x_0, y_1), (y_1, x_0), (x_i, y_0), (y_0, x_i), (y_0, y_1), (y_1, y_0)\} \neq \emptyset$ for any natural number i.

We consider two possible cases:

Case 1. If $(x_0, x_1, \ldots, x_i)$ is an induced subdigraph of $\vec{G}$.
In this case we have $i \in \{1, 2\}$ because if $i \geq 3$ then $G[\{x_0, x_1, x_2, x_3\}]$ is an induced subgraph of G isomorphic to F_1, contradicting that G is $\mathcal{F}$-free.

When $i = 1$, we have $D_1 \subset \vec{G}$ and hence $F_1 \subset G$ (notice that the underlying graph of D_1 is isomorphic to F_1). Since G has no induced subgraph isomorphic to F_1 we have $A(D) \cap \{(x_0, y_1), (y_1, x_0), (y_0, x_1), (x_1, y_0), (y_0, y_1), (y_1, y_0)\} \neq \emptyset$ and (a) holds.

When $i = 2$, we have $D_2 \subset \vec{G}$ and hence $F_2 \subset G$ (notice that the underlying graph of D_2 is isomorphic to F_2). Since G has no induced subgraph isomorphic to F_2, we have $A(D) \cap \{(x_0, y_1), (y_1, x_0), (y_0, x_2), (x_2, y_0), (y_0, y_1), (y_1, y_0)\} \neq \emptyset$ and (a) holds.

Case 2. If $(x_0, \ldots, x_i)$ is not an induced subdigraph of $\vec{G}$. (i.e. there exists r, s $\{r, s\} \subseteq \{0, \ldots, i\}$ $|r - s| \geq 2$ such that $\{(x_r, x_s), (x_s, x_r)\} \cap A(D) \neq \emptyset$).

Let $j, k \in \{0, \ldots, i\}$ such that $k - j = \max\{r - s \mid s < r, \{(x_r, x_s), (x_s, x_r)\} \cap A(D) \neq \emptyset\}$; the choice of k and j implies that the undirected path $(x_0, \ldots, x_j, x_k, x_{k+1}, \ldots, x_i)$ is an induced subgraph of G. Since G has no induced subgraph isomorphic to F_1 (notice that F_1 is the undirected path of length 3), we have that the length of the undirected path $(x_0, x_1, \ldots, x_j, x_k, x_{k+1}, \ldots, x_i)$ is one or two. We will analyze the two cases:

Case 2.1. The length of $(x_0, \ldots, x_j, x_k, \ldots, x_i)$ is one.
In this case $j = 0$, $k = i = 1$ and the underlying graph of $D[\{y_0, x_0, x_1, y_1\}]$ is isomorphic to F_1. Now since G has no induced subgraph isomorphic to F_1, we conclude that $A(D) \cap \{(x_0, y_1), (y_1, x_0), (x_i, y_1), (y_1, x_i), (y_1, y_0), (y_0, y_1)\} \neq \emptyset$.

Case 2.2. The length of $(x_0, \ldots, x_j, x_k, \ldots, x_i)$ is two.
In this case $j = 0$, $k = i - 1$ or $j = 1$ and $k = i$; in any case the underlying graph of $D[\{x_0, \ldots, x_j, x_k, \ldots, x_i\} \cup \{y_0, y_1\}]$ is isomorphic to F_2. The choice of j and k, and the fact that G has no induced subgraph isomorphic to F_2 imply that $A(D) \cap \{(x_0, y_1), (y_1, x_0), (x_i, y_0), (y_0, x_i), (y_0, y_1), (y_1, y_0)\} \neq \emptyset$. So Proposition (a) is proved. Hence it follows from Corollary 1 that any maximal independent set of H meets every non-augmentable in H path of H. We conclude (i) implies (ii).

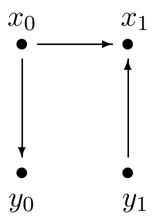
Now let G be a graph satisfying property (ii). If G contains an induced subgraph isomorphic to F_1, say $V(F_1) = \{y_0, x_0, x_1, y_1\}$, $E(F_1) = \{y_0x_0, x_0x_1, x_1y_1\}$. Then considereing the orientation $\vec{G}$ of G (where $V(\vec{G}) = V(G)$ and $A(\vec{G}) = \{(x_0, y_0), (x_0, x_1), (y_1, x_1)\} \cup \{(y, z), (z, y) \mid |\{y, z\} \cap V(F_1)| \leq 1\}$), we have: $H = \vec{G}[\{y_0, x_0, x_1, y_1\}]$ is an induced subdigraph of $\vec{G}$; $T_H = (x_0, x_1)$ is a non-augmentable in H path of H and $I_H = \{y_0, y_1\}$ is a maximal independent set of H such that $T_H \cap I_H = \emptyset$ contradicting the assertion (ii).

If G contains an induced subgraph isomorphic to F_2, say $F_2 = G[\{y_0, x_0, x_1, x_2, y_1\}]$, $V(F_2) = \{y_0, x_0, x_1, x_2, y_1\}$, $E(F_2) = \{y_0x_0, y_0x_1, y_1x_1, y_1x_2, x_0x_1, x_1x_2\}$. Then considering the orientation $\vec{G}$ of G where $V(\vec{G}) = V(G)$

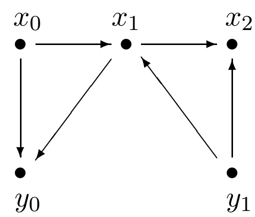
$$A(\vec{G}) = \{(y_0, x_0), (y_0, x_1), (y_1, x_1), (y_1, x_2), (x_0, x_1), (x_1, x_2)\} \cup \{(y, z), (z, y) \mid |\{y, z\} \cap V(F_2)| \leq 1\}$$

we have: $H = \vec{G}[\{x_0, x_1, x_2, y_0, y_1\}]$ is an induced subdigraph of $\vec{G}$, $T_H = (x_0, x_1, x_2)$ is a non-augmentable in H path of H and $I_H = \{y_0, y_1\}$ is a maximal independent set of H such that $T_H \cap I_H = \emptyset$ contradicting (ii). ■

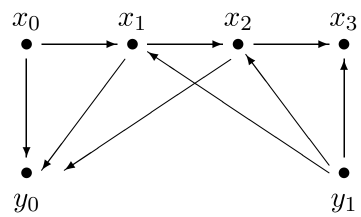
Observation 1. *Notice that D_i contains no induced subdigraph isomorphic to D_j, for each j, $1 \leq j < i$; and D_i is a digraph with a non-augmentable in D_i path namely $T = (x_0, \ldots, x_i)$ and a maximal independent set $\alpha = \{y_0, y_1\}$ such that $T \cap \alpha = \emptyset$.*



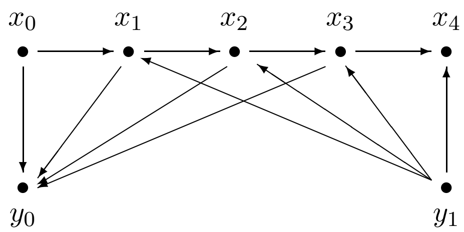
D_1



D_2



D_3



D_4

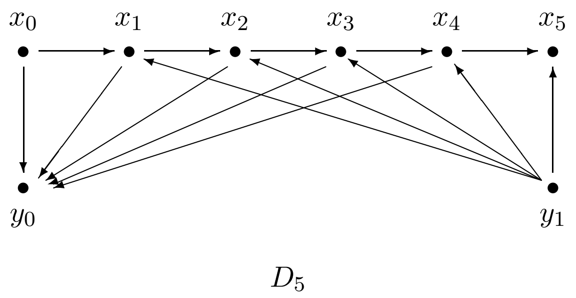


Figure 1

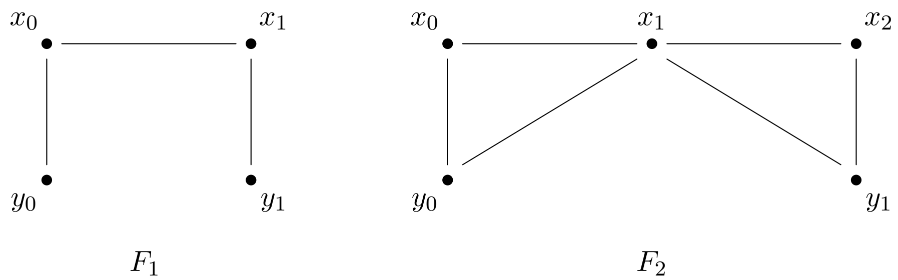


Figure 2

References

[1] C. Berge, Graphs (North-Holland, 1985).

[2] H. Galeana-Sánchez and H.A. Rincón-Mejía, *Independent sets which meet all longest paths*, Discrete Math. **152** (1996) 141–145.

[3] P.A. Grillet, *Maximal chains and antichains*, Fund. Math. **65** (1969) 157–167.

[4] J.M. Laborde, C. Payan and N.H. Huang, *Independent sets and longest directed paths in digraphs*, in: Graphs and Other Combinatorial Topics. Proceedings of the Third Czechoslovak Symposium of Graph Theory (1982) 173–177.

Received 16 January 1998
Revised 5 June 1998