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ON THE SIMPLEX GRAPH OPERATOR

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Abstract

A simplex of a graph G is a subgraph of G which is a complete graph. The simplex graph $\mathrm{Simp}(G)$ of G is the graph whose vertex set is the set of all simplices of G and in which two vertices are adjacent if and only if they have a non-empty intersection. The simplex graph operator is the operator which to every graph G assigns its simplex graph $\mathrm{Simp}(G)$. The paper studies graphs which are fixed in this operator and gives a partial answer to a problem suggested by E. Prisner.

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In [1], page 131, E. Prisner posed the problem whether there are infinite Simp-periodic graphs other than those consisting of isolated vertices. This paper is a contribution to that problem. We consider undirected graphs without loops and multiple edges.

A simplex in a graph G is a subgraph of G which is a complete graph. (It need not be maximal, hence this concept is broader than that of a clique). If a simplex has k vertices, it is called a k-simplex. Also a 1-simplex is considered; it consists of one vertex. The simplex graph $\mathrm{Simp}(G)$ of G is the graph whose vertex set is the set of all simplices of G and in which two vertices are adjacent if and only if they have a non-empty intersection (as simplices). The simplex graph operator is the operator which assigns to every graph G its simplex graph $\mathrm{Simp}(G)$. A graph G is said to be Simp-fixed, if it is a fixpoint of the simplex graph operator, i.e. if $\mathrm{Simp}(G) \cong G$.

The graph G is said to be Simp-periodic, if it is a fixpoint of some iteration of the simplex graph operator.

The mentioned problem from [1] concerns Simp-periodic graphs, but we shall treat only Simp-fixed graphs. Obviously every graph consisting of isolated vertices (regular graph of degree 0) is Simp-fixed and also the empty graph (in which both the vertex set and the edge set are empty) is Simp-fixed. No other finite graph is Simp-fixed. Namely, the set of simplices of G includes all 1-simplices and their number is equal to the number of vertices of G. If G has at least one edge, it has, moreover, k-simplices for $k \geq 2$ and thus the vertex set of $\mathrm{Simp}(G)$ has more elements than the vertex set of G and $\mathrm{Simp}(G)$ cannot be isomorphic to G. Therefore there is a question whether an infinite graph exists which is Simp-fixed and has at least one edge.

The first theorem will have a preparatory character.

Theorem 1. *Let G be a Simp-fixed graph. Then no vertex of G has a finite degree greater than one.*

Proof. Suppose the contrary. Let r be the least integer greater than one such that G contains a vertex of degree r. Let a vertex v_0 have the degree r. By $v_1, \ldots, v_r$ we denote the vertices adjacent to v_0. As G is Simp-fixed, there exist simplices $S_0, S_1, \ldots, S_r$ in G to which the vertices $v_0, v_1, \ldots, v_r$ correspond; the simplex S_0 has non-empty intersections with all the simplices $S_1, \ldots, S_r$. Suppose that S_0 is a k-simplex for $k \geq 2$. Then it contains two distinct vertices w_1, w_2. If both w_1, w_2 have degree 1, then S_0 is a 2-simplex forming a connected component of G. Then G must contain a connected component whose image in the operator Simp is S_0; but a 2-simplex is not a simplex graph for any graph. Therefore at least one of the vertices w_1, w_2, say w_1, has degree greater than one. As r is the minimum of such degrees, the degree of w_1 is at least r. The vertex w_1 is incident to at least $r-1$ edges distinct from w_1w_2; these edges with their end vertices form 2-simplices having a non-empty intersection with S_0. Further such simplices are 1-simplices consisting of w_1 and consisting of w_2. There are at least $r+1$ simplices having non-empty intersections with S_0 and thus the degree of v_0 is at least $r+1$, which is a contradiction. We have proved that S_0 is a 1-simplex; let it consist of a vertex S_0. The vertex S_0 cannot have degree 0 or 1, because so would have also v_0. Therefore the degree of S_0 is at least r. Each edge incident with S_0 forms a 2- simplex. As the degree of v_0 is r, no k-simplices for $k \geq 3$ containing S_0 exist; the neighbours of S_0 form an independent set and their number, i.e. the degree of S_0, is exactly r. Any two of the mentioned 2-simplices have a common vertex S_0 and thus the neighbours of v_0 form an r-simplex. We have proved that a vertex of G with degree r has the property that its neighbours form an r-simplex. But then this must hold for S_0, too, which is a contradiction. This proves the assertion. ■

At considerations concerning infinite cardinal numbers we shall suppose the validity of Axiom of Choice and the existence of well-ordering of cardinal numbers which follows from it. As usual, by $\aleph_0$ we denote the cardinality of the set of positive integers, by $\aleph_{\alpha+1}$ for a positive integer α we denote the cardinal number immediately following after $\aleph_\alpha$. By $\aleph_\omega$ we denote the least cardinal number which is greater then $\aleph_\alpha$ for every non-negative integer α. It is well-known $\aleph_\omega = \sum_{\alpha<\omega} \aleph_\alpha$.

Theorem 2. *Any graph G which contains at least one edge and whose vertex set has cardinality less that $\aleph_\omega$ is not Simp-fixed.*

Proof. Suppose that there exists a Simp-fixed graph G having at least one edge. Then G contains vertices of non-zero degrees. If the maximum degree of a vertex of G is 1, then G contains at least one connected component which is a 2-simplex. The existence of such connected component was excluded in the proof of Theorem 1. Therefore G contains at least one vertex of degree grather than 1. According to Theorem 1 such a degree cannot be finite. Thus G contains a vertex v_0 of infinite degree r. If $r \geq \aleph_\omega$, then also $|V(G)| \geq \aleph_\omega$ and the assertion is true. Thus suppose $r = \aleph_\alpha$ for some non-negative integer α. The edges incident with v_0 together with their end vertices form 2-simplices. Any two of these 2-simplices have a common vertex v_0 and, as G is Simp-fixed, vertices of an $\aleph_\alpha$-simplex S_1 in G correspond to them. Choose a vertex v_1 in S_1 and consider all simplices which are subgraphs of S and contain v_1. Their number is $\exp \aleph_\alpha$ and any two of them have a common vertex v_1; therefore vertices of an $(\exp \aleph_\alpha)$-simplex S_2 in G correspond to them. We can proceed further, constructing always S_{n+1} from S_n. We have $|V(S_2)| = \exp \aleph_\alpha \geq \aleph_{\alpha+1}$, $|V(S_3)| = \exp \aleph_\alpha \geq \exp \aleph_{\alpha+1} \geq \aleph_{\alpha+2}$ etc., in general $|V(S_n)| \geq \aleph_{\alpha+n+1}$. Therefore the vertex set of G contains subsets of all cardinalities which are less that $\aleph_\omega$ and hence its cardinality is at least $\aleph_\omega$. ■

A further theorem concerns a more general question.

Theorem 3. *Let G be a connected graph such that the cardinalities of $V(G)$ and of $V(\mathrm{Simp}(\mathrm{Simp}(G)))$ are equal. Then the cardinality of $V(G)$ is 0, 1 or a limit cardinal number.*

Proof. At the beginning of this paper we have written that for a finite graph G having at least one edge always $|V(\mathrm{Simp}(G))| > |V(G)|$. Thus suppose that $|V(G)|$ is equal to some isolated infinite cardinal number $\aleph_{\beta+1}$, where β is an ordinal number. If G contains a vertex of degree $\aleph_{\beta+1}$, then $\mathrm{Simp}(G)$ contains an $\aleph_{\beta+1}$-simplex and $\mathrm{Simp}(\mathrm{Simp}(G))$ contains an $(\exp \aleph_{\beta+1})$-simplex and thus $|V(\mathrm{Simp}(\mathrm{Simp}(G)))| \geq \exp \aleph_{\beta+1} > \aleph_{\beta+1}$. Hence all vertices of G must have degrees less than $\aleph_{\beta+1}$, i.e. less than or equal to $\aleph_\beta$. Choose a vertex v of G and for each non-negative integer k by $N_k(v)$ denote the set of all vertices whose distance from v in G is equal to k. As G is connected, the union of $N_k(v)$ for all non-negative integers k is $V(G)$. By induction we prove that $|N_k(v)| \leq \aleph_\beta$ for each k. For $k = 0$ we have $N_0(v) = \{v\}$ and $|N_0(v)| = 1 < \aleph_\beta$. Now suppose that the assertion is true for some k. Each vertex of $N_{k+1}(v)$ is adjacent to a vertex of $N_k(v)$ and the cardinality of $N_{k+1}(v)$ cannot exceed the cardinality of the set of edges joining vertices of $N_k(v)$ with vertices of $N_{k+1}(v)$. As $|N_k(v)| \leq \aleph_\beta$ and each vertex of $N_k(v)$ has degree at most $\aleph_\beta$, there are at most $\aleph_\beta$ such edges and $|N_{k+1}(v)| \leq \aleph_\beta$. And then $V(G)$ is the union of $\aleph_0$ disjoint sets of cardinalities at most $\aleph_\beta$, hence also $|V(G)| \leq \aleph_\beta < \aleph_{\beta+1}$, which is a contradiction. This proves the assertion. ∎

Note that in this case also the limit cardinal number $\aleph_0$ may occur. This theorem has importance for Simp-periodic graphs. By Simp^k we denote the k-th iteration of Simp, where k is a positive integer. From the inequality $|V(G)| \leq |V(\mathrm{Simp}(G))|$ it is clear that if $\mathrm{Simp}^k(G) \cong G$, then $|V(G)| = |V(\mathrm{Simp}(\mathrm{Simp}(G)))|$, the number k being an arbitrary positive integer, and the following corollary holds.

Corollary. *Let G be a connected Simp-periodic graph. Then the cardinality of $V(G)$ is 0, 1 or a limit cardinal number.* ∎

The last theorem will concern locally finite graphs. Remember the well-known fact that a connected infinite locally finite graph has always a countable vertex set.

Theorem 4. *Let G be an infinite locally finite graph. Then so is* $\mathrm{Simp}(G)$.

Proof. Let S be a simplex in G; as G is locally finite, S is finite. Each vertex v of S can be contained only in a finite number of simplices of G, because this number cannot exceed the number of all subsets of the set of neighbours of v. As also S is finite, the set of all simplices having non-empty intersections with S is finite: the vertex of $\mathrm{Simp}(G)$ coressponding to S has a finite degree. As S was chosen arbitrarily, the graph $\mathrm{Simp}(G)$ is locally finite. ■

References

[1] E. Prisner, *Graph dynamics*, Longman House, Burnt Mill, Harlow, Essex 1995.

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