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SHORT CYCLES OF LOW WEIGHT IN NORMAL PLANE MAPS WITH MINIMUM DEGREE 5

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Abstract

In this note, precise upper bounds are determined for the minimum degree-sum of the vertices of a 4-cycle and a 5-cycle in a plane triangulation with minimum degree 5: $w(C_4) \leq 25$ and $w(C_5) \leq 30$. These hold because a normal plane map with minimum degree 5 must contain a 4-star with $w(K_{1,4}) \leq 30$. These results answer a question posed by Kotzig in 1979 and recent questions of Jendrol' and Madaras.

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The *weight* of a subgraph in a plane map M is the sum of the degrees (in M) of its vertices. By $w(S)$, we denote the minimum weight of a subgraph isomorphic to S in M. By M_5 or T_5 we mean a connected plane map with minimum degree 5 and each face having size at least 3 (that is, a normal plane map) or exactly 3 (that is, a triangulation), respectively. As conjectured by Kotzig [4] for each T_5 and proved in [1] for each M_5, $w(C_3) \leq 17$, and this bound is precise. Also, Kotzig [5] announced that $25 \leq w(C_4) \leq 26$ for each T_5. Jendrol' and Madaras [3] proved that $w(C_4) \leq 35$,

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$w(C_5) \le 45$ and $w(K_{1,4}) \le 39$ for each T_5 and $w(K_{1,3}) \le 23$, which bound is best possible, and $w(K_{1,4}) \le 45$ for each M_5.

Our main result is:

Theorem 1. *Each normal plane map with minimum degree* 5 *contains a* 4*-star with weight at most* 30 *with a* 5*-vertex as its centre.*

This clearly implies:

Corollary 2. *Each plane triangulation with minimum degree* 5 *contains a* 4*-cycle with weight at most* 25 *and a* 5*-cycle with weight at most* 30*.*

The bounds in Theorem 1 and Corollary 2 are all precise, as the following examples show. Take any polyhedron in which every vertex is of type 5.6^2 or 6^3, such as the Archimedean solid in which every vertex is incident with a 5-face and two 6-faces. Truncate all the vertices to obtain a graph in which every vertex has type 3.10.12 or 3.12^2. Cap each 10-face and 12-face by putting a new vertex inside it and joining it to all the boundary vertices. We have obtained a triangulation with minimum degree 5 in which the neighbours of every 5-vertex v have degrees (in cyclic order round v) $(5,5,10,5,12)$ or $(5,5,12,5,12)$. This graph clearly has $w(C_4) = 25$ and $w(C_5) = w(K_{1,4}) = 30$.

It follows that our results above completely solve the problems raised by Kotzig [5] and Jendrol' and Madaras [3]. In the proof below, we use some ideas from our unpublished manuscript [2].

We shall use the following terminology. The number of edges incident with a vertex v or $r(f)$ respectively, and $v_1, \dots, v_{d(v)}$ denote the neighbours of v, in cyclic order round v. If $d(v_i) = 5$ then v_i is a *strong*, *semiweak* or *weak* neighbour of v according as none, one or both of v_{i-1}, v_{i+1} have degree 5, and v_i is *twice weak* if $d(v_j) = 5$ whenever $|j-i| \le 2$ (modulo $d(v)$). A *k-vertex* is a vertex v with $d(v) = k$, and a *$>k$-vertex* has $d(v) > k$, etc.

***Proof of Theorem* 1.** It suffices to prove the theorem for triangulations, since adding an extra edge to a normal plane map with minimum degree 5 cannot create a new 4-star with a 5-vertex as its centre, nor can it reduce the weight of any existing 4-star. So suppose that $G = (V, E, F)$ is a triangulation that is a counterexample to Theorem 1. Since G is a triangulation, $2|E| = 3|F|$, and so Euler's formula $|V| - |E| + |F| = 2$ implies

$$\sum_{v \in V} (d(v) - 6) = -12. \qquad (1)$$

Assign a *charge* $\mu(v) = d(v) - 6$ to each vertex $v \in V$, so that only 5-vertices have negative charge. Using the properties of G as a counterexample, we define a local redistribution of charges, preserving their sum, such that the *new charge* $\mu'(v)$ is non-negative for all $v \in V$. This will contradict the fact that the sum of the new charges is, by (1), equal to -12. The technique of discharging is often used in solving structural and colouring problems on plane graphs.

Our discharging rules are as follows.

Rule 1. (a) Each vertex v of degree 7 sends $\frac{1}{3}$ to each strong neighbour and $\frac{1}{6}$ to each semiweak neighbour.

(b) Each vertex v with degree 8, 9 or $\geq$12 first gives a "basic" contribution of $\frac{\mu(v)}{d(v)} = \frac{d(v)-6}{d(v)}$ to each neighbouring vertex v_i. Then each neighbour v_i with $d(v_i) > 5$ shares the charge just received equally between v_{i-1} and v_{i+1}.

(c) Each 10-vertex or 11-vertex v first gives a "basic" $\frac{2}{5}$ to each neighbour. Then, whenever $d(v_i) > 5$, v_i transfers $\frac{1}{10}$ of v's donation to each 5-vertex in $\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}$.

Rule 2. If $d(v) = 11$ then v gives a "supplementary" $\frac{1}{10}$ to each twice weak neighbour.

Rule 3. If v is 5-vertex adjacent to an 11-vertex w, say $w = v_5$, and if $d(v_1) = d(v_4) = 5$, then v gives back to v_5 the following:

(a) $\frac{1}{2}$ if both v_2 and v_3 have degree $\geq$9;

(b) $\frac{1}{4}$ if at least one of v_2, v_3 has degree exactly 8.

We must prove that $\mu'(v) \geq 0$ for each vertex v. If $d(v) \notin \{5, 7, 11\}$, then, by Rule 1 (b) and (c), v distributes its own original charge of $\mu(v) = d(v) - 6$ to its neigbours in equal shares, and possibly participates in transferring the others' charges, so that $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{d(v)-6}{d(v)} = 0$. We deal with the remaining values of $d(v)$ in three cases.

Case 1. $d(v) = 11$. Then $\mu(v) = d(v) - 6 = 5$. If v has a neighbour v_i with $d(v_i) \geq 6$, then none of $v_{i-2}, \ldots, v_{i+2}$ is twice weak and so none of them receives a supplementary $\frac{1}{10}$ from v by Rule 2. Thus $\mu'(v) \geq 5 - 11 \times \frac{2}{5} - 6 \times \frac{1}{10} = 0$. So we may assume that all neighbours of v have degree 5.

Each edge v_iv_{i+1} lies in two triangles, say $v_iv_{i+1}v$ and $v_iv_{i+1}w_i$. If $d(w_i) = 8$ for some i, then v receives $\frac{1}{4}$ by Rule 3(b) from each of v_i and

v_{i+1}, so that $\mu'(v) \geq 5 + 2 \times \frac{1}{4} - 11 \times \frac{1}{2} = 0$. So we may assume that $d(w_i) \neq 8$, for each i.

If $d(w_{i-1}) \geq 9$ and $d(w_i) \geq 9$ for some i, then v_i gives back $\frac{1}{2}$ to v by Rule 3(a), and we are done. Also, it is impossible that $d(w_{i-1}) \leq 7$ and $d(w_i) \leq 7$ for any i, since by hypothesis there is no 4-star with weight ≤ 30 centered at v_i. Therefore, for each i, one of $d(w_{i-1})$ and $d(w_i)$ is at most 7 and the other is at least 9. But this cannot hold for all i modulo 11, since 11 is odd.

Case 2. $d(v) = 7$. Then $\mu(v) = d(v) - 6 = 1$. By Rule 1(a), no weak neighbour receives anything from v, and so there are at most four receivers. If there are exactly four, then at least two are semiweak and so receive $\frac{1}{6}$ each, with a total expenditure by v of at most $2 \times \frac{1}{6} + 2 \times \frac{1}{3} = 1$. Otherwise, v gives at most $3 \times \frac{1}{3} = 1$.

neighbour:	strong	semiweak	weak
7:	1/3	1/6	0
8:	1/2	3/8	1/4
9:	2/3	1/2	1/3
10:	$\geq 3/5$	$\geq 1/2$	$\geq 2/5$
11:	$\geq 3/5$	$\geq 1/2$	1/2
≥ 12 :	≥ 1	$\geq 3/4$	$\geq 1/2$

Table 1. Donations to 5-vertices by Rules 1 and 2

Case 3. $d(v) = 5$. Then $\mu(v) = d(v) - 6 = -1$. The amounts of charge received by v from its neighbours by Rules 1 and 2 are summarized in Table 1. However, v may give back charge to some 11-vertices by Rule 3.

Suppose Rule 3(a) applies to v, so that v's neighbours $v_1, \ldots, v_5$ have degrees $(5, \geq 9, \geq 9, 5, 11)$. Then v is a semiweak neighbour of each of v_2 and v_3, so that it receives at least $\frac{1}{2}$ from each of them by Table 1, and gives nothing back to either of them by Rule 3. It also receives at least $\frac{1}{2}$ from v_5 by Table 1, and gives back exactly $\frac{1}{2}$ to v_5 by Rule 3(a). We deduce that $\mu'(v) \geq 0$.

From now on, we may assume that Rule 3(a) does not apply to v. Suppose Rule 3(b) applies. Because v is not the centre of a 4-star with weight ≤ 30, v's neighbours have degrees $(5, 8, \geq 8, 5, 11)$. Thus v is a semiweak neighbour of v_2 and v_3 and so it receives at least $\frac{3}{8}$ from each of them by Table 1, and gives nothing back. It also receives at least $\frac{1}{2}$ from v_5 by Table 1, and gives $\frac{1}{4}$ back. Thus $\mu'(v) \geq 0$.

So we may suppose that Rule 3 does not apply to v at all, and the amount that v receives from its neighbours is at least that given in Table 1. Because of the absence of 4-stars with weight ≤ 30, the degree-sequence of v's neighbours, in nondecreasing order, must be one of the following.

$(5, 5, 5, \geq 11, \geq 11)$: Then each $\geq$11-vertex gives $\geq \frac{1}{2}$ to v by Table 1.

$(5, 5, 6, \geq 10, \geq 10)$: If each of the two $\geq$10-neighbours gives $\geq \frac{1}{2}$ to v, we are done.

Suppose there is a 10-vertex, say v_1, giving $\frac{2}{5}$ to v. Then v must be a twice weak neighbour of v_1 by Rule 1(c). W.l.o.g., suppose that $d(v_2) = d(v_5) = 5$ and $d(v_3) = 6$. If $d(v_4) \geq 12$ then v_4 gives $\geq \frac{3}{4}$ to v by Table 1, so that $\mu'(v) \geq -1 + \frac{2}{5} + \frac{3}{4} > 0$. So we may assume $10 \leq d(v_4) \leq 11$; note that v is not a twice weak neighbour of v_4. Let u be the vertex (other than v) adjacent to v_4 and v_5. Since v_5 has two 5-neighbours other than u (because v is a twice weak neighbour of v_1), and also has a 10-neighbour v_1, it follows that $d(u) > 5$. Then Rule 1(c) ensures that v receives $\frac{1}{10}$ from v_4 via each of v_3 and u, so that $\mu'(v) \geq -1 + 2 \times \frac{2}{5} + 2 \times \frac{1}{10} = 0$.

$(5, 5, 7, \geq 9, \geq 9)$: If v is weak for neither of the $\geq$9-neighbours then each of them gives $\geq \frac{1}{2}$, and we are done by Table 1. Otherwise, v is weak for a $\geq$9-neighbour, giving $\geq \frac{1}{3}$, and semiweak for the other two neighbours of degree 7 and ≥ 9, giving $\geq \frac{1}{6} + \frac{1}{2}$ in total.

$(5, 5, \geq 8, \geq 8, \geq 8)$: If v is weak for none of the three $\geq$ 8-neighbours, then v receives $\geq 3 \times \frac{3}{8} > 1$ in total. Otherwise, v is weak for one of them and semiweak for the other two, so that receives $\geq \frac{1}{4} + 2 \times \frac{3}{8} = 1$ in total.

$(5, 6, 6, \geq 9, \geq 9)$: Each $\geq$9-neigbour gives $\geq \frac{1}{2}$.

$(5, 6, \geq 7, \geq 8, \geq 8)$: For each of the three $\geq$7-neighbours, v is semiweak or strong; for at least one of them, v is strong. By Table 1, v thus receives either $\geq \frac{1}{6} + \frac{3}{8} + \frac{1}{2} > 1$ or $\geq \frac{3}{8} + \frac{3}{8} + \frac{1}{3} > 1$.

$(5, \geq 7, \geq 7, \geq 7, \geq 7)$: For at least two $\geq$7-neighbours, v is strong; for the others, semiweak. Thus, $\mu'(v) \geq -1 + 2 \times \frac{1}{3} + 2 \times \frac{1}{6} = 0$.

$(\geq 6, \geq 6, \geq 6, \geq 8, \geq 8)$: $\mu'(v) \geq -1 + 2 \times \frac{1}{2} = 0$.

$(\geq 6, \geq 6, \geq 7, \geq 7, \geq 7)$: $\mu'(v) \geq -1 + 3 \times \frac{1}{3} = 0$.

Thus we have proved $\mu'(v) \geq 0$ for every $v \in V$ and $f \in F$, which contradicts (1) and completes the proof of Theorem 1. ■

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