

SHORT CYCLES OF LOW WEIGHT IN NORMAL PLANE MAPS WITH MINIMUM DEGREE 5

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Abstract

In this note, precise upper bounds are determined for the minimum degree-sum of the vertices of a 4-cycle and a 5-cycle in a plane triangulation with minimum degree 5: $w(C_4) \leq 25$ and $w(C_5) \leq 30$. These hold because a normal plane map with minimum degree 5 must contain a 4-star with $w(K_{1,4}) \leq 30$. These results answer a question posed by Kotzig in 1979 and recent questions of Jendrol' and Madaras.

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The *weight* of a subgraph in a plane map M is the sum of the degrees (in M) of its vertices. By $w(S)$, we denote the minimum weight of a subgraph isomorphic to S in M . By M_5 or T_5 we mean a connected plane map with minimum degree 5 and each face having size at least 3 (that is, a normal plane map) or exactly 3 (that is, a triangulation), respectively. As conjectured by Kotzig [4] for each T_5 and proved in [1] for each M_5 , $w(C_3) \leq 17$, and this bound is precise. Also, Kotzig [5] announced that $25 \leq w(C_4) \leq 26$ for each T_5 . Jendrol' and Madaras [3] proved that $w(C_4) \leq 35$,

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$w(C_5) \leq 45$ and $w(K_{1,4}) \leq 39$ for each T_5 and $w(K_{1,3}) \leq 23$, which bound is best possible, and $w(K_{1,4}) \leq 45$ for each M_5 .

Our main result is:

Theorem 1. *Each normal plane map with minimum degree 5 contains a 4-star with weight at most 30 with a 5-vertex as its centre.*

This clearly implies:

Corollary 2. *Each plane triangulation with minimum degree 5 contains a 4-cycle with weight at most 25 and a 5-cycle with weight at most 30.*

The bounds in Theorem 1 and Corollary 2 are all precise, as the following examples show. Take any polyhedron in which every vertex is of type 5.6^2 or 6^3 , such as the Archimedean solid in which every vertex is incident with a 5-face and two 6-faces. Truncate all the vertices to obtain a graph in which every vertex has type $3.10.12$ or 3.12^2 . Cap each 10-face and 12-face by putting a new vertex inside it and joining it to all the boundary vertices. We have obtained a triangulation with minimum degree 5 in which the neighbours of every 5-vertex v have degrees (in cyclic order round v) $(5, 5, 10, 5, 12)$ or $(5, 5, 12, 5, 12)$. This graph clearly has $w(C_4) = 25$ and $w(C_5) = w(K_{1,4}) = 30$.

It follows that our results above completely solve the problems raised by Kotzig [5] and Jendrol' and Madaras [3]. In the proof below, we use some ideas from our unpublished manuscript [2].

We shall use the following terminology. The number of edges incident with a vertex v or $r(f)$ respectively, and $v_1, \dots, v_{d(v)}$ denote the neighbours of v , in cyclic order round v . If $d(v_i) = 5$ then v_i is a *strong*, *semiweak* or *weak* neighbour of v according as none, one or both of v_{i-1}, v_{i+1} have degree 5, and v_i is *twice weak* if $d(v_j) = 5$ whenever $|j-i| \leq 2$ (modulo $d(v)$). A k -vertex is a vertex v with $d(v) = k$, and a $>k$ -vertex has $d(v) > k$, etc.

Proof of Theorem 1. It suffices to prove the theorem for triangulations, since adding an extra edge to a normal plane map with minimum degree 5 cannot create a new 4-star with a 5-vertex as its centre, nor can it reduce the weight of any existing 4-star. So suppose that $G = (V, E, F)$ is a triangulation that is a counterexample to Theorem 1. Since G is a triangulation, $2|E| = 3|F|$, and so Euler's formula $|V| - |E| + |F| = 2$ implies

$$(1) \quad \sum_{v \in V} (d(v) - 6) = -12.$$

Assign a *charge* $\mu(v) = d(v) - 6$ to each vertex $v \in V$, so that only 5-vertices have negative charge. Using the properties of G as a counterexample, we define a local redistribution of charges, preserving their sum, such that the *new charge* $\mu'(v)$ is non-negative for all $v \in V$. This will contradict the fact that the sum of the new charges is, by (1), equal to -12 . The technique of discharging is often used in solving structural and colouring problems on plane graphs.

Our discharging rules are as follows.

Rule 1. (a) Each vertex v of degree 7 sends $\frac{1}{3}$ to each strong neighbour and $\frac{1}{6}$ to each semiweak neighbour.

(b) Each vertex v with degree 8, 9 or ≥ 12 first gives a “basic” contribution of $\frac{\mu(v)}{d(v)} = \frac{d(v)-6}{d(v)}$ to each neighbouring vertex v_i . Then each neighbour v_i with $d(v_i) > 5$ shares the charge just received equally between v_{i-1} and v_{i+1} .

(c) Each 10-vertex or 11-vertex v first gives a “basic” $\frac{2}{5}$ to each neighbour. Then, whenever $d(v_i) > 5$, v_i transfers $\frac{1}{10}$ of v ’s donation to each 5-vertex in $\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}$.

Rule 2. If $d(v) = 11$ then v gives a “supplementary” $\frac{1}{10}$ to each twice weak neighbour.

Rule 3. If v is 5-vertex adjacent to an 11-vertex w , say $w = v_5$, and if $d(v_1) = d(v_4) = 5$, then v gives back to v_5 the following:

- (a) $\frac{1}{2}$ if both v_2 and v_3 have degree ≥ 9 ;
- (b) $\frac{1}{4}$ if at least one of v_2, v_3 has degree exactly 8.

We must prove that $\mu'(v) \geq 0$ for each vertex v . If $d(v) \notin \{5, 7, 11\}$, then, by Rule 1 (b) and (c), v distributes its own original charge of $\mu(v) = d(v) - 6$ to its neighbours in equal shares, and possibly participates in transferring the others’ charges, so that $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{d(v)-6}{d(v)} = 0$. We deal with the remaining values of $d(v)$ in three cases.

Case 1. $d(v) = 11$. Then $\mu(v) = d(v) - 6 = 5$. If v has a neighbour v_i with $d(v_i) \geq 6$, then none of v_{i-2}, \dots, v_{i+2} is twice weak and so none of them receives a supplementary $\frac{1}{10}$ from v by Rule 2. Thus $\mu'(v) \geq 5 - 11 \times \frac{2}{5} - 6 \times \frac{1}{10} = 0$. So we may assume that all neighbours of v have degree 5.

Each edge $v_i v_{i+1}$ lies in two triangles, say $v_i v_{i+1} v$ and $v_i v_{i+1} w_i$. If $d(w_i) = 8$ for some i , then v receives $\frac{1}{4}$ by Rule 3(b) from each of v_i and

v_{i+1} , so that $\mu'(v) \geq 5 + 2 \times \frac{1}{4} - 11 \times \frac{1}{2} = 0$. So we may assume that $d(w_i) \neq 8$, for each i .

If $d(w_{i-1}) \geq 9$ and $d(w_i) \geq 9$ for some i , then v_i gives back $\frac{1}{2}$ to v by Rule 3(a), and we are done. Also, it is impossible that $d(w_{i-1}) \leq 7$ and $d(w_i) \leq 7$ for any i , since by hypothesis there is no 4-star with weight ≤ 30 centered at v_i . Therefore, for each i , one of $d(w_{i-1})$ and $d(w_i)$ is at most 7 and the other is at least 9. But this cannot hold for all i modulo 11, since 11 is odd.

Case 2. $d(v) = 7$. Then $\mu(v) = d(v) - 6 = 1$. By Rule 1(a), no weak neighbour receives anything from v , and so there are at most four receivers. If there are exactly four, then at least two are semiweak and so receive $\frac{1}{6}$ each, with a total expenditure by v of at most $2 \times \frac{1}{6} + 2 \times \frac{1}{3} = 1$. Otherwise, v gives at most $3 \times \frac{1}{3} = 1$.

neighbour:	strong	semiweak	weak
7:	1/3	1/6	0
8:	1/2	3/8	1/4
9:	2/3	1/2	1/3
10:	$\geq 3/5$	$\geq 1/2$	$\geq 2/5$
11:	$\geq 3/5$	$\geq 1/2$	1/2
≥ 12 :	≥ 1	$\geq 3/4$	$\geq 1/2$

Table 1. Donations to 5-vertices by Rules 1 and 2

Case 3. $d(v) = 5$. Then $\mu(v) = d(v) - 6 = -1$. The amounts of charge received by v from its neighbours by Rules 1 and 2 are summarized in Table 1. However, v may give back charge to some 11-vertices by Rule 3.

Suppose Rule 3(a) applies to v , so that v 's neighbours v_1, \dots, v_5 have degrees $(5, \geq 9, \geq 9, 5, 11)$. Then v is a semiweak neighbour of each of v_2 and v_3 , so that it receives at least $\frac{1}{2}$ from each of them by Table 1, and gives nothing back to either of them by Rule 3. It also receives at least $\frac{1}{2}$ from v_5 by Table 1, and gives back exactly $\frac{1}{2}$ to v_5 by Rule 3(a). We deduce that $\mu'(v) \geq 0$.

From now on, we may assume that Rule 3(a) does not apply to v . Suppose Rule 3(b) applies. Because v is not the centre of a 4-star with weight ≤ 30 , v 's neighbours have degrees $(5, 8, \geq 8, 5, 11)$. Thus v is a semiweak neighbour of v_2 and v_3 and so it receives at least $\frac{3}{8}$ from each of them by Table 1, and gives nothing back. It also receives at least $\frac{1}{2}$ from v_5 by Table 1, and gives $\frac{1}{4}$ back. Thus $\mu'(v) \geq 0$.

So we may suppose that Rule 3 does not apply to v at all, and the amount that v receives from its neighbours is at least that given in Table 1. Because of the absence of 4-stars with weight ≤ 30 , the degree-sequence of v 's neighbours, in nondecreasing order, must be one of the following.

(5, 5, 5, ≥ 11 , ≥ 11): Then each ≥ 11 -vertex gives $\geq \frac{1}{2}$ to v by Table 1.

(5, 5, 6, ≥ 10 , ≥ 10): If each of the two ≥ 10 -neighbours gives $\geq \frac{1}{2}$ to v , we are done.

Suppose there is a 10-vertex, say v_1 , giving $\frac{2}{5}$ to v . Then v must be a twice weak neighbour of v_1 by Rule 1(c). W.l.o.g., suppose that $d(v_2) = d(v_5) = 5$ and $d(v_3) = 6$. If $d(v_4) \geq 12$ then v_4 gives $\geq \frac{3}{4}$ to v by Table 1, so that $\mu'(v) \geq -1 + \frac{2}{5} + \frac{3}{4} > 0$. So we may assume $10 \leq d(v_4) \leq 11$; note that v is not a twice weak neighbour of v_4 . Let u be the vertex (other than v) adjacent to v_4 and v_5 . Since v_5 has two 5-neighbours other than u (because v is a twice weak neighbour of v_1), and also has a 10-neighbour v_1 , it follows that $d(u) > 5$. Then Rule 1(c) ensures that v receives $\frac{1}{10}$ from v_4 via each of v_3 and u , so that $\mu'(v) \geq -1 + 2 \times \frac{2}{5} + 2 \times \frac{1}{10} = 0$.

(5, 5, 7, ≥ 9 , ≥ 9): If v is weak for neither of the ≥ 9 -neighbours then each of them gives $\geq \frac{1}{2}$, and we are done by Table 1. Otherwise, v is weak for a ≥ 9 -neighbour, giving $\geq \frac{1}{3}$, and semiweak for the other two neighbours of degree 7 and ≥ 9 , giving $\geq \frac{1}{6} + \frac{1}{2}$ in total.

(5, 5, ≥ 8 , ≥ 8 , ≥ 8): If v is weak for none of the three ≥ 8 -neighbours, then v receives $\geq 3 \times \frac{3}{8} > 1$ in total. Otherwise, v is weak for one of them and semiweak for the other two, so that receives $\geq \frac{1}{4} + 2 \times \frac{3}{8} = 1$ in total.

(5, 6, 6, ≥ 9 , ≥ 9): Each ≥ 9 -neighbour gives $\geq \frac{1}{2}$.

(5, 6, ≥ 7 , ≥ 8 , ≥ 8): For each of the three ≥ 7 -neighbours, v is semiweak or strong; for at least one of them, v is strong. By Table 1, v thus receives either $\geq \frac{1}{6} + \frac{3}{8} + \frac{1}{2} > 1$ or $\geq \frac{3}{8} + \frac{3}{8} + \frac{1}{3} > 1$.

(5, ≥ 7 , ≥ 7 , ≥ 7 , ≥ 7): For at least two ≥ 7 -neighbours, v is strong; for the others, semiweak. Thus, $\mu'(v) \geq -1 + 2 \times \frac{1}{3} + 2 \times \frac{1}{6} = 0$.

(≥ 6 , ≥ 6 , ≥ 6 , ≥ 8 , ≥ 8): $\mu'(v) \geq -1 + 2 \times \frac{1}{2} = 0$.

(≥ 6 , ≥ 6 , ≥ 7 , ≥ 7 , ≥ 7): $\mu'(v) \geq -1 + 3 \times \frac{1}{3} = 0$.

Thus we have proved $\mu'(v) \geq 0$ for every $v \in V$ and $f \in F$, which contradicts (1) and completes the proof of Theorem 1. ■

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