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A PATH(OLOGICAL) PARTITION PROBLEM

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Abstract

Let $\tau(G)$ denote the number of vertices in a longest path of the graph G and let k_1 and k_2 be positive integers such that $\tau(G) = k_1+k_2$. The question at hand is whether the vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that $\tau(G[V_1]) \leq k_1$ and $\tau(G[V_2]) \leq k_2$. We show that several classes of graphs have this partition property.

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1. Introduction

Let G be a graph. We denote the number of vertices of G by $v(G)$ and the number of vertices in a longest path (which need not be an induced path) in G by $\tau(G)$. If S is any subset of the vertex set $V(G)$, we denote the subgraph of G induced by S by $G[S]$. We denote the distance between two vertices v and w by $d(v, w)$, and we define the *distance between a vertex x*

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of G and a subset S of $V(G)$ by $d(x, S) = min\{d(x, v)|v \in S\}$. The *open* (*closed*) *neighbourhood* of a vertex v is defined, as usual, as the set of vertices $N(v) = \{u \in V(G)|uv \in E(G)\}$ ($N[v] = N(v) \cup \{v\}$, respectively).

Given any pair of positive integers (k_1, k_2), we say that G is (k_1, k_2)-*partitionable* if there exisits a partition $\{V_1, V_2\}$ of $V(G)$ into two subsets V_1 and V_2 such that $\tau(G[V_1]) \leq k_1$ and $\tau(G[V_2]) \leq k_2$. If G can be (k_1, k_2)-partitioned for every pair of positive integers (k_1, k_2) satisfying $k_1 + k_2 = \tau(G)$, we say that G is τ-*partitionable.*

Similar partition concepts can be defined for other parameters. For example, we say a graph G is Δ-*partitionable* (where $\Delta(G)$ denotes the maximum degree of G) if, for any pair of positive integers (k_1, k_2) satisfying $k_1 + k_2 \geq \Delta(G) - 1$, there exists a partition $\{V_1, V_2\}$ of $V(G)$ such that $\Delta(G[V_1]) \leq k_1$ and $\Delta G([V_2]) \leq k_2$. Lovász proved in [10] that every graph G is Δ-partitionable. Stiebitz proved in [12] a dual type of partition result with respect to δ (the minimum degree).

The main aim of this paper is to prove results supporting the following conjecture:

Conjecture 1. *Every graph is τ-partitionable.*

This problem is stated as Problem 1 in [1]. Its similarity to Lovász's theorem is underlined by this formulation since both are stated in terms of reducuble bounds for some additive hereditary properties—see [1] for details.

This problem is also related to an open problem on kernels described in [11]. Conjecture 1 was discussed by Lovász and Mihók in 1981 in Szeged and treated in the theses [7] and [13]. A short description of these problems and their relationship to our problem can be found in [3]. The directed version of Conjecture 1 has been presented by Laborde et al in [9].

The k-chromatic number $\chi_k(G)$ of a graph G is defined in [4] as the smallest number of sets in a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that $\tau(G[V_i]) \leq k$ for each i. This is clearly related to our problem. In fact, the upper bound $\chi_k(G) \leq \lfloor(\tau(G) - 1 - k)/2\rfloor + 2$ given in Theorem 2 of [4] can be improved to $\chi_k(G) \leq \lceil\tau(G)/k\rceil$ if Conjecture 1 is true.

In Section 3 we shall consider graphs that are (k_1, k_2)-partitionable for specific integers k_1 and k_2, and in Section 4 we shall show that various classes of graphs are τ-partitionable.

Note that a graph is τ-partitionable if each of its (connected) components is τ-partitionable; therefore we shall only consider connected graphs in the sequel.

2. Preliminary Results

A graph G is called k-τ-saturated if $\tau(G) \leq k$ and $\tau(G+e) > k$ for every $e \in E(\overline{G})$. (Such graphs are called $\mathcal{W}_{k+1}$-maximal in [2] but here we follow the terminology of [8].) Note that, for a graph G with $\tau(G) = k$, one can add edges to obtain a graph H such that H is k-τ-saturated. Our motivation for studying such graphs lies of course in the following equivalence: *Every graph G with $\tau(G) = k$ is τ-partitionable if and only if every k-τ-saturated graph is τ-partitionable.* Therefore we need to consider Conjecture 1 only for k-τ-saturated graphs.

The *join* of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is the graph obtained from the (disjoint) union of G_1 and G_2 by adding all possible edges joining a vertex from G_1 and a vertex from G_2. If it is possible to write a graph G as $G = G_1 + G_2$, then G is called *decomposable*; otherwise it is called *indecomposable*.

The following useful result concerning decomposable k-τ-saturated graphs appears (as Theorem 6) in [2].

Lemma 2.1. *If G is a k-τ-saturated graph which is not complete and $G = G_1+G_2$ with $1 \leq v(G_1) \leq v(G_2)$, then G_1 is a complete graph with $v(G_1) \leq \frac{k}{2}$ and $\tau(G_2) \leq k+1-2v(G_1)$.*

Corollary 2.2. *A k-τ-saturated graph G is decomposable if and only if $\Delta(G) = v(G) - 1$.*

We shall also need the following relationship between $\tau(G)$ and $\beta(G)$, the vertex independence number of G.

Lemma 2.3. *If G is any graph, then*

$$\tau(G) \leq 2[v(G) - \beta(G)] + 1.$$

Proof. Let S be an independent set of vertices in G such that

$$|S| = \beta(G).$$

Then

$$|V(G-S)| = v(G) - \beta(G).$$

If P is any path in G, then

$$|V(P) \cap S| \leq |V(P) \cap (V-S)| + 1.$$

Therefore

$$\begin{aligned}\tau(G) &\leq |V(P) \cap S| + |V(P) \cap (V - S)| \\ &\leq 2v(G - S) + 1 \\ &\leq 2(v(G) - \beta(G)) + 1.\end{aligned}$$ ∎

In Section 4 we shall prove that graphs whose blocks satisfy certain conditions are τ-partitionable. In order to do so, we shall need the following two lemmas.

Lemma 2.4. *Let G be a graph such that every cyclic block of G is Hamiltonian and let (k_1, k_2) be a pair of positive integers satisfying $k_1 + k_2 = \tau(G)$. If G has a block A such that $v(A) > k_1$, then $v(B) \leq k_2$ for every other block B of G.*

Proof. If B is any block of G other than A, then there is a path in G containing all the vertices of A and all the vertices of B. ∎

Our next result rephrases Problem 2.30 of [5]. In it, two blocks of a graph are called *incident* if they have a common vertex.

Lemma 2.5. *The blocks of every graph can be partitioned into two sets in such a way that no two incident blocks are in the same partition class.*

Proof. Let A be any block. Put A in one class, all blocks incident with A in the other class and continue in the obvious way. ∎

3. Graphs that are (k_1, k_2)-Partitionable for Specific k_1 and k_2

Proposition 3.1. *Let G be any graph with $\tau(G) = k \geq 1$. Then G can be $(k-1, 1)$-partitioned.*

Proof. Let V_2 be a maximal independent set of vertices of G and let $V_1 = V(G) - V_2$. Then, clearly, $\tau(G[V_1]) \leq k - 1$ and $\tau(G[V_2]) \leq 1$. ∎

Theorem 3.2. *Let G be a graph with $\tau(G) = k$. If*

$$k_2 \leq k_1 \leq k \leq \lceil \frac{k_1 + 1}{2} \rceil + k_2,$$

then G has a (k_1, k_2)-partition.

Proof. Let W_0 be any subset of $V(G)$ with $\tau(G[W_0]) = k_1$. Let

$$W_i = \{v \in V(G) | d(v, W_0) = i\}.$$

If $\tau(G[W_i]) \geq k_2 + 1$ for some $i \geq 1$, then

$$\tau(G) \geq \lceil \frac{k_1 + 1}{2} \rceil + k_2 + 1 > k = \tau(G).$$

Therefore

$$\tau(G[W_i]) \leq k_2 \text{ for each } i \geq 1.$$

Now put

$$V_1 = \bigcup_{i \text{ even}} W_i \text{ and } V_2 = \bigcup_{i \text{ odd}} W_i.$$

Then

$$\tau(G[V_1]) \leq k_1 \text{ and } \tau([G[V_2]) \leq k_2.$$ ■

Corollary 3.3. *Let G be a graph and suppose that*

$$k_i \geq \frac{2(\tau(G) - 1)}{3} \textit{ for } i = 1, 2.$$

Then G is (k_1, k_2)-partitionable.

Theorem 3.4. *Let G be a graph and suppose that $\tau(G) = k_1 + k_2$. If $k_2 \leq 4$, then G is (k_1, k_2)-partitionable.*

Proof. We use an algorithm to construct V_1 and V_2 if $k_2 = 4$. The algorithm begins with $V_1 = V(G)$ and $V_2 = \emptyset$.

1. Let

$$X = \{v \in V_1 \mid v \text{ is an end-vertex of a path in } V_1 \text{ with } k_1 + 1 \text{ vertices}\}.$$

If $X = \emptyset$ we are done. If $X \neq \emptyset$, we proceed to Step 2.

2. If no vertex of X is adjacent to any vertex of V_2 (as is the case in the beginning), then move any vertex of X to V_2 and return to Step 1.

3. If some vertex v of X is adjacent to a vertex of V_2, then move v to V_2. Now there are two possibilities:

3.1. If v is adjacent to only one vertex of V_2, return to Step 1. (Note that there is still no path on five vertices in V_2, since v cannot be an end-vertex of such a path in V_2.)

3.2. If v is adjacent to more than one vertex of V_2, then there is now a cycle C in V_2. (No vertex of X is adjacent to vertices of two different components of V_2, since a "new" component of V_2 is only begun when there are no vertices in X that are adjacent to vertices of V_2.) This cycle will be either a C_3 or a C_4. Let K be the component of V_2 containing C. Now return all the vertices of $K - C$ to V_1 and then return to Step 1. Note that, if at some stage a 4-cycle appears in V_2, then, from that stage onward, no vertex that is adjacent to any vertex of that 4-cycle will ever be in X.

Suppose that at some stage a 3-cycle abc appears in V_2 and, at some later stage, a vertex x that is adjacent to a vertex of this 3-cycle is moved from X to V_2. Then:

If x is adjacent to only one vertex of this 3-cycle, say to a, then no vertex adjacent to b or c will be in X hereafter, since each of b and c is now an end-vertex of a P_4 in V_2.

If, on the other hand, x is adjacent to more than one vertex of the 3-cycle, then there is now a 4-cycle in V_2, and thereafter no vertex adjacent to a, b or c will be in X.

At some stage X will become empty so that $\tau(G[V_1]) \leq k_1$ while $\tau(G[V_2])$ remains at most 4.

Similar algorithms can be described to handle the cases $k_2 = 2$ and 3; for $k_2 = 1$ the result is already proven in Proposition 3.1. ■

The result of this theorem can be extended to include more cases but our approach becomes too elaborate then. The cases in which $k_2 \leq 5$ have been treated in [7] and [13] too (in terms of the existence of kernels); also with elaborate proofs.

Theorem 3.5. *Let G be a graph with $k_1 + k_2 = \tau(G)$. If*

$$2[v(G) - \beta(G)] + 1 - k_1 \leq \tau(G),$$

then G is (k_1, k_2)-partitionable.

Proof. Let S be an independent set of vertices in G with

$$|S| = \beta(G).$$

If $v(G) - \beta(G) \leq k_1$, let

$$V_1 = V(G) - S \text{ and } V_2 = S.$$

If $v(G) - \beta(G) > k_1$, let V_1 be any subset of $V(G) - S$ with $|V_1| = k_1$ and let $V_2 = V(G) - V_1$.
Then

$$\tau(G[V_1]) \leq k_1$$

and, by Lemma 2.3,

$$\begin{aligned}\tau(G[V_2]) &\leq 2[|V_2| - \beta(G[V_2])] + 1\\ &= 2[v(G) - k_1 - \beta(G)] + 1 \text{ (since } \beta(G[V_2]) = \beta(G))\\ &= 2v(G) - k_1 - 2\beta(G) + 1 - k_1 \leq \tau(G) - k_1\\ &= k_2.\end{aligned}$$

■

Theorem 3.6. *If G is a graph with $\Delta(G) \leq 3$, then G is $(2,2)$-partitionable.*

Proof. By Lovász' Theorem (see [10]) G is Δ-partitionable. Hence there exists a partition $\{V_1, V_2\}$ of $V(G)$ such that $\Delta(G[V_1]) \leq 1$ and $\Delta G([V_2]) \leq 1$. This partition satisfies our requirements since $\tau(G[V_1]) \leq 2$ and $\tau G([V_2]) \leq 2$. ■

Our next result also gives a class of (k_1, k_2)-partitionable graphs; this time it assumes the existence of a suitable cycle.

Theorem 3.7. *Let G be a graph with $\tau(G) = k_1 + k_2$ and with $k_1 \geq k_2$. If G contains a cycle of length k_1, then G is (k_1, k_2)-partitionable.*

Proof. Let C be a cycle of length k_1 in G. Put

$$W_0 = V(C) \text{ and } W_i = \{v \in V(G) - V(C))|d(v, W_0) = i\}.$$

Now consider any vertex $v_i \in W_i$. Then there is a path $v_i v_{i-1} \dots v_1 v_0$ in G with $v_j \in W_j$ for $j = 0, 1, \dots, i$. Therefore $\tau(G[W_i]) \leq k_2$ for all $i \geq 1$. Now put

$$V_1 = \bigcup_{i \text{ even}} W_i \text{ and } V_2 = \bigcup_{i \text{ odd}} W_i.$$

Then $\tau(G[V_i]) \leq k_i$ for $i = 1, 2$. ■

Theorem 3.8. *Let G be a graph with $\tau(G) = k_1 + k_2$ and with $k_1 \geq k_2$. If G has a vertex v such that $\tau(G - N[v]) \leq k_2$, then G has a (k_1, k_2)-partition.*

Proof. If $\tau(N(v)) \leq k_1$, then $(N(v), G - N(v))$ is a (k_1, k_2)-partition of G. If $\tau(N(v)) > k_1$, then v together with any path of order $k_1 - 1$ in $N(v)$ form a cycle of length k_1 in G. Hence it follows from Theorem 3.7 that G has a (k_1, k_2)-partition in this case. ■

Corollary 3.9. *Let G be a graph with $\tau(G) = k_1 + k_2$ and with $k_1 \geq k_2$. If $\Delta(G) \geq v(G) - k_2 - 1$, then G has a (k_1, k_2)-partition.*

Proof. Let G be a vertex of G of degree at least $v(G) - k_2 - 1$. Then $|G - N[v]| \leq k_2$, and the result follows from Theorem 3.8. ■

Let $c(G)$ denote the length of a longest cycle in G. Since Conjecture 1 is true for all trees (indeed, it is trivially true for all bipartite graphs), we need only consider graphs that contain cycles. Following the idea of the proofs of Theorem 3.2 and Theorem 3.7 we can now prove

Theorem 3.10. *Let G be a graph containing a cycle and suppose that t is any integer such that $c(G) \leq t$. Then G is $(t-1, t-1)$-partitionable.*

Proof. Let v be any vertex of G and put

$$W_0 = \{v\} \text{ and } W_i = \{x \in V(G) | d(x, v) = i\}.$$

If some W_j contains a path P of order t, then there are internally disjoint paths from some vertex in some W_i, $i < j$ to the endvertices of P, that is, $c(G) > t$. Therefore $\tau(G[W_j]) \leq t - 1$ for all $j \geq 1$. Now put

$$V_1 = \bigcup_{i \text{ even}} W_i \text{ and } V_2 = \bigcup_{i \text{ odd}} W_i$$

to complete the proof. ■

It seems plausible to approach Conjecture 1 by proving it for blocks and proving that, if all blocks of G are τ-partitionable then so is G. We only have a partial result supporting this approach; it shows how some graphs can be (k_1, k_2)-partitioned using the partitions of its blocks. If G is a graph and B is a subgraph of G, we denote the sets of vertices $V(B) \cap N(x)$ and $V(B) \cap N[x]$ by $N_B(x)$ and $N_B[x]$, respectively.

Theorem 3.11. *Let G be a graph with $\tau(G) \leq k_1 + k_2$ and with $k_1 \geq k_2$. Suppose that every block of G has a (k_1, k_2)-partition and that for every block B of G and every cutvertex x of G which is in B we have that $\tau(B[N_B(x)]) \leq k_2$ and that $\tau(B - N_B[x]) \leq k_2$. Then G has a (k_1, k_2)-partition.*

Proof. The result follows by induction on the number of blocks of G; it is clearly true if G consists of only one block.

Hence suppose it is true for all graphs with at most $r - 1$ blocks and let G be a graph with $r \geq 2$ blocks. Let B be an endblock of G and let x be the cutvertex of G which is in B. Then, by the induction hypothesis, the graph induced by $V(G) - V(B - x)$ has a (k_1, k_2)-partition, say $\{V_1', V_2'\}$.

If $x \in V_1'$, we can define the desired partition of $V(G)$ by $V_1 = V_1' \cup V(B - N_B[x])$ and $V_2 = V_2' \cup N_B(x)$. On the other hand, if $x \in V_2'$, the partition of $V(G)$ defined by $V_1 = V_1' \cup N_B(x)$ and $V_2 = V_2' \cup V(B - N_B[x])$ has the required properties. ■

4. Graphs that are τ-Partitionable

Several classes of graphs are easily seen to be τ-partitionable: for example complete graphs, bipartite graphs (since they are $(1, 1)$-partitionable) and any graph which has a Hamiltonian path. In this latter case we note that $v(G) = \tau(G)$. We also have

Proposition 4.1. *If G is a graph with $v(G) \leq \tau(G) + 1$, then G is τ-partitionable.*

Proof. Let $\tau(G) = k = k_1 + k_2$ with $k_1 \geq k_2$. If $v(G) = k$, then G is obviously (k_1, k_2)-partitionable. We therefore assume that $v(G) = k + 1$. Then G does not have a Hamiltonian cycle. Therefore, by the well-known sufficient condition of Dirac for a graph to have a Hamiltonian cycle (see [6]), $\delta(G) < \frac{k+1}{2}$. Hence $k_1 \geq \frac{k}{2} \geq \delta(G)$. Now let v be a vertex of G with $\deg v = d \leq k_1$, let V_1 consist of the d neighbours of v together with any $k_1 - d$ other vertices of G and let $V_2 = G - V_1$. Then V_2 consists of v together with k_2 non-neighbours of v and hence $\tau(G[V_2]) \leq k_2$. Also, $\tau(G[(V_1]) \leq |V_1| = k_1$. ■

In the next two theorems we use the remarks following Theorem 3.4; the first one follows directly from them.

Theorem 4.2. *If G is a graph with $\tau(G) \leq 11$, then G is τ-partitionable.*

Corollary 4.3. *If G is a graph of order at most 13, then G is τ-partitionable.*

Proof. If $\tau(G) \leq 11$, the result follows from Theorem 4.2. If $\tau(G) = 12$ or 13, the result follows from Proposition 4.1. ■

Theorem 4.4. *If G is a graph containing a cycle and $c(G) \leq 7$, then G is τ-partitionable.*

Proof. Let $\tau(G) = k_1 + k_2$ with $k_1 \geq k_2$. If $k_2 \leq 5$, then G is (k_1, k_2)-partitionable by the remarks following Theorem 3.4. Otherwise $k_1 \geq k_2 \geq 6$ so that G is (k_1, k_2)-partitionable by Theorem 3.10. ■

Theorem 4.5. *Suppose G is a graph such that every cyclic block of G is Hamiltonian. Then G is τ-partitionable.*

Proof. Let

$$\tau(G) = k = k_1 + k_2 \text{ and suppose that } k_1 \geq k_2.$$

Case (i). $v(B) \leq k_2$ for every block B of G.
By Lemma 2.5 we can partition the blocks of G in such a way that no two incident blocks are in the same class.

Case (ii). There is a block A of G such that $v(A) > k_1$.
Then $v(B) \leq k_2$ for every other block B of G by Lemma 2.4. Put k_1 of the vertices of A in V_1 and the other vertices of A in V_2. Then we put all the remaining vertices of each block incident with a vertex which is in V_1 (V_2) in V_2 (V_1 respectively). Continue in the obvious way.

Case (iii). There is a block A such that $v(A) \geq k_2$ and $v(B) \leq k_1$ for every block B of G.
Put k_2 vertices of A in V_2, and the remaining vertices of A in V_1. Then put all the remaining vertices of each block of G incident with a vertex of $A \cap V_2$ in V_1. Now put the remaining vertices of each block B that is incident with a vertex of $A \cap V_1$ as follows:

Let $\{a\} = V(B) \cap (A \cap V_1)$. If $v(B) \leq k_2 + 1$, put all the vertices of $B - a$ in V_2. If $v(B) > k_2 + 1$, put k_2 of the vertices of $B - a$ in V_2, and the rest in V_1.

Carry on distributing in this way: If two blocks share a cut vertex in V_2, one of the blocks will be completely in V_1, while if two blocks share a cut vertex in V_1, neither of the two blocks will have more than k_2 vertices in V_2. Now

$$\tau(G[V_2]) \leq k_2$$

since each path in $G[V_2]$ lies in a single block of G and we did not put more than k_2 vertices of any block in V_2.

Also, $\tau(G[V_1]) \leq k_1$ for if P is a path in $G[V_1]$ and $v(P) > k_1$, then P has vertices in at least two different blocks of G. Let

$$H = \cup\{V(B) \mid V(B) \cap V(P) \neq \emptyset\}.$$

Then at least one endblock B of H has at least k_2 vertices in V_2. Let Q be a path containing all the vertices of B, and all the vertices of P. Then $v(Q) > k_1 + k_2$.

Since B has at least k_2 vertices in V_2 and $\tau(G) = k_1 + k_2$, it follows that P has at most k_1 vertices in V_1. ■

A graph G is called a *block graph* if every block of G is complete. As a special case of Theorem 4.5, we have

Corollary 4.6. *If G is a block graph, then G is τ-partitionable.*

Corollary 4.7. *If G is a graph without even cycles, then G is τ-partitionable.*

Proof. If G has no even cycles, then every cyclic block of G is an odd cycle. Hence G is τ-partitionable by Theorem 4.5. ■

The next two results show that if the maximum degree of a graph is either small or large compared to its order, the graph is τ-partitionable.

Theorem 4.8. *If G is a graph with $\Delta(G) \leq 3$, then G is τ-partitionable.*

Proof. Let (k_1, k_2) be any pair of positive integers satisfying $k_1+k_2 = \tau(G)$ and $k_1 \geq k_2$. If $k_2 \leq 4$, then the required partition can be obtained from Theorem 3.4. Otherwise, it can be obtained from Theorem 3.6. ■

Theorem 4.9. *If G is a graph with $\Delta(G) \geq v(G) - 7$, then G is τ-partitionable.*

Proof. Let (k_1, k_2) be any pair of postive integers satisfying $k_1+k_2 = \tau(G)$. If $k_2 \leq 5$, it follows from the remarks following Theorem 3.4 that G has a (k_1, k_2)-partition. If $k_2 \geq 6$, it follows from Corollary 3.9 that G has a (k_1, k_2)-partition. ■

Finally we can now prove

Theorem 4.10. *Every decomposable graph is τ-partitionable.*

Proof. Let G be a decomposable graph with $\tau(G) = k$. Then G is a subgraph of some k-τ-saturated graph G^* which is also decomposable. By Corollary 2.2, $\Delta(G^*) = v(G^*) - 1$. Therefore, by Theorem 4.9, G^* is τ-partitionable, and hence G is τ-partitionable. ■

One last remark about the problem in general: If G is a graph such that for every pair of positive integers k_1 and k_2 for which $\tau(G) = k_1 + k_2$ the vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that $\tau(G[V_i])$ is *equal* to k_i for $i = 1,\ 2$, then G is clearly τ-partitionable. We do not know if the converse of this implication is also true.

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