# ON GENERATING SNARKS 

Busiso P. Chisala<br>Department of Mathematical Sciences, Chancellor College<br>P.O. Box 280, Zomba, Malawi


#### Abstract

We discuss the construction of snarks (that is, cyclically 4-edge connected cubic graphs of girth at least five which are not 3-edge colourable) by using what we call colourable snark units and a welding process.


Keywords: snarks, cubic graphs, sirth, edge colouring.
1991 Mathematics Subject Classification: 05C15.

## 1. Introduction

We define a pendant to be a connected graph all of whose vertices have degree either 3 or 1 . By a process we call welding, two such pendants will often produce a snark. In the reverse direction, excising a pendant from a snark produces pendants called snark units. We characterise snark units that can be 3-edge coloured and use these to generate other snarks, and close by showing that snarks exist with the property that (1) they have a 1 -factor whose corresponding 2 -factor has precisely two odd cycles, and (2) for all such 1-factors, the distance between the two odd cycles can be arbitrarily large.

We shall consider the problem of 3-edge colouring of a cubic graph $G$ in terms of a nowhere zero flow in the additive 2-group $\Gamma=\mathbf{Z}_{2} \times \mathbf{Z}_{2}$ (see Jaeger's article [3] for further details). We will denote the three nonzero elements of this group by $x, y$ and $z$, and say that a cubic graph colourable when it is 3-edge colourable, and uncolourable otherwise. Unless otherwise stated, all graphs will be cubic.

An edge cut of $G$ is the set of edges between some proper subset $U$ of $V(G)$ and its complement $\bar{U}$. The edge cut is cyclic if both the induced subgraphs $G[U]$ and $G[\bar{U}]$ contain cycles. We shall say that $G$ is $k$-edge
connected (cyclically $k$-edge connected) if each (cyclic) edge cut of $G$ has order at least $k$. A snark is a cyclically 4 -edge connected cubic graph of girth at least five which is not 3 -edge colourable. Generally, we will follow the terminology of Bondy and Murty's text [1].

## 2. Snark Units

Following Isaacs' paper on snarks [4], we shall call a graph with all vertices of degree 1 or 3 a pendant, and more precisely, a $k$-pendant if it has $k$ vertices of degree one. We will also refer to the edges on the vertices of degree one as free or pendant edges.

To sever an edge $u v$ of a graph $G$ is to introduce an new vertex $w$ on this edge and replace it by the two edges $u w, w v$. Severing all the edges of a $k$-edge cut produces two $k$-pendants $G_{1}$ and $G_{2}$. We shall say that $G_{2}$ is the result of excising $G_{1}$ from $G$ and that $G_{1}$ is the complement of $G_{2}$. To weld together two $k$-pendants with respect to a pairing of their free edges reverses this construction.

A rather special case is the following two 5-pendants from the Petersen graph:



Figure 1
We shall call the first pendant, which is simply a tree, $L_{3}$, and the second, is its complement $U_{7}$.

The edges of the 5 -pendant $U_{7}$ naturally fall into three sets: the pair on the left, that on the right and the single free edge in the middle. We label these $I=\left\{i_{1}, i_{2}\right\}, O=\left\{o_{1}, o_{2}\right\}$ and $\{c\}$. More generally, given any cubic graph $G$, if $e_{1}=u v, e_{2}=v w \in E(G)$ are incident edges, then $e_{1}, e_{2}$, the other two edges at $u$ and $w$ and the third edge at $v$ determines a subgraph isomorphic to $L_{3}$. Excision of this pendant yields a 5 -pendant $H$ whose free edges we label $i_{1}, i_{2}, c, o_{1}, o_{2}$ as above.

Proposition 1. Let $H$ be a 5-pendant produced by excising an $L_{3}$ from an uncolourable cubic graph $G$. Suppose that
(i) $H$ is connected, and
(ii) $H$ admits a colouring $\gamma: E(H) \rightarrow \Gamma$,
then setting $\gamma(I)=\gamma\left(i_{1}\right)+\gamma\left(i_{2}\right)$ and $\gamma(O)=\gamma\left(o_{1}\right)+\gamma\left(o_{2}\right)$, we have

$$
\{\gamma(I), \gamma(O)\}=\{0, \gamma(c)\} .
$$

In particular, one of $\gamma(I), \gamma(O)$ is zero, and the other agrees with the colour at $c$.
Proof. Using the same labels $i_{1}, i_{2}, c, o_{1}, o_{2}$ on the free edges of $L_{3}$, we see that for any colouring $\sigma$ on $L_{3}$, if $\sigma(c)=x$, then $\{\sigma(I), \sigma(O)\}=\{y, z\}$. Since $G$ is not colourable, any colouring on the $L_{3}$ is incompatible with the possible colourings on its complement $H$. So if a colouring $\gamma$ on $H$ satisfies $\gamma(c)=x$, then the only possible non-zero value for $\gamma(I)$ and $\gamma(O)$ is $x=\gamma(c)$. Since $H$ is connected, $\gamma(I)+\gamma(O)=\gamma(c)$ and the result follows.
Of course, $\gamma(I)=0$ simply means that the colours on the two $i$-edges are the same. We shall call any connected, colourable 5 -pendant together with a labelling of its free edges $i_{1}, i_{2}, c, o_{1}, o_{2}$ which has the property that for any colouring, either $\gamma(I)=0$ or $\gamma(O)=0$, a colourable snark unit, or CSU. It is not apparent (but true), that there are complements of $L_{3}$ subgraphs of uncolourable graphs that remain uncolourable, we will give an example of this is Section 3 below.

It is readily checked that $U_{7}$ is colourable. From its symmetry, there is a colouring in which $\gamma(I)$ is zero, and one where it is $\gamma(c)$. We will call a CSU symmetrical if it shares this property with $U_{7}$, and give an example in Section 5 of a non-symmetrical CSU.

Welding an $L_{3}$ (or indeed any 5-pendant that admits the same colouring property) onto a CSU by identifying correspondingly labelled edges yields an uncolourable cubic graph. From $U_{7}$ this can only be the Petersen graph, so not only is $U_{7}$ the unique complement of any $L_{3}$ in the Petersen graph, it is also the CSU of least order.

If $(G, M)$ is a cubic graph together with a matching (or 1-factor), the associated 2-factor is a disjoint union of cycles. Since $|V(G)|$ is even, there is an even number of cycles of odd length among these. We will say that $G$ has a Tait number $n$ if the smallest number of cycles of odd length in any 2 -factor of $G$ is $2 n$, and write $\tau(G)=n$. In particular, $\tau(G)=0$
iff $G$ admits a colouring. A minimal matching is one that yields $2 \tau(G)$ odd cycles. Referring to the description of $L_{3}$ above for notation, we have the following characterisation.

Proposition 2. The complement $H$ of an $L_{3}$ in an uncolourable graph $G$ is a CSU if and only if $\tau(G)=1$ and $G$ has a matching in which the two odd cycles of the associated 2-factor are bridged by one of the edges $e_{1}, e_{2}$ of the $L_{3}$.

Proof. Let $\gamma$ be a colouring on $H$ in which $\gamma(c)=x$, and $\gamma(I)=0$. Up to a renaming of the $o$-edges, the only possible assignment is:

$$
\gamma(c)=x ; \quad\left(\gamma\left(i_{1}\right), \gamma\left(i_{2}\right)\right)=(\alpha, \alpha) ; \quad\left(\gamma\left(o_{1}\right), \gamma\left(o_{2}\right)\right)=(y, z),
$$

for some colour $\alpha$.
We claim that we can further assume that $\alpha=x$ : if, say $\alpha=y$, then consider the alternating $y$ - $x$ path beginning at $i_{1}$. It terminates at one of $c, o_{1}$ or $i_{2}$. It cannot end at $c$, since switching colours along such a path would give colouring $\gamma^{*}(c)=y ;\left(\gamma^{*}\left(i_{1}\right), \gamma^{*}\left(i_{2}\right)\right)=(x, y)$ and $\left(\gamma^{*}\left(o_{1}\right), \gamma^{*}\left(o_{2}\right)\right)=(y, z)$, which is inadmissible for a CSU. This path also cannot end at $o_{1}$ for the same reason, so it terminates at the other input $i_{2}$. Switching colours gives the desired result. Clearly, the same argument holds if $\alpha=z$.

We also conclude that the alternating $x-y$ path (resp. $x-z$ ) path from $c$ ends at $o_{1}$ (resp. $o_{2}$ ). With the assumptions above, we see that in $H$, there are precisely two $x-y$ paths that are not cycles of even length: that between $c$ and $o_{1}$, and the one between the edges $i_{1}$ and $i_{2}$.

Welding the $L_{3}$ and considering the matching $M$ in $H$ induced by the edges coloured $z$, we see that $M^{*}=M \cup\{P Q\}$ is a 1 -factor in $G$, as illustrated below by the bold edges:


Figure 2

The cycle on $i_{1}, i_{2}$ has odd length, as does the cycle on $c$ and $o_{1}$, and these are the only odd cycles in the 2 -factor given by this matching. Further they are bridged by matching edge $P Q$.

In the reverse direction, suppose that $\tau(G)=1$, and that for a minimal matching, the two odd cycles are bridged by an edge. This defines an $L_{3}$ by the choice of two edges from one of the cycles, and three from the second, as in the figure above.

## 3. Linear Constructions

If $G$ is a snark, then the girth condition implies that the distance (in $H$ ) between $i$-edges and that between the $c$-edge and the $o$-edges must be at least 5 . On the other hand, it is also clear that a minimal cyclic edge cut of the weld of a CSU $H$ and $L_{3}$ uses at most one (interior) edge from $L_{3}$, so this weld is cyclically 4 -edge connected provided that the $H$ is cyclically 3 -edge connected.

We shall term a CSU with these two properties proper. Evidently, welding the $L_{3}$ on a proper CSU produces a snark. By the order of a CSU, we shall mean the number of its vertices of degree 3 . Thus, $U_{7}$ has order 7 .

In view of Proposition 1, we obtain CSU's of any order $k$ provided there is a snark of order $k+3$ with $\tau=1$ which becomes colourable on removing a single edge. From [5] we have that there are no snarks of orders 12,14 or 16 , so the next order of a proper CSU after $U_{7}$ is 15 , which can be obtained from either of the two snarks of order 18.

In this Section, we show how to produce (proper) CSU's of any odd order $\geq 15$ starting from $U_{7}$. The following figure shows a CSU of order 15 created by welding two $U_{7}$ 's, introducing a new vertex on one of the interfacing edges and attaching a free edge $c$ there:


Figure 3

The $c$-edges of the two $U_{7}$ 's become edge $e_{3}$. Suppose that $\gamma(c)=x$. Then edges $\left\{\gamma\left(e_{1}\right), \gamma\left(e_{2}\right)\right\}=\{y, z\}$. From the possible colourings for a $U_{7}$ 's $o$ - and $i$ - edges, it follows that $\gamma\left(e_{4}\right)$ is either $y$ or $z$ and that $\gamma\left(e_{3}\right)=x$, so one of $\gamma(I), \gamma(O)$ is zero.

From this CSU, we can produce another of order 17 by subdividing edges $e_{1}, e_{4}$ and introducing a new edge between these new vertices. A further subdivision of $e_{2}$ and one of the new edges replacing $e_{4}$ then gives a unit of order 19.

An order 21 unit is made from three $U_{7}$ 's in sequence, with the first two welded at the $c$-edges.

The argument above shows that for any $n \geq 2$, we can produce a CSU of order $7 n+(n-1)=8 n-1$ by using a sequence of $n$ copies of $U_{7}$ linked as illustrated below for $n=4$.


Figure 4
The edges marked $x$ can be successively shown to carry this colour, starting from edge $c$.

The subdivision process permits a further $4(n-1)=4 n-4$ vertex additions. For $n \geq 3$ we have $4 n-4 \geq 8$, so that we obtain CSU's of each odd order $\geq 15$. It is clear that all these pendants are cyclically 3 -edge-connected and by the constructions, satisfy the distance condition, hence they are all proper CSU's. Welding $L_{3}$ 's on them gives an alternative construction to Isaacs' snarks of all even orders $\geq 18$ with $\tau=1$.

## 4. Cyclic Constructions

Given two CSU's $H_{1}$, $H_{2}$, let the only admissible weldings be
(1) the $o$-edges of $H_{1}$ to the $i$-edges of $H_{2}$ and
(2) the $c$-edges of the two.

Starting with $n$ (possibly different) CSU's $H_{1}, \ldots, H_{n}$, let the graph $G$ be the result of effecting a set of admissible welds between the $H_{i}$ 's.

Let $G^{*}$ be the graph whose vertices consist of
(1) a set $\left\{g_{1}, \ldots, g_{n}\right\}$ in one-to-one correspondence with the units $G_{i}$;,
(2) one vertex $f_{j}$ for each end vertex of a free edge $c_{j}$ of a unit $G_{j}$ that is not welded to any other edge;
(3) one vertex $d_{j}$ for each pair of o- or $i$ - edges of a unit $G_{j}$ that is not welded to any other pair from another unit.
The edge set of $G^{*}$ is given by: $g_{i} g_{j} \in E\left(G^{*}\right)$ iff the pendant $G_{i}$ is welded to the pendant $G_{j}$ in $G$. For all vertices $g_{j}, f_{j}$ and $d_{j}$, we also demand that $f_{j} g_{j}, d_{j} g_{j} \in E\left(G^{*}\right)$. Evidently, $G^{*}$ is a $k$-pendant for some $k \geq 0$.

We also equip $G^{*}$ with a weight function $w: E\left(G^{*}\right) \rightarrow\{1,2\}$ defined by

$$
\begin{aligned}
& w\left(g_{i} g_{j}\right)=\text { the number of edges welded between } G_{i}, G_{j} ; \\
& w\left(f_{j} g_{j}\right)=1 ; \\
& w\left(d_{j} g_{j}\right)=2 \text { for each relevant index } j .
\end{aligned}
$$

In contrast to the weight functions in [2], this weight function has an odd sum at each vertex of degree 3 .

If $G$ has a colouring $\gamma$, there is an induced flow $\gamma^{*}$ on $G^{*}$ given by taking the sum of the colours on the edges in $G$ that the edge in $G^{*}$ arises from: this flow is zero on edges arising from pairs with the same colour.

If we take $n$ copies of $U_{7}$ and weld then together in a cycle, so that the edge $o_{1}, o_{2}$ of the $k$-th CSU are welded to the edges $i_{1}, i_{2}$ of the $(k+1)$ st unit, and the $o$-edges of the $n$-th unit are welded to the $i$-edges of the first, we obtain an $n$-pendant $U_{7}^{n}$ with free edges $c_{1}, c_{2}, \ldots, c_{n}$. These are all admissible welds.

Proposition 3. If $n \geq 3$ is odd, then $U_{7}^{n}$ is uncolourable.
Proof. Taking $G=U_{7}^{n}$, we see that $G^{*}$ is an $n$-pendant consisting of a free edge at each vertex of an $n$-cycle. Its weight function takes the value 2 around the cycle, and 1 on the free edges. If $G$ was coloured, then for each $U_{7}$, exactly one of its $o-/ i$ - pair of edges receives a colour sum 0 . Thus $\gamma^{*}(e)=0$ for a set of edges in $G^{*}$ that forms a matching of the vertices in the cycle. Since this is only possible if the cycle has even length, $G$ is uncolourable.
For example, welding an $L_{3}$ on $U_{7}^{5}$ gives an example of a snark where the excision of this $L_{3}$ does not produce a CSU.

In the construction above, if we identify the three end vertices of $U_{7}^{3}$ together, we obtain the following uncolourable graph $G_{3}$ on 22 vertices, for which $G_{3}^{*}=K_{4}$ (see Figure 5).


Figure 5
$G_{3}$ has girth 5 , is cyclically 5 -edge connected and hence is a snark.
We can turn this construction around. Starting with any cubic graph $G$ and a (1,2)-weight function $w$ such that the sum over the edges incident at any vertex is odd, the set of edges where $w(e)=2$ forms a disjoint union of cycles $\Sigma$. Replacing each such cycle with a cycle of CSU's as above produces a graph $H$ such that $H^{*}=G$. By Proposition 3, if at least one of the cycles in $\Sigma$ has odd length, $H$ is uncolourable.

As with Isaacs' $J$ class, $G_{3}$ is the smallest in an infinite family $\left\{G_{2 k+1}\right\}$, with members produced from the prisms on $2 k+1$-cycles together with the weight function that is 2 only on one of the copies of this cycle. All are cyclically 5-edge connected. A more general method is to take an existing snark together with any 1-factor. This matching determines the edges assigned weight 2 from which a snark may be built.

## 5. Asymmetric CSU's

The constructions in the last section carry over if we replace the unit $U_{7}$ by any symmetric one. It is not hard to characterise symmetric units in terms of the possible alternating paths between the $i$ - / o- edges. The following is an example of a unit that is not symmetric: the pair of edges that accepts a sum flow of zero is distinguished.

The first diagram in the following figure shows $G_{5}$, the snark on 40 vertices of the family mentioned in Section 3.

The bold subgraph in that diagram shows an $L_{3}$ whose complement is a CSU we will call $U_{37}$ : the second diagram is of the associated colouring $\gamma^{*}$ on
the prism $G_{5}^{*}$, with the corresponding $L_{3}$ excised. The bold edge indicates the two input edges $i_{1}, i_{2}$. The edges labelled 0 are those where the sum flow on the corresponding $U_{7}$ 's $i$-/ o- pair of edges is zero. In essence, this is a zero flow at the 'double edge' given by the two inputs. It is easily checked that this colouring extends to $U_{37}$.


Figure 6
To show that $U_{37}$ is not symmetrical, it suffices to note that setting $\gamma\left(i_{1}\right)=y$, $\gamma\left(i_{2}\right)=z$ (while fixing $\gamma(c)=x$ ) would permit an extension of $\gamma$ to a colouring of the outer cycle $U_{7}^{5}$, which is not possible. So only $\gamma(I)=0$ is possible.

## 6. Tait Number One

In this Section, we give constructions of snarks with $\tau=1$ that have the property that in a minimal matching, the two cycles of odd length are not bridged by an edge. For $n \geq 2$, we denote by $L_{n}$ the tree with the $2 n+2$ vertices in the set $\left\{v_{1}, v_{2}, \ldots, v_{n}, w_{1}, w_{2}, \ldots, w_{n}, z_{1}, z_{n}\right\}$. The edge set consists of $v_{i} w_{i}, i=1, \ldots, n$ together with $v_{1} z_{1}, v_{n} z_{n}$ and $v_{i} v_{i+1}$ for $1 \leq i \leq n-1$. So $L_{n}$ is an $(n+2)$-pendant: the 5 -pendant $L_{3}$ is then an example.

Let $G$ be the cubic graph that is the weld of the 6 -pendant $H$ on the left and the $L_{4}$ beside it in the following figure. The boxes represent two copies of the 'directed' CSU in Section 5. The arc within each box indicates
the distinguished pair of edges that always carry the same colour in any colouring of the unit. In essence, there is no longer a distinction between the unit's $o$-edges and the $c$-edge.


Figure 7

The welding is along equivalently labelled edges. In view of the property of the directed unit, the following is readily checked by considering alternating paths.

Claim 1. The 6-pendant $H$ is colourable, and in every such colouring $\gamma$, we have $\gamma\left(i_{1}\right)=\gamma\left(i_{2}\right), \gamma\left(o_{1}\right)=\gamma\left(o_{2}\right)$ and $\gamma\left(c_{1}\right)=\gamma\left(c_{2}\right)$. Consequently, $G$ is uncolourable.
In fact, the colours at the $o-, i$ - and $c$-edges of $H$ can be arbitrarily assigned, so may be taken to be all equal to $x$.

Claim 2. $\tau(G)=1$. In any minimal matching, the two odd cycles are linked by a path of length three.

Colouring all the pendants on $H$ by $x$, let $M$ be the matching of the $z$ coloured edges (shown above as the bold edges). This can be completed to a matching $M^{*}=M \cup\left\{f_{1}, f_{3}\right\}$ of $G$ in which the only odd cycles are the two $x-y$ paths between $i_{1}$ and $i_{2}$ and that between $o_{1}$ and $o_{2}$. The path between $c_{1}$ and $c_{2}$ together with edge $f_{2}$ becomes a cycle of even length. This is clearly a minimal matching with the requisite property.

By Proposition 2, we need only show that there is no edge whose removal leaves a colourable graph. We concentrate on the two units numbered 1
and 2. Removal of any one of $f_{1}, f_{2}, f_{3}$ produces a subgraph with either or both of these two units intact, hence uncolourable. The same holds if we remove an edge within either of the units: the other remains unaffected.

The remaining case is of the other edges in $H$. With respect to unit 1 , it is readily checked that removal of any of $e_{1}, e_{2}$ or $e_{3}$ does not affect unit 2, unless it is together with one of $e_{1}^{\prime}, e_{2}^{\prime}, e_{3}^{\prime}$. In any event, the shortest path between these edges is evidently 3 . No other single edge will do. Notice that the resulting graphs are cyclically 4 -edge-connected snarks.

We close this Section by showing that, perhaps not surprisingly, 'anything goes'. If a cubic graph is uncolourable with Tait number one, then either it becomes colourable on removal of exactly one edge, as covered in Proposition 2, or else removal of two edges suffices. In this latter case, among all matchings, there is one where the distance between the two odd cycles, measured by the length of a path from a vertex on one to a vertex on the other, is minimised.

Proposition 4. Snarks with Tait number 1 exist for which the distance between the two odd cycles is any integer $\geq 3$.

Proof. Since the first and last edge of such a minimal length path must belong to the matching, the distance cannot be 2 . The example in figure 7 has distance 3 . An example with distance 4 (resp. 5) is produced by excising the two numbered CSU's and replacing the resulting subgraph by the pendants in the figure below, and the $L_{4}$ by an $L_{5}$ (resp. $L_{6}$ )


Figure 8

These pendants illustrate the difference between even and odd distance cases. If the bold edges are coloured $z$, there is a unique way of ensuring that edges $e_{1}, e_{2}, e_{3}$ (and edges $e_{1}^{\prime}, e_{2}^{\prime}, e_{3}^{\prime}$ ) have a sum flow of zero. In the even cases, $e_{1}$ and $e_{1}^{\prime}$ are both coloured $z$, and $c_{n}$ is the edge adjacent to $e_{1}^{\prime}$; in the odd case, it is the edge adjacent to $e_{3}^{\prime}$ that is labelled $c_{n}$. The edges $c_{2}, c_{3}, \ldots, c_{n-1}$ are all coloured $z$, and assigned arbitrarily.

The labelling of the $c$-edges is the same as on the corresponding $L_{5}$ and $L_{6}$ pendants, on which edges $f_{1}, f_{4}$ (resp. $f_{5}$ ) are given colour $z$. The rest of the edges lie on a single alternating $x-y$ path between $c_{1}$ and $c_{3}$ (resp. $c_{4}$ ). The assertion about distance and the fact that these are all snarks should be clear.

## References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (American Elsevier, New York, 1976).
[2] B. Jackson, On cycle Covers, cycle Decompositions and Euler Tours of Graphs, Preprint (1993).
[3] F. Jaeger, Nowhere-zero Flow Problems, in: Graph Theory 3, edited by L.W, Beincke and R.J. Wilson (Academic Press Ltd., New York, 1988).
[4] R. Isaacs, Infinite families of non-trivial trivalent graphs which are not Tait colorable, Amer. Math. Monthly 82 (1975) 221-239.
[5] J.J. Watkins and R.J. Wilson, A Survey of snarks, in: Graph Theory, Combinatorics and Applications, Vol. 2, Proceedings of the Sixth Quadrennial International Conference on the Theory and Applications of Graphs, Y. Alavi et. al. (eds) (John Wiley \& Sons, 1991) 1129-1144.

