# SHORT CYCLES OF LOW WEIGHT IN NORMAL PLANE MAPS WITH MINIMUM DEGREE 5 

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#### Abstract

In this note, precise upper bounds are determined for the minimum degree-sum of the vertices of a 4 -cycle and a 5 -cycle in a plane triangulation with minimum degree 5: $w\left(C_{4}\right) \leq 25$ and $w\left(C_{5}\right) \leq 30$. These hold because a normal plane map with minimum degree 5 must contain a 4 -star with $w\left(K_{1,4}\right) \leq 30$. These results answer a question posed by Kotzig in 1979 and recent questions of Jendrol' and Madaras.


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The weight of a subgraph in a plane map $M$ is the sum of the degrees (in $M$ ) of its vertices. By $w(S)$, we denote the minimum weight of a subgraph isomorphic to $S$ in $M$. By $M_{5}$ or $T_{5}$ we mean a connected plane map with minimum degree 5 and each face having size at least 3 (that is, a normal plane map) or exactly 3 (that is, a triangulation), respectively. As conjectured by Kotzig [4] for each $T_{5}$ and proved in [1] for each $M_{5}$, $w\left(C_{3}\right) \leq 17$, and this bound is precise. Also, Kotzig [5] announced that $25 \leq$ $w\left(C_{4}\right) \leq 26$ for each $T_{5}$. Jendrol' and Madaras [3] proved that $w\left(C_{4}\right) \leq 35$,

[^0]$w\left(C_{5}\right) \leq 45$ and $w\left(K_{1,4}\right) \leq 39$ for each $T_{5}$ and $w\left(K_{1,3}\right) \leq 23$, which bound is best possible, and $w\left(K_{1,4}\right) \leq 45$ for each $M_{5}$.

Our main result is:

Theorem 1. Each normal plane map with minimum degree 5 contains a 4 -star with weight at most 30 with a 5 -vertex as its centre.

This clearly implies:
Corollary 2. Each plane triangulation with minimum degree 5 contains a 4 -cycle with weight at most 25 and a 5-cycle with weight at most 30 .

The bounds in Theorem 1 and Corollary 2 are all precise, as the following examples show. Take any polyhedron in which every vertex is of type $5.6^{2}$ or $6^{3}$, such as the Archimedean solid in which every vertex is incident with a 5 -face and two 6 -faces. Truncate all the vertices to obtain a graph in which every vertex has type 3.10 .12 or $3.12^{2}$. Cap each 10 -face and 12 -face by putting a new vertex inside it and joining it to all the boundary vertices. We have obtained a triangulation with minimum degree 5 in which the neighbours of every 5 -vertex $v$ have degrees (in cyclic order round $v$ ) $(5,5,10,5,12)$ or $(5,5,12,5,12)$. This graph clearly has $w\left(C_{4}\right)=25$ and $w\left(C_{5}\right)=w\left(K_{1,4}\right)=30$.

It follows that our results above completely solve the problems raised by Kotzig [5] and Jendrol' and Madaras [3]. In the proof below, we use some ideas from our unpublished manuscript [2].

We shall use the following terminology. The number of edges incident with a vertex $v$ or $r(f)$ respectively, and $v_{1}, \ldots, v_{d(v)}$ denote the neighbours of $v$, in cyclic order round $v$. If $d\left(v_{i}\right)=5$ then $v_{i}$ is a strong, semiweak or weak neighbour of $v$ according as none, one or both of $v_{i-1}, v_{i+1}$ have degree 5 , and $v_{i}$ is twice weak if $d\left(v_{j}\right)=5$ whenever $|j-i| \leq 2(\operatorname{modulo} d(v))$. A $k$-vertex is a vertex $v$ with $d(v)=k$, and a $>k$-vertex has $d(v)>k$, etc.

Proof of Theorem 1. It suffices to prove the theorem for triangulations, since adding an extra edge to a normal plane map with minimum degree 5 cannot create a new 4 -star with a 5 -vertex as its centre, nor can it reduce the weight of any existing 4 -star. So suppose that $G=(V, E, F)$ is a triangulation that is a counterexample to Theorem 1. Since $G$ is a triangulation, $2|E|=3|F|$, and so Euler's formula $|V|-|E|+|F|=2$ implies

$$
\begin{equation*}
\sum_{v \in V}(d(v)-6)=-12 \tag{1}
\end{equation*}
$$

Assign a charge $\mu(v)=d(v)-6$ to each vertex $v \in V$, so that only 5 -vertices have negative charge. Using the properties of $G$ as a counterexample, we define a local redistribution of charges, preserving their sum, such that the new charge $\mu^{\prime}(v)$ is non-negative for all $v \in V$. This will contradict the fact that the sum of the new charges is, by (1), equal to -12 . The technique of discharging is often used in solving structural and colouring problems on plane graphs.

Our discharging rules are as follows.
Rule 1. (a) Each vertex $v$ of degree 7 sends $\frac{1}{3}$ to each strong neighbour and $\frac{1}{6}$ to each semiweak neighbour.
(b) Each vertex $v$ with degree 8,9 or $\geq 12$ first gives a "basic" contribution of $\frac{\mu(v)}{d(v)}=\frac{d(v)-6}{d(v)}$ to each neighbouring vertex $v_{i}$. Then each neighbour $v_{i}$ with $d\left(v_{i}\right)>5$ shares the charge just received equally between $v_{i-1}$ and $v_{i+1}$.
(c) Each 10 -vertex or 11 -vertex $v$ first gives a "basic" $\frac{2}{5}$ to each neighbour. Then, whenever $d\left(v_{i}\right)>5, v_{i}$ transfers $\frac{1}{10}$ of $v$ 's donation to each 5 -vertex in $\left\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\right\}$.

Rule 2. If $d(v)=11$ then $v$ gives a "supplementary" $\frac{1}{10}$ to each twice weak neighbour.

Rule 3. If $v$ is 5 -vertex adjacent to an 11 -vertex $w$, say $w=v_{5}$, and if $d\left(v_{1}\right)=d\left(v_{4}\right)=5$, then $v$ gives back to $v_{5}$ the following:
(a) $\frac{1}{2}$ if both $v_{2}$ and $v_{3}$ have degree $\geq 9$;
(b) $\frac{1}{4}$ if at least one of $v_{2}, v_{3}$ has degree exactly 8 .

We must prove that $\mu^{\prime}(v) \geq 0$ for each vertex $v$. If $d(v) \notin\{5,7,11\}$, then, by Rule 1 (b) and (c), $v$ distributes its own original charge of $\mu(v)=d(v)-6$ to its neigbours in equal shares, and possibly participates in transferring the others' charges, so that $\mu^{\prime}(v) \geq d(v)-6-d(v) \times \frac{d(v)-6}{d(v)}=0$. We deal with the remaining values of $d(v)$ in three cases.

Case 1. $d(v)=11$. Then $\mu(v)=d(v)-6=5$. If $v$ has a neighbour $v_{i}$ with $d\left(v_{i}\right) \geq 6$, then none of $v_{i-2}, \ldots, v_{i+2}$ is twice weak and so none of them receives a supplementary $\frac{1}{10}$ from $v$ by Rule 2 . Thus $\mu^{\prime}(v) \geq 5-11 \times$ $\frac{2}{5}-6 \times \frac{1}{10}=0$. So we may assume that all neighbours of $v$ have degree 5 .

Each edge $v_{i} v_{i+1}$ lies in two triangles, say $v_{i} v_{i+1} v$ and $v_{i} v_{i+1} w_{i}$. If $d\left(w_{i}\right)=8$ for some $i$, then $v$ receives $\frac{1}{4}$ by Rule 3(b) from each of $v_{i}$ and
$v_{i+1}$, so that $\mu^{\prime}(v) \geq 5+2 \times \frac{1}{4}-11 \times \frac{1}{2}=0$. So we may assume that $d\left(w_{i}\right) \neq 8$, for each $i$.

If $d\left(w_{i-1}\right) \geq 9$ and $d\left(w_{i}\right) \geq 9$ for some $i$, then $v_{i}$ gives back $\frac{1}{2}$ to $v$ by Rule $3(\mathrm{a})$, and we are done. Also, it is impossible that $d\left(w_{i-1}\right) \leq 7$ and $d\left(w_{i}\right) \leq 7$ for any $i$, since by hypothesis there is no 4 -star with weight $\leq 30$ centered at $v_{i}$. Therefore, for each $i$, one of $d\left(w_{i-1}\right)$ and $d\left(w_{i}\right)$ is at most 7 and the other is at least 9 . But this cannot hold for all $i$ modulo 11 , since 11 is odd.

Case 2. $d(v)=7$. Then $\mu(v)=d(v)-6=1$. By Rule 1(a), no weak neighbour receives anything from $v$, and so there are at most four receivers. If there are exactly four, then at least two are semiweak and so receive $\frac{1}{6}$ each, with a total expenditure by $v$ of at most $2 \times \frac{1}{6}+2 \times \frac{1}{3}=1$. Otherwise, $v$ gives at most $3 \times \frac{1}{3}=1$.

| neighbour: | strong | semiweak | weak |
| :---: | :---: | :---: | :---: |
| $7:$ | $1 / 3$ | $1 / 6$ | 0 |
| $8:$ | $1 / 2$ | $3 / 8$ | $1 / 4$ |
| $9:$ | $2 / 3$ | $1 / 2$ | $1 / 3$ |
| $10:$ | $\geq 3 / 5$ | $\geq 1 / 2$ | $\geq 2 / 5$ |
| $11:$ | $\geq 3 / 5$ | $\geq 1 / 2$ | $1 / 2$ |
| $\geq 12:$ | $\geq 1$ | $\geq 3 / 4$ | $\geq 1 / 2$ |

Table 1. Donations to 5 -vertices by Rules 1 and 2
Case 3. $\quad d(v)=5$. Then $\mu(v)=d(v)-6=-1$. The amounts of charge received by $v$ from its neighbours by Rules 1 and 2 are summarized in Table 1. However, $v$ may give back charge to some 11 -vertices by Rule 3.

Suppose Rule 3(a) applies to $v$, so that $v$ 's neighbours $v_{1}, \ldots, v_{5}$ have degrees $(5, \geq 9, \geq 9,5,11)$. Then $v$ is a semiweak neighbour of each of $v_{2}$ and $v_{3}$, so that it receives at least $\frac{1}{2}$ from each of them by Table 1 , and gives nothing back to either of them by Rule 3 . It also receives at least $\frac{1}{2}$ from $v_{5}$ by Table 1 , and gives back exactly $\frac{1}{2}$ to $v_{5}$ by Rule $3(\mathrm{a})$. We deduce that $\mu^{\prime}(v) \geq 0$.

From now on, we may assume that Rule 3(a) does not apply to $v$. Suppose Rule 3(b) applies. Because $v$ is not the centre of a 4 -star with weight $\leq 30$, $v$ 's neighbours have degrees $(5,8, \geq 8,5,11)$. Thus $v$ is a semiweak neighbour of $v_{2}$ and $v_{3}$ and so it receives at least $\frac{3}{8}$ from each of them by Table 1, and gives nothing back. It also receives at least $\frac{1}{2}$ from $v_{5}$ by Table 1, and gives $\frac{1}{4}$ back. Thus $\mu^{\prime}(v) \geq 0$.

So we may suppose that Rule 3 does not apply to $v$ at all, and the amount that $v$ receives from its neighbours is at least that given in Table 1. Because of the absence of 4 -stars with weight $\leq 30$, the degree-sequence of $v$ 's neighbours, in nondecreasing order, must be one of the following.
( $5,5,5, \geq 11, \geq 11$ ): Then each $\geq 11$-vertex gives $\geq \frac{1}{2}$ to $v$ by Table 1 .
$(5,5,6, \geq 10, \geq 10)$ : If each of the two $\geq 10$-neighbours gives $\geq \frac{1}{2}$ to $v$, we are done.

Suppose there is a 10 -vertex, say $v_{1}$, giving $\frac{2}{5}$ to $v$. Then $v$ must be a twice weak neighbour of $v_{1}$ by Rule 1(c). W.l.o.g., suppose that $d\left(v_{2}\right)=$ $d\left(v_{5}\right)=5$ and $d\left(v_{3}\right)=6$. If $d\left(v_{4}\right) \geq 12$ then $v_{4}$ gives $\geq \frac{3}{4}$ to $v$ by Table 1 , so that $\mu^{\prime}(v) \geq-1+\frac{2}{5}+\frac{3}{4}>0$. So we may assume $10 \leq d\left(v_{4}\right) \leq 11$; note that $v$ is not a twice weak neighbour of $v_{4}$. Let $u$ be the vertex (other than $v$ ) adjacent to $v_{4}$ and $v_{5}$. Since $v_{5}$ has two 5 -neighbours other than $u$ (because $v$ is a twice weak neighbour of $v_{1}$ ), and also has a 10 -neighbour $v_{1}$, it follows that $d(u)>5$. Then Rule $1(\mathrm{c})$ ensures that $v$ receives $\frac{1}{10}$ from $v_{4}$ via each of $v_{3}$ and $u$, so that $\mu^{\prime}(v) \geq-1+2 \times \frac{2}{5}+2 \times \frac{1}{10}=0$.
$(5,5,7, \geq 9, \geq 9)$ : If $v$ is weak for neither of the $\geq 9$-neighbours then each of them gives $\geq \frac{1}{2}$, and we are done by Table 1. Otherwise, $v$ is weak for a $\geq 9$-neighbour, giving $\geq \frac{1}{3}$, and semiweak for the other two neighbours of degree 7 and $\geq 9$, giving $\geq \frac{1}{6}+\frac{1}{2}$ in total.
( $5,5, \geq 8, \geq 8, \geq 8$ ): If $v$ is weak for none of the three $\geq 8$-neighbours, then $v$ receives $\geq 3 \times \frac{3}{8}>1$ in total. Otherwise, $v$ is weak for one of them and semiweak for the other two, so that receives $\geq \frac{1}{4}+2 \times \frac{3}{8}=1$ in total.
( $5,6,6, \geq 9, \geq 9$ ): Each $\geq 9$-neigbour gives $\geq \frac{1}{2}$.
( $5,6, \geq 7, \geq 8, \geq 8$ ): For each of the three $\geq 7$-neighbours, $v$ is semiweak or strong; for at least one of them, $v$ is strong. By Table $1, v$ thus receives either $\geq \frac{1}{6}+\frac{3}{8}+\frac{1}{2}>1$ or $\geq \frac{3}{8}+\frac{3}{8}+\frac{1}{3}>1$.
( $5, \geq 7, \geq 7, \geq 7, \geq 7$ ): For at least two $\geq 7$-neighbours, $v$ is strong; for the others, semiweak. Thus, $\mu^{\prime}(v) \geq-1+2 \times \frac{1}{3}+2 \times \frac{1}{6}=0$.

$$
\begin{aligned}
& (\geq 6, \geq 6, \geq 6, \geq 8, \geq 8): \mu^{\prime}(v) \geq-1+2 \times \frac{1}{2}=0 . \\
& (\geq 6, \geq 6, \geq 7, \geq 7, \geq 7): \mu^{\prime}(v) \geq-1+3 \times \frac{1}{3}=0 .
\end{aligned}
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Thus we have proved $\mu^{\prime}(v) \geq 0$ for every $v \in V$ and $f \in F$, which contradicts (1) and completes the proof of Theorem 1.

## References

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