# AN INEQUALITY CHAIN OF DOMINATION <br> PARAMETERS FOR TREES 

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#### Abstract

We prove that the smallest cardinality of a maximal packing in any tree is at most the cardinality of an $R$-annihilated set. As a corollary to this result we point out that a set of parameters of trees involving packing, perfect neighbourhood, $R$-annihilated, irredundant and dominating sets is totally ordered. The class of trees for which all these parameters are equal is described and we give an example of a tree in which most of them are distinct.


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## 1. Introduction

This paper is concerned with the relative values of certain graph parameters for trees. These parameters involve several types of vertex subsets $X$ of a simple graph $G$, including dominating, irredundant, packing, perfect
neighbourhood and $R$-annihilated subsets. Our first task is the definition of such sets and to observe that each may be characterized in terms of a certain partition of the vertex set $V$ of $G$ induced by $X$.

We denote by $N(X)$ ( $N[X]$ ) the open (closed) neighbourhood of the set $X$. As usual $N(\{x\})$ and $N[\{x\}]$ will be abbreviated to $N(x)$ and $N[x]$. For $A, B \subseteq V$, we say that $A$ dominates $B$, written $A \succ B$, (or $B$ is dominated by $A$ ) if $B \subseteq N[A]$.

The private neighbourhood $p n(x, X)$ of $x$ in $X$ is defined by

$$
p n(x, X)=N[x]-N[X-\{x\}] .
$$

An element $u$ of $p n(x, X)$ is called a private neighbour of $x$ relative to $X$ and is one of two types. Either $u$ is an isolate of $G[X]$, in which case $u=x$, or $u \in V-X$ and is adjacent to precisely one vertex of $X$. The latter type is called an external private neighbour (epn) of $X$.

The concept of private neighbourhood enables us to define from $X$, a partition $\mathcal{P}(X)=Z_{X} \cup Y_{X} \cup E_{X} \cup F_{X} \cup C_{X} \cup R_{X}$ (disjoint union) of $V$, where:

$$
\begin{aligned}
Z_{X} & =\{x \in X \mid x \text { is isolated in } G[X]\}, \\
Y_{X} & =X-Z_{X}, \\
E_{X} & =\left\{v \in V-X \mid v \text { is an epn of some } y \in Y_{X}\right\}, \\
F_{X} & =\left\{v \in V-X \mid v \text { is an epn of some } z \in Z_{X}\right\}, \\
C_{X} & =\{v \in V-X| | N(v) \cap X \mid \geq 2\}, \\
\text { and } \quad R_{X} & =V-N[X] .
\end{aligned}
$$

When the basic subset $X$ is clear from the context, we will omit the subscripts $X$.

We now give the definition of four of the above mentioned types of vertex subsets $X$ and the characterization of each in terms of $\mathcal{P}(X)$.
$X$ is dominating if $N[X]=V$ (i.e., if $R=\emptyset$ ); $X$ is irredundant if for all $x \in X, p n(x, X) \neq \emptyset$ (i.e. if $E \cap N(y) \neq \emptyset$ for each $y \in Y$ ); $X$ is a packing if for all distinct $x_{1}, x_{2} \in X, N\left[x_{1}\right] \cap N\left[x_{2}\right]=\emptyset$ (i.e., $C \cup Y=\emptyset$ ), and $X$ is a perfect neighbourhood set (abbr. PN-set) if $\phi(X)=\bigcup_{x \in X} p n(x, X) \succ V$ (i.e., if $Z \cup E \cup F \succ V)$. A vertex $v$ is called an $X$-perfect vertex if $v \in Z \cup E \cup F$.

In order to motivate the definition of the next principal property, we first state a condition given in [3] for an irredundant set to be maximal. We need one additional concept about private neighbourhoods. For $x \in X$ and
$u \in V-X, u$ annihilates $x$ (or $x$ is annihilated by $u$ ) if $\emptyset \neq p n(x, X) \subseteq N[u]$. Observe that if $u$ annihilates $x$, then $p n(x, X \cup\{u\})=\emptyset$, i.e., (informally) addition of $u$ to $X$ destroys (or annihilates) the private neighbourhood of $x$. Let

$$
A_{X}=\{u \in V-X \mid u \text { annihilates some } x \in X\} .
$$

We write $A$ for $A_{X}$, if the basic subset $X$ is clear. For $U \subseteq V-X$, define $X$ to be $U$-annihilated if $U \subseteq A$. We can now state a condition for an irredundant set $X$ to be maximal in terms of the partition $\mathcal{P}(X)$.

Theorem 1. [3] The set $X$ is maximal irredundant if and only if $X$ is irredundant and $N[R]$-annihilated.

We observe that the class of $N[R]$-annihilated sets (such sets have also been called external redundant sets $([3,4]))$ is contained in the larger class of $R$ annihilated sets (abbreviated Ra-sets), which is a class of sets of principal interest in this work. We will also consider sets which are both $R$-annihilated and irredundant, that is, Rai-sets.

Notice that for each $z \in Z$ and $r \in R, z \in p n(z, X)-N[r]$ and so $z$ is not annihilated by $r$. Thus any vertex of $X$ which is annihilated by $r \in R$, is necessarily in $Y$.

The parameters considered in this paper are $\gamma(G), i(G), \theta(G), \theta_{i}(G)$, $r a(G), \operatorname{rai}(G), \operatorname{er}(G)$ and $\operatorname{ir}(G)$, which are the smallest cardinalities of dominating sets, independent dominating sets, PN -sets, independent PN -sets, Ra-sets, Rai-sets, external redundant and maximal irredundant sets, respectively; $\rho_{L}(G)(\rho(G))$ which is the smallest (largest) cardinality of a maximal packing and $\gamma_{2}(G)$ which is the smallest cardinality of $X$ such that each vertex of $V$ is within distance two of $X$, i.e., such that $X 2$-dominates $G$.

We abbreviate $\gamma(G)$ to $\gamma$ etc. when the graph $G$ involved is clear. Further, for example, a dominating set (maximal irredundant set) of minimum cardinality $\gamma(G)(i r(G))$ will be called a $\gamma$-set (an $i r$-set).

The following inequalities are immediately implied by the definitions, Theorem 1 and the well-known inequalities $i r \leq \gamma \leq i$.

Proposition 2. For any graph $G$,

$$
\gamma_{2} \leq\left\{\begin{array}{c}
r a \leq\left\{\begin{array}{c}
r a i \\
e r
\end{array}\right\} \leq i r \\
\theta \leq \theta_{i} \leq \rho_{L} \leq \rho
\end{array}\right\} \leq \gamma \leq i
$$

In Section 2 we prove our principal result, namely that for any tree $T$, $\rho_{L}(T) \leq r a(T)$ and hence establish a longer total order for trees than those given in Proposition 2 for general graphs. The trees for which all parameters in the total order are equal, are presented in Section 3 and finally an example with most of the parameters unequal, is given in Section 4.

This research evolved from attempts to prove the conjecture of Fricke, Haynes, Hedetniemi and Henning [8] that $\theta \leq i r$ for arbitrary graphs. This was shown to be false by Favaron and Puech [7]. However, Cockayne, Hedetniemi, Hedetniemi and Mynhardt [5] and Cockayne and Mynhardt [6] established the inequality for trees and claw-free graphs, respectively. Favaron and Puech [7,12] observed that the proof for trees actually establishes the stronger result $\theta_{i} \leq r a i($ see Proposition 2) and found other classes of graphs (including claw-free graphs and chordal graphs) for which this latter inequality holds.

In [2] the present authors observed that some other known results concerning the parameter $i r$ may be strengthened to theorems concerning $R$ annihilation. In the same paper other classes of graphs (defined by degree conditions) for which $\theta_{i} \leq r a i$, were found. It was further proved that for some of these classes, the even stronger inequality (see Proposition 2) $\rho_{L} \leq$ rai holds. This was the motivation for the main theorem in the present paper.

An area of research that has received considerable attention is the study of classes of graphs for which some of the above-mentioned parameters are equal (or not equal). For any two graph theoretical parameters $\lambda$ and $\mu$, we define $G$ to be a $(\lambda, \mu)$-graph if $\lambda(G)=\mu(G)$ and a $(\lambda, \mu)$-tree if, in addition, $G$ is a tree. In general, if $\lambda$ and $\mu$ are domination parameters, the class of $(\lambda, \mu)$-graphs is very difficult to characterise. For trees, however, some success has been achieved. For example, $(\gamma, i)$-trees were characterised by the present authors in [1]. More relevant to the present paper, Meir and Moon [11] showed that $\rho=\gamma$ for all trees, Hartnell [9] characterised ( $\rho_{L}, \rho$ )trees while Topp and Volkmann [13] characterised $\left(\gamma_{2}, \rho\right)$-trees. (They actually proved a more general result but we only mention the relevant part here.) In Section 3 we show that the two classes of trees described by Hartnell and Topp and Volkmann are the same and that it is in fact precisely the class of $\left(\gamma_{2}, i\right)$-trees. We also give a different description of these trees.

References to further work on domination, irredundance and packing may be found in the comprehensive bibliography of the book by Haynes, Hedetniemi and Slater [10].

## 2. The Main Result

In order to prove the principal theorem, further notation and structures are now defined. Let $X$ be an Ra-set of any graph $G$. For $x \in X, B_{x}$ denotes the set of epns of $x$ and $B=\bigcup_{x \in X} B_{x}(=E \cup F)$.

We now define a partition of $X \cup B \cup R$ into exactly $|X|$ non-empty subsets. With each $x \in X$ we associate a subset $R_{x}$ of $R$ sequentially. Suppose $X=\left\{x_{1}, \ldots, x_{p}\right\}$. Let

$$
R_{x_{1}}=\left\{r \in R \mid r \text { annihilates } x_{1}\right\}
$$

and for $j=2, \ldots, p$, let

$$
R_{x_{j}}=\left\{r \in R-\bigcup_{k=1}^{j-1} R_{x_{k}} \mid r \text { annihilates } x_{j}\right\} .
$$

Observe that the $R_{x_{j}}$ 's are disjoint and that the $R$-annihilation property implies that $R_{x}=\emptyset$ for $x \in Z$ and $\bigcup_{x \in X} R_{x}=R$. It now follows that

$$
X \cup B \cup R=\bigcup_{x \in X}\left(\{x\} \cup B_{x} \cup R_{x}\right) \quad \text { (disjoint union), }
$$

which defines the required partition.
Next, for $x \in X$ let $D_{x}=\{x\} \cup B_{x} \cup R_{x}$, for each component $Q$ of $G[X]$ let $D_{Q}=\bigcup_{x \in Q} D_{x}$ and observe that $Q \neq Q^{\prime}$ implies that $D_{Q} \cap D_{Q^{\prime}}=\emptyset$. From this point onwards $G$ is a tree $T$. Contract each set $D_{Q}$ to a single vertex $d_{Q}$. This forms a tree $f(T)$ with vertex set $C \cup\left(\bigcup_{\text {components of } G[X]}\left\{d_{Q}\right\}\right)$. Each leaf of $f(T)$ is a vertex $d_{Q}$, since each $c \in C$ is adjacent in $T$ to at least two sets $D_{Q}$ (by the tree property).

Having defined the above structures, we now prove the following important preliminary result.

Theorem 3. For each Ra-set $X$ of a tree $T$, there exists a maximal packing $P$ of $T$ such that $P \cap C_{X}=\emptyset$.

Proof. We use induction on $n$, the order of $T$, and observe that the statement holds for any tree $T$ and Ra-set $X$ for which $C_{X}=\emptyset$, which includes all Ra-sets of $K_{1}$ and $K_{2}$.

Now suppose that the conclusion holds for all trees of order less than $n$ and let $X$ be an Ra-set of an $n$-vertex tree $T$ with $C=C_{X} \neq \emptyset$. The contracted tree $f(T)$ has a leaf $l$ such that at most one vertex at distance two from $l$ is not a leaf (e.g., let $l$ be an endvertex of a longest path of $f(T)$ ). Let $Q$ be the component of $T$ such that $d_{Q}=l$. In $T$ the set $D_{Q}$ is linked to $V-D_{Q}$ by exactly one edge $u t$ with $u \in D_{Q}$ and $t=V-D_{Q}$. There are now two cases to consider.

Case 1. $t \notin C$.
Consider $T^{\prime}=T\left[V^{\prime}\right]$ where $V^{\prime}=V-D_{Q}$. The set $X \cap V^{\prime}$ is an $R$-annihilated set of the tree $T^{\prime}$ and the set $C^{\prime} \subseteq V^{\prime}$ of vertices adjacent in $T^{\prime}$ to at least two vertices of $X \cap V^{\prime}$, is equal to $C$. By the induction hypothesis, there exists a maximal packing $P^{\prime}$ of $T^{\prime}$ with $P^{\prime} \cap C^{\prime}=P^{\prime} \cap C=\emptyset$. Let $w$ be a vertex of $P^{\prime}$ whose distance $d(w, t)$ is minimum and observe that the maximality of $P^{\prime}$ implies that $0 \leq d(w, t) \leq 2$. The required packing $P$ is now formed in one of the following ways. Let $N_{2}[w]$ denote the set of all vertices at distance at most two from $w$.
(i) If $D_{Q}-N_{2}[w]=\emptyset$, then $P=P^{\prime}$.

Otherwise:
(ii) If $d(w, t)=2$, then let $P^{\prime \prime}$ be a maximal packing of $T\left[D_{Q}\right]$ which contains $u$ and set $P=P^{\prime} \cup P^{\prime \prime}$.
(iii) If $d(w, t)=1$, then let $P^{\prime \prime}$ be a maximal packing of $T\left[D_{Q}\right]$ which does not contain $u$ (e.g., which contains a neighbour of $u$ in $D_{Q}$ ) and set $P=P^{\prime} \cup P^{\prime \prime}$.
(iv) If $d(w, t)=0$ (i.e., $w=t$ ), then let $P^{\prime \prime}$ be a maximal packing of $T\left[D_{Q}-N_{2}[w]\right]$ and set $P=P^{\prime} \cup P^{\prime \prime}$.
In each of these four situations, $P$ is a maximal packing of $T$ with $P \cap C=\emptyset$.
Case 2. $t \in C$.
Let $k+1(\geq 2)$ be the degree of $t$ in $f(T)$ and let $d_{Q_{j}}, j=1, \ldots, k$ be leaves of $f(T)$ adjacent to $t$, where $d_{Q_{j}}$ is the contraction of $D_{Q_{j}}$ in $T$. The $k$ subscripts $j$ are chosen so that $\left|D_{Q_{j}}\right|>1$ for $1 \leq j \leq s$ and $\left|D_{Q_{j}}\right|=1$ for $s+1 \leq j \leq k$, where $s$ is possibly equal to 0 or to $k$. For $1 \leq j \leq k$, let $u_{j}$ be the (unique) neighbour of $t$ in $Q_{j}$ and if $s>0$, then for $1 \leq j \leq s$, let $v_{j}$ be a vertex of $D_{Q_{j}}$ at distance two from $t$.

The set $V^{\prime}=V-\left(\bigcup_{j=1}^{k} D_{Q_{j}} \cup\{t\}\right)$ induces a subtree $T^{\prime}$ of $T$. The set $X^{\prime}=X \cap V^{\prime}$ is an Ra-set of $T^{\prime}$ and the set $C^{\prime} \subseteq V^{\prime}$ of vertices adjacent in
$T^{\prime}$ to at least two vertices $X \cap V^{\prime}$, is equal to $C-\{t\}$. By the induction hypothesis, $T^{\prime}$ has a maximal packing $P^{\prime}$ such that $P^{\prime} \cap C^{\prime}=P^{\prime} \cap C=\emptyset$.

Let $w \in P^{\prime}$ such that $d(w, t)$ is minimum. By the maximality of $P^{\prime}$, $1 \leq d(w, t) \leq 3$. There are now two subcases to consider.

Subcase (i). $2 \leq d(w, t) \leq 3$.
Let $P_{1}$ be a maximal packing of $D_{Q_{1}}$ containing $u_{1}$ and when $s \geq 2$, for $2 \leq$ $j \leq s$ let $P_{j}$ be a maximal packing of $D_{Q_{j}}$ containing $v_{j}$. Then $P^{\prime \prime}=\bigcup_{j=1}^{s} P_{j}$ is a maximal packing of $T\left[\bigcup_{j=1}^{k} D_{Q_{j}} \cup\{t\}\right]$ which does not contain $t$. Set $P=P^{\prime} \cup P^{\prime \prime}$.

Subcase (ii). $d(w, t)=1$.
If $s=0$, then let $P=P^{\prime}$. If $s>0$, then let $P_{j}$ be a maximal packing of $T\left[D_{Q_{j}}\right]$ containing $v_{j}$ for $1 \leq j \leq s$. Then $P^{\prime \prime}=\bigcup_{j=1}^{s} P_{j}$ is a maximal packing of $T\left[\bigcup_{j=1}^{k} D_{Q_{j}} \cup\{t\}\right]$ not dominating $t$. Set $P=P^{\prime} \cup P^{\prime \prime}$.

In each situation in the subcases, the constructed set $P$ is a maximal packing of $T$ such that $P \cap C=\emptyset$. The induction proof is complete.

Theorem 4. For any tree, $\rho_{L} \leq r a$.
Proof. Let $X$ be an $r a$-set of a tree $T$ and $P$ a maximal packing of $T$ whose existence is guaranteed by Theorem 3, i.e., such that $P \cap C=\emptyset$. The $R$-annihilation property implies that for each $x \in X$ with $R_{x} \neq \emptyset$, every vertex of $R_{x}$ dominates $B_{x}$. We deduce that for each $x \in X, T\left[D_{x}\right]$ has diameter at most two and so $\left|P \cap D_{x}\right| \leq 1$. Therefore

$$
\rho_{L} \leq|P|=|P \cap C|+\left|P \cap\left(\bigcup_{x \in X} D_{x}\right)\right| \leq|X|=r a .
$$

Corollary 5. For any tree,

$$
\gamma_{2} \leq \theta \leq \theta_{i} \leq \rho_{L} \leq r a \leq\left\{\begin{array}{c}
r a i \\
e r
\end{array}\right\} \leq i r \leq \gamma=\rho \leq i .
$$

Proof. Immediate from Proposition 2, Theorem 4 and the result of Meir and Moon [11] that $\rho=\gamma$ for trees.

## 3. The Class of $\left(\gamma_{2}, i\right)$-Trees

A vertex of a tree $T$ is said to be remote if it is adjacent to a leaf, and to be a branch vertex if it has degree at least three. The set of branch vertices of $T$ is denoted by $B(T)$ and the set of leaves by $L(T)$. A path $P$ in $T$ is said to be a $v-L$ path if $P$ is a path from $v$ to a leaf of $T$. With every branch vertex $v$ of $T$ of degree $d$ and $N(v)=\left\{x_{1}, x_{2}, \ldots, x_{d}\right\}$ we associate $d$ integers $l_{1}, l_{2}, \ldots l_{d}$, where $l_{j}$ is the length of a shortest $v-L$ path containing $x_{j}$. Without loss of generality we assume that $l_{1} \leq l_{2} \leq \ldots \leq l_{d}$. We define the following types of branch vertices:
type 1: $l_{1}=1$ and $l_{j} \in\{1,4\}$ for each $j=2, \ldots, d$,
type 2: $l_{1}=2$ and $l_{j}=3$ for each $j=2, \ldots, d$.
We characterise $\left(\gamma_{2}, i\right)$-trees in terms of branch vertices of types 1 and 2 . We begin with the following characterisation of $\left(\rho_{L}, \rho\right)$-trees by Hartnell [9].

Theorem 6 [9]. A tree with at least three vertices is a $\left(\rho_{L}, \rho\right)$-tree if and only if no two remote vertices are adjacent and every non-remote vertex is adjacent to exactly one remote vertex.

A special case of a more general result by Topp and Volkmann [13] provides a characterisation of $\left(\gamma_{2}, \rho\right)$-trees:

Theorem 7 [13]. A tree $T$ satisfies $\gamma_{2}(T)=\rho(T)=n$ if and only if either
(1) $T$ has diameter at most two (in which case $\gamma_{2}(T)=\rho(T)=1$ )
or
(2) there exists a decomposition (partition of the vertex set) of $T$ into $n$ subgraphs $T_{1}, \ldots, T_{n}$ in such a way that:
(a) $T_{j}$ is a tree of diameter two $(j=1, \ldots, n)$, and
(b) if $T_{0}$ is the subgraph of $T$ induced by the edges which do not belong to $T_{1}, \ldots, T_{n}$, then for each $j \in\{1, \ldots, n\}$ there exists $u_{j} \in V\left(T_{j}\right)-$ $V\left(T_{0}\right)$ such that $d_{T}\left(u_{j}, V\left(T_{0}\right)\right)=2$.
Since any $\left(\gamma_{2}, \rho\right)$-tree is a $\left(\rho_{L}, \rho\right)$-tree, if $\mathcal{T}_{2}^{\prime}\left(\mathcal{T}_{3}\right.$, respectively) is the class of trees described in Theorem 6 (Theorem 7), then $\mathcal{T}_{3}-\left\{P_{1}, P_{2}\right\} \subseteq \mathcal{T}_{2}^{\prime}$. We show that $\mathcal{T}_{2}=\mathcal{T}_{2}^{\prime} \cup\left\{P_{1}, P_{2}\right\}$ and $\mathcal{T}_{3}$ are equal to the following class of trees, for which membership is decided only by properties of branch vertices. Let $T \in \mathcal{T}_{1}$ if and only if $T \in\left\{P_{1}, P_{2}, P_{3}, P_{6}\right\}$ or $B(T) \neq \emptyset$ and each branch vertex of $T$ is of type 1 or type 2. (See Figure 1, in which $u_{j}$ is of type 1 and $v_{j}$ of type 2 for $j=1,2,3$.)


Figure 1. A tree in $\mathcal{T}_{1}$
Theorem 8. Let $T$ be a tree. The following conditions are equivalent:
(a) $T \in \mathcal{T}_{1}$.
(b) $T$ is a $\left(\rho_{L}, \rho\right)$-tree.
(c) $T$ is a $\left(\gamma_{2}, \rho\right)$-tree.
(d) $T$ is a $\left(\gamma_{2}, i\right)$-tree.

Proof. Since the theorem obviously holds for $P_{1}$ and $P_{2}$ we only consider trees with at least three vertices.
(a) $\Longrightarrow(\mathrm{b})$. Let $T \in \mathcal{T}_{1}$. If $T$ is a path (i.e., $\left.T \in\left\{P_{3}, P_{6}\right\}\right)$ the result is easy to check, so suppose $B(T) \neq \emptyset$ and let $u_{1}, u_{2}$ be two remote vertices of $T$. Suppose contrary to the statement of Theorem 6 that $u_{1}$ and $u_{2}$ are adjacent, with $v_{j}(j=1,2)$ a leaf adjacent to $u_{j}$. If $\left\{u_{1}, u_{2}\right\} \cap B(T)=\emptyset$, then $T \cong P_{4}$, a contradiction. Hence we may assume that $u_{1} \in B(T)$. But $d\left(u_{1}, v_{1}\right)=1$ and $d\left(u_{1}, v_{2}\right)=2$ so that $l_{1}\left(u_{1}\right)=1$ and $l_{j}\left(u_{1}\right)=2$ for some $j$ and therefore $u_{1}$ is not a type 1 or a type 2 branch vertex. Thus $T$ contains no adjacent remote vertices. Next, let $u$ be a non-remote vertex. If $u$ is a leaf then obviously $u$ is adjacent to exactly one remote vertex, so assume that $\operatorname{deg} u \geq 2$. If $u \in B(T)$, then $u$ is of type 2 (since $u$ is not remote) and thus $u$ is adjacent to exactly one remote vertex as required. Hence suppose $\operatorname{deg} u=2$. If $u$ is adjacent to two remote vertices $u_{1}$ and $u_{2}$, and $\left\{u_{1}, u_{2}\right\} \cap B(T)=\emptyset$, then $T \cong P_{5}$, a contradiction. On the other hand, if (say) $u_{1} \in B(T)$, then $l_{1}\left(u_{1}\right)=1$ and (if $v_{2}$ is a leaf adjacent to $u_{2}$ ) $d\left(u_{1}, v_{2}\right)=3$, i.e., $l_{j}\left(u_{1}\right)=3$ for some $j$ with $2 \leq j \leq \operatorname{deg} u_{1}$, a contradiction. Hence $u$ is adjacent to at most one remote vertex. If $u$ is not adjacent to any remote vertex, let $N(u)=\left\{x_{1}, x_{2}\right\}$ and let $u_{1}$ (say) be a branch vertex
at minimum distance from $u$ such that $x_{1}$ lies on the $u-u_{1}$ path in $T$, while $u_{2}$ is a remote vertex at minimum distance from $u$ with $x_{2}$ on the $u-u_{2}$ path. Note that $d\left(u, u_{2}\right) \geq 2$ and either $u_{1}=x_{1}$ with $u_{1}$ of type 2 (since $u_{1}$ is not a remote vertex by hypothesis), or $d\left(u, u_{1}\right) \geq 2$. In the former case the shortest $u_{1}-L$ path in $T$ through $u$ contains $u_{2}$ and has length at least four, contradicting $u_{1}$ being of type 2 . In the latter case, if $w$ is the neighbour of $u_{1}$ on the $u_{1}-u$ path, then the shortest $u_{1}-L$ path through $w$ contains both $u$ and $u_{2}$ and hence has length at least five, which is also impossible. Therefore $u$ is adjacent to exactly one remote vertex and we have proved that $\mathcal{T}_{1} \subseteq \mathcal{T}_{2}$.
(b) $\Longrightarrow$ (a). Let $T \in \mathcal{T}_{2}^{\prime}$ and note that if $T$ is a path, then $T \cong P_{3}$ or $P_{6}$ and the result holds. We may thus assume that $B(T) \neq \emptyset$. Let $u$ be any vertex in $B(T)$ and suppose firstly that $u$ is remote. Let $N(u)=\left\{x_{1}, \ldots, x_{d}\right\}$ with $d=\operatorname{deg} u \geq 3$. Without loss of generality we may assume that $x_{1}, \ldots, x_{t}$ are leaves and $x_{t+1}, \ldots, x_{d}$ are non-leaves, for some $1 \leq t \leq d$. If $t=d$, then $u$ is of type 1 and we are done, so suppose $t<d$. Then for any $s$ with $t<s \leq d$, $\operatorname{deg} x_{s} \geq 2$ and by hypothesis, $x_{s}$ is not remote. For any $w \in N\left(x_{s}\right)-\{u\}, w$ is not remote (since the neighbour $u$ of $x_{s}$ is remote) and there exists $v \in N(w)-\left\{x_{s}\right\}$ such that $v$ is remote. But then the length of the shortest $u-L$ path through $x_{s}$ is equal to four and it follows that $u$ is of type 1 . Now suppose that $u$ is not remote. Then $u$ is adjacent to exactly one remote vertex, say $x_{1}$, so that $l_{1}(u)=2$. For any $j=\{2, \ldots, d\}, x_{j}$ is not remote and $N\left(x_{j}\right)-\{u\}$ contains a remote vertex $w$. Hence the length of a shortest $u-L$ path through $x_{j}$ is equal to three and so $u$ is of type 2 . We have thus shown that $\mathcal{T}_{1}=\mathcal{T}_{2}$.
(b) $\Longrightarrow$ (c). Consider any $T \in \mathcal{T}_{2}^{\prime}$ and denote the remote vertices of $T$ by $M$. Then $\bigcup_{m \in M} N[m]$ is a partition of $V(T)$ since no two remote vertices of $T$ are adjacent and each non-remote vertex is adjacent to exactly one remote vertex. Let $T_{m}$ be the subtree of $T$ induced by $N[m]$. Clearly, $\operatorname{diam}\left(T_{m}\right)=2$. Further, any edge joining a vertex of $T_{m}$ to a vertex of $T_{s}$, $m \neq s$, joins a non-remote vertex of $T$ to another non-remote vertex. Let $T_{0}$ be the subgraph of $T$ induced by these edges and let $v$ be a leaf of $T$ adjacent to $m$. Then clearly $d_{T}\left(v, V\left(T_{0}\right)\right)=2$. We have thus shown that conditions 2(a) and (b) of Theorem 7 are satisfied. Finally, $P_{1}$ and $P_{2}$ satisfy condition 1 of Theorem 7.
$(\mathrm{c}) \Longrightarrow(\mathrm{b})$. This is obvious and it follows that $\mathcal{T}_{2}=\mathcal{T}_{3}$.
$(\mathrm{c}) \Longrightarrow(\mathrm{d})$. Since $\rho=\gamma$ for all trees (Meir and Moon [11]) we only need
to show that $i=\gamma$ for any tree $T \in \mathcal{T}_{1}=\mathcal{T}_{3}$. Let $\left\{T_{m} \mid m \in M\right\}$ be the decomposition of $T$ as defined in the previous paragraph; by Theorem 7, $\rho(T)=|M|$. But $M$ is independent and dominating, hence $i(T) \leq|M|$ and the desired result follows.
(d) $\Longrightarrow$ (c). Obvious.

Corollary 9. Let $\lambda$ be any of the parameters $\gamma_{2}, \theta, \theta_{i}, \rho_{L}$ and let $\mu$ be any of the parameters $\rho, \gamma, i$. Then $T$ is $a(\lambda, \mu)$-tree if and only if $T \in \mathcal{T}_{1}=$ $\mathcal{T}_{2}=\mathcal{T}_{3}$.

Let $\beta$ denote the independence number, i.e., the cardinality of a maximum independent set, of a graph. (The notation $\beta_{0}$ or $\alpha$ is also sometimes used.) We conclude this section by showing that the class of $\left(\gamma_{2}, \beta\right)$-trees is not particularly interesting. We obtain this as a corollary to the following result which holds for general graphs.

Proposition 10. Let $\lambda$ be any of the parameters $\gamma_{2}, \theta, \theta_{i}$ and $\rho_{L}$. Then $G$ is a connected $(\lambda, \beta)$-graph if and only if $G$ is complete.
Proof. If $G$ is complete, then $\lambda(G)=\beta(G)=1$. Conversely, suppose $\rho_{L}(G)=\beta(G)$ but $G$ is not complete. Let $X$ be any $\rho_{L}$-set of $G$. We claim that $\bigcup_{x \in X} N[x]$ is a partition of $V(G)$. Indeed, by the packing property, $N\left[x_{1}\right] \cap N\left[x_{2}\right]=\emptyset$ for distinct $x_{1}, x_{2} \in X$; moreover, if $y \in V(G)-N[X]$, then since $X \cup\{y\}$ is independent we have $\beta(G) \geq \rho_{L}(G)+1$, a contradiction. Now suppose that there exist two non-adjacent neighbours $u$ and $v$ of some vertex $x \in X$. Since $(X-\{x\}) \cup\{u, v\}$ is independent, we have $\beta(G) \geq$ $\rho_{L}(G)+1$, a contradiction. Therefore $G[N[x]]$ is complete for each $x \in X$. Since $G$ is not complete by assumption, it follows that $|X| \geq 2$. Moreover, since $G$ is connected, there exists a neighbour $w$ of some $x \in X$ such that $X^{\prime}=\left\{x^{\prime} \in X: N(w) \cap N\left[x^{\prime}\right] \neq \emptyset\right\}$ satisfies $\left|X^{\prime}\right| \geq 2$. (Note that $x \in X^{\prime}$.) But then $\left(X-X^{\prime}\right) \cup\{w\}$ is a maximal packing of cardinality less than $\rho_{L}$, a contradiction. The result now follows from Proposition 2.

Corollary 11. Let $\lambda$ be any of the parameters $\gamma_{2}, \theta, \theta_{i}$ and $\rho_{L}$. Then $T$ is $a(\lambda, \beta)$-tree if and only if $T \in\left\{K_{1}, K_{2}\right\}$.

## 4. A Tree with Distinct Parameter Values

Consider the tree $T$ in Figure 2. Let $A=\left\{a_{j}: 1 \leq j \leq 13\right\}, B=\left\{b_{j}: 1 \leq\right.$ $j \leq 12\}, C=\left\{c_{j}: 1 \leq j \leq 23\right\}, D=\left\{d_{j}: 1 \leq j \leq 10\right\}, E=\left\{e_{j}: 1 \leq j \leq 10\right\}$,
$F=\left\{b_{1}, w\right\} \cup D \cup E$ and $B^{*}=B-\left\{b_{1}\right\}$. We illustrate that the two total orders given in Corollary 5 can be strict by showing that $\gamma_{2}(T)=16$, $\theta(T)=17, \theta_{i}(T)=18, \rho_{L}(T)=19, \operatorname{ra}(T)=23, \operatorname{rai}(T)=\operatorname{er}(T)=24$, $\operatorname{ir}(T)=25, \gamma(T)=26$ and $i(T)=27$. (It is also possible to obtain trees $T_{1}$ and $T_{2}$ in which all these parameters are distinct and $\operatorname{er}\left(T_{1}\right)<\operatorname{rai}\left(T_{1}\right)$, while $\operatorname{rai}\left(T_{2}\right)<\operatorname{er}\left(T_{2}\right)$. This shows that er and rai are incomparable even for trees. However, we do not exhibit examples of such trees here.)


Figure 2. A tree with distinct parameter values

Recall that any maximal packing of a graph $G$ is an independent PN-set, while any PN-set 2-dominates $G$. Also, any maximal irredundant set is external redundant and any external redundant set of $G$ is $R$-annihilated. (All these relationships are direct consequences of the definitions.)

Let $X$ be any 2 -dominating set of $T$. In order to 2 -dominate the leaves of $T[A]$ and $T[C],|A \cap X| \geq 4$ and $|C \cap X| \geq 6$. To 2-dominate $b_{6}, b_{10}$ and $b_{12},\left|B^{*} \cap X\right| \geq 2$ and if $\left|B^{*} \cap X\right|=2$, then $B^{*} \cap X=\left\{b_{4}, b_{8}\right\}$. Since
the leaves of $T[F]$ are 2-dominated, $|F \cap X| \geq 4$, and if $|F \cap X|=4$, then $F \cap X=\left\{d_{1}, d_{6}, e_{1}, e_{6}\right\}$. It follows that $|X| \geq 4+6+2+4=16$. Let

$$
X_{0}=\left\{a_{2}, a_{5}, a_{9}, a_{12}, c_{2}, c_{5}, c_{9}, c_{14}, c_{19}, c_{22}\right\}
$$

and

$$
X_{1}=\left\{b_{4}, b_{8}, d_{1}, d_{6}, e_{1}, e_{6}\right\} .
$$

Then $X_{2}=X_{0} \cup X_{1}$ is a 2-dominating set of $T$ and hence $\gamma_{2}(T)=16$.
Suppose furthermore that $X$ is a PN-set. If $\left|X \cap\left(B^{*} \cup F\right)\right|=6$, then, as shown above, $X \cap\left(B^{*} \cup F\right)=X_{1}$. But then $w$ is not $X$-perfect and $b_{1}$ is not dominated by an $X$-perfect vertex. Thus $\left|X \cap\left(B^{*} \cup F\right)\right| \geq 7$ and $|X| \geq 17$. Since $X_{2} \cup\left\{b_{1}\right\}$ is a PN-set, it follows that $\theta(T)=17$.

Moreover, if $X$ is an independent PN -set, then in order to 2-dominate the leaves of $T[F]$, we need at least six vertices in $F \cap X$, and if $|F \cap X|=6$, then no vertex in $B^{*}$ is 2-dominated by any vertex of $F \cap X$. It follows that $|X| \geq 4+6+2+6=18$. On the other hand, $X_{0} \cup\left\{b_{4}, b_{8}, e_{1}, e_{8}, e_{10}, d_{3}, d_{5}, d_{6}\right\}$ is an independent PN-set of $T$ and hence $\theta_{i}(T)=18$.

Suppose that in addition, $X$ is a maximal packing of $T$. In particular, since $X$ is independent, $|F \cap X| \geq 6$ and if $|F \cap X|=6$, then by the above analysis, $b_{2}, b_{6}, b_{10}$ and $b_{12}$ are 2-dominated by $X \cap B^{*}$. Therefore $\left|B^{*} \cap X\right| \geq 2$ and if $\left|B^{*} \cap X\right|=2$, then $B^{*} \cap X=\left\{b_{4}, b_{8}\right\}$. But then $b_{7} \in N\left[b_{4}\right] \cap N\left[b_{8}\right]$, a contradiction. Therefore $|X| \geq 19$. Since $X=X_{0} \cup\left\{b_{4}, b_{10}, b_{12}, d_{1}, d_{8}, d_{10}, e_{3}, e_{5}, e_{6}\right\}$ is a maximal packing, it follows that $\rho_{L}(T)=19$.

Let $Y$ be any $R$-annihilated set of $T$. Define

$$
\begin{aligned}
Y_{a} & =\left\{a_{1}, a_{4}, a_{8}, a_{11}\right\} \\
Y_{b} & =\left\{b_{3}, b_{4}, b_{8}, b_{9}\right\} \\
Y_{c} & =\left\{c_{1}, c_{4}, c_{8}, c_{12}, c_{13}, c_{18}, c_{21}\right\} \\
Y_{f} & =\left\{d_{2}, d_{4}, d_{7}, d_{9}, e_{2}, e_{4}, e_{7}, e_{9}\right\} .
\end{aligned}
$$

For each $y \in Y$, if $|p n(y, Y)| \geq 2$, then (since $T$ is a tree) no vertex in $R_{Y}$ annihilates $y$. Hence if $d_{1} \in Y$, then $\left\{d_{2}, d_{3}, d_{4}, d_{5}\right\} \cap Y \neq \emptyset$. If $d_{1} \notin Y$, then clearly $\left\{d_{2}, d_{3}\right\} \cap Y \neq \emptyset$ and $\left\{d_{4}, d_{5}\right\} \cap Y \neq \emptyset$. A similar argument holds with respect to $d_{6}$ and it follows that $|Y \cap D| \geq 4$. Similarly, $|Y \cap E| \geq 4$ and $|Y \cap B| \geq 4$. Moreover, $|Y \cap B|=4$ if and only if $Y \cap B=Y_{b}$ or $\left(Y_{b}-\left\{b_{9}\right\}\right) \cup\left\{b_{11}\right\}$, and $|Y \cap A| \geq 4$ with $|Y \cap A|=4$ if and only if $Y \cap A=Y_{a}$. Let $C_{j}=\left\{c_{j}, c_{j+1}, c_{j+2}\right\}$ for $j \in\{1,4,18,21\}$ and $C_{j}=$ $\left\{c_{j}, c_{j+1}, c_{j+2}, c_{j+3}\right\}$ for $j \in\{8,13\}$. Since $Y$ is 2-dominating, $Y \cap C_{j} \neq \emptyset$ for
each $j \in\{1,4,8,13,18,21\}$ and hence $|Y \cap C| \geq 6$. However, if $|Y \cap C|=6$, then $c_{k}$ and $c_{l}$ with $\left\{c_{k}\right\}=Y \cap C_{8}$ and $\left\{c_{l}\right\}=Y \cap C_{13}$ are isolated in $Y$. If $k=8$, then $c_{10}$ does not annihilate any $y \in Y$, and obviously $k \notin\{10,11\}$. Hence $k=9$. Similarly, $l=14$. But then $c_{12}$ does not annihilate any $y \in Y$. Thus $|Y \cap C| \geq 7$ and as in the case of $C_{8}$ and $C_{13}$, it can be shown that no vertex of $Y \cap C_{j}(j \in\{1,4,18,21\})$ is isolated in $Y$. Therefore $|Y \cap C|=7$ if and only if $Y \cap C=Y_{c}$. Therefore $|Y| \geq 23$. Since

$$
Y_{0}=Y_{a} \cup Y_{b} \cup Y_{c} \cup Y_{f}
$$

is an $R$-annihilated set, it follows that $r a(T)=23$.
Let $Y$ moreover be irredundant. Then $|Y \cap C| \geq 8$, for otherwise $Y \cap C=$ $Y_{c}$ and $c_{12}$ is redundant in $Y$. Thus $|Y| \geq 24$. However,

$$
Y_{1}=\left(Y_{0}-\left\{c_{8}, c_{12}\right\}\right) \cup\left\{c_{5}, c_{9}, c_{14}\right\}
$$

is an $R$-annihilated irredundant set, so that $\operatorname{rai}(T)=24$.
Furthermore, if $Y$ is maximal irredundant, then as above, $|Y \cap C| \geq 8$. If $|Y| \leq 24$, then $|Y \cap B|=4$ and so $Y \cap B=Y_{b}$ or $\left(Y_{b}-\left\{b_{9}\right\}\right) \cup\left\{b_{11}\right\}$. But then $Y \cup\{w\}$ is irredundant since $b_{1} \in p n(w, Y \cup\{w\})$, a contradiction. Hence $|Y \cap B| \geq 5$ and thus $|Y| \geq 25$. On the other hand,

$$
Y_{2}=Y_{1} \cup\{w\}
$$

is a maximal irredundant set and so $\operatorname{ir}(T)=25$.
Now suppose that $Y$ is an external redundant (i.e., $N[R]$-annihilated) set but not necessarily irredundant. Then $|Y| \geq r a(T)=23$. If $|Y|=23$, then $|Y \cap B|=4$ with $Y \cap B=Y_{b}$ or $\left(Y_{b}-\left\{b_{9}\right\}\right) \cup\left\{b_{11}\right\},|Y \cap A|=4$ with $Y \cap A=Y_{a}$ and $|Y \cap C|=7$ with $Y \cap C=Y_{c}$. Moreover, $|Y \cap F|=8$ and $b_{1} \in R_{Y}$. But then $w \in N[R]$ does not annihilate any $y \in Y$, a contradiction from which it follows that $|Y| \geq 24$. Since $Y_{0} \cup\{w\}$ is external redundant, we have that $\operatorname{er}(T)=24$.

Let $I$ be any dominating set of $T$. Clearly, $|I \cap A| \geq 5,|I \cap B| \geq 5$, $|I \cap D| \geq 4,|I \cap E| \geq 4$ and $|I \cap C| \geq 8$. Moreover, if $|I \cap C|=8$, then $\left\{c_{9}, c_{14}\right\} \subseteq I$ and $\left\{c_{8}, c_{13}\right\} \cap I \neq \emptyset$. Thus $I$ is not independent and $|I| \geq 26$. Since

$$
I_{0}=X_{0} \cup Y_{f} \cup\left\{a_{7}, b_{1}, b_{4}, b_{6}, b_{9}, b_{12}, c_{8}, c_{17}\right\}
$$

dominates $T$, it follows that $\gamma(T)=26$.
As explained above, if $I$ is independent, then $|I|>26$. On the other hand,

$$
I_{1}=\left(I_{0}-\left\{c_{8}\right\}\right) \cup\left\{c_{7}, c_{12}\right\}
$$

is an independent dominating set and hence $i(T)=27$. This completes the discussion of the tree $T$.

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