

THE CHROMATICITY OF A FAMILY OF
2-CONNECTED 3-CHROMATIC GRAPHS WITH
FIVE TRIANGLES AND CYCLOMATIC NUMBER SIX

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Abstract

In this note, all chromatic equivalence classes for 2-connected 3-chromatic graphs with five triangles and cyclomatic number six are described. New families of chromatically unique graphs of order n are presented for each $n \geq 8$. This is a generalization of a result stated in [5]. Moreover, a proof for the conjecture posed in [5] is given.

Keywords: chromatically equivalent graphs, chromatic polynomial, chromatically unique graphs, cyclomatic number.

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1. INTRODUCTION

The graphs which we consider here are finite, undirected, simple and loopless. Let G be a graph, $V(G)$ its vertex set, $E(G)$ its edge set, $\chi(G)$ its chromatic number and $P(G, \lambda)$ its chromatic polynomial. Two graphs G and H are said to be *chromatically equivalent*, or in short χ -equivalent, written $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is said to be *chromatically unique*, or in short χ -unique, if for any graph H satisfying $H \sim G$, we have $H \cong G$, i.e. H is isomorphic to G . A family of all nonisomorphic chromatically equivalent graphs is called a χ -equivalence class.

A *wheel* W_n is a graph of order n , $n \geq 4$, obtained by the join of K_1 and C_{n-1} . Any edge incident with the central vertex in W_n is called a *spoke* of the wheel. For any two integers n, k with $n \geq 4$ and $n - 1 \geq k \geq 1$, let $W(n, k)$ denote the graph of order n obtained from a wheel W_n by deleting

all but k consecutive spokes. It is known that the graphs $W(n, 1)$ ($n \geq 4$) and $W(n, 2)$ ($n \geq 4$) are χ -unique. Chao and Whitehead [1] showed that the graphs $W(n, 3)$ ($n \geq 5$) and $W(n, 4)$ ($n \geq 6$) are χ -unique, while $W(7, 5)$ is not. Then Koch and Teo [3] showed that all graphs $W(n, 5)$ ($n \geq 8$) are χ -unique. Recently Li and Whitehead [5] showed that all graphs $W(n, 6)$ ($n \geq 8$) are χ -unique. This is a solution to one of the problems stated in [2] (see Problem 2 [2]). They also described two additional families of chromatically unique graphs. The family of graphs they studied consists of 2-connected 3-chromatic graphs with five triangles and cyclomatic number six. In this paper, all classes of χ -equivalent graphs of order at least 8 for this family are described. In particular, a complete characterization of chromatically unique graphs for the family is presented. Also a proof for the conjecture posed in [5] is given.

2. KNOWN RESULTS

In computing chromatic polynomials, we make use of Whitney's reduction formula given in [6]. The formula is

$$P(G, \lambda) = P(G_{-e}, \lambda) - P(G/_e, \lambda)$$

or equivalently

$$P(G_{-e}, \lambda) = P(G, \lambda) + P(G/_e, \lambda)$$

where G_{-e} is the graph obtained from G by deleting an edge e and $G/_e$ is the graph obtained from G by contracting the edge e .

We also make use of the overlapping formula given in [6]. The formula is

$$P(G, \lambda) = P(H, \lambda)P(F, \lambda)/P(K_p, \lambda)$$

where G is a gluing of two disjoint graphs H and F over their complete subgraph K_p for $p \geq 1$.

Moreover, we shall use the known results for χ -equivalent graphs presented in Lemma 1. For a graph F , let $I_G(F)$ denote the number of induced subgraphs of G which are isomorphic to F .

Lemma 1 [3]. *Let G and H be two χ -equivalent graphs. Then*

- (i) $|V(G)| = |V(H)|$,
- (ii) $|E(G)| = |E(H)|$,
- (iii) $\chi(G) = \chi(H)$,
- (iv) $I_G(C_3) = I_H(C_3)$,

- (v) $I_G(C_4) - 2I_G(K_4) = I_H(C_4) - 2I_H(K_4)$,
- (vi) G is connected iff H is connected,
- (vii) G is 2-connected iff H is 2-connected.

3. RESULTS

Next we consider the following 2-connected pairwise nonisomorphic graphs $X_i(n)$, shortly denoted by X_i , each of order n , $n \geq 8$, presented in Figure 1.

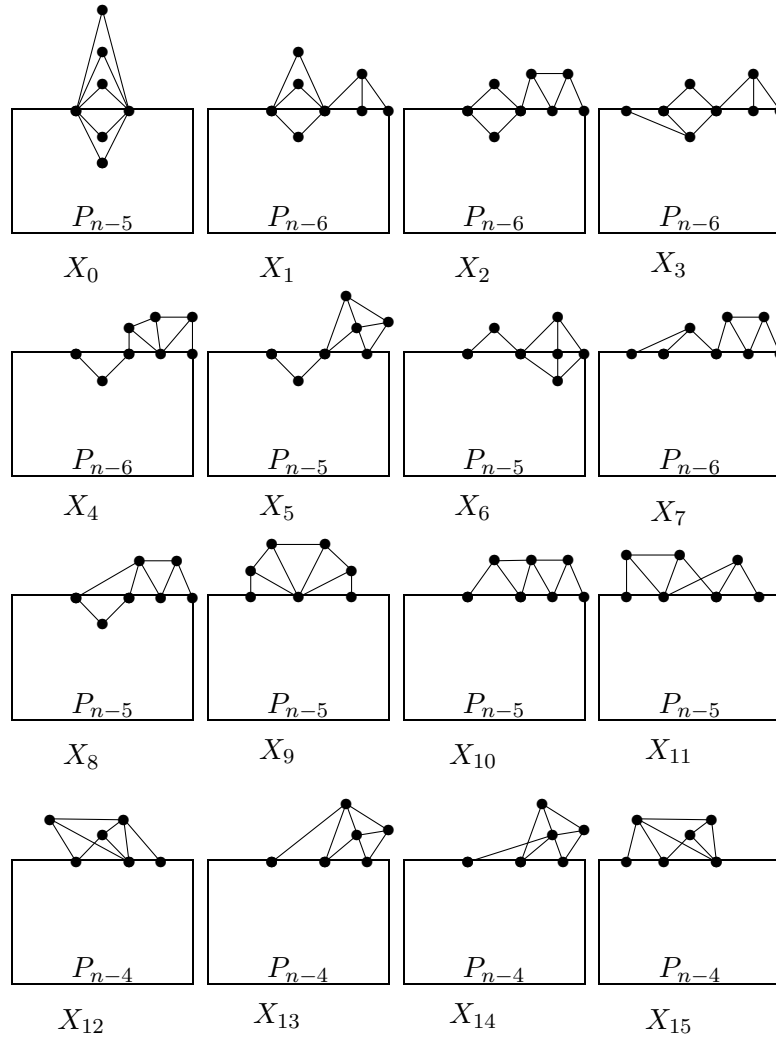


Figure 1

Thin lines denote here paths, filled circle — vertices, and bold lines — edges of a graph. Checking the degree sequences of these graphs one can easily note that they are pairwise nonisomorphic.

First we prove the following lemma.

Lemma 2. $X_{15} \sim X_{14}$, and $X_i \not\sim X_j$ for other pairs $i, j = 0, \dots, 13$ and $i \neq j$.

Proof. By using Whitney's reduction formula we have:

$$\begin{aligned} P(X_0, \lambda) &= (\lambda - 2)^5 P(C_{n-5}, \lambda), \\ P(X_1, \lambda) &= (\lambda - 2)^4 (P(C_{n-4}, \lambda) - P(C_{n-5}, \lambda)), \\ P(X_2, \lambda) &= (\lambda - 2)^3 ((\lambda - 3)P(C_{n-4}, \lambda) + P(C_{n-5}, \lambda)), \\ P(X_3, \lambda) &= (\lambda - 2)^3 (P(C_{n-3}, \lambda) - 2P(C_{n-4}, \lambda) + P(C_{n-5}, \lambda)), \\ P(X_4, \lambda) &= (\lambda - 2)^2 ((\lambda^2 - 5\lambda + 7)P(C_{n-4}, \lambda) - P(C_{n-5}, \lambda)), \\ P(X_5, \lambda) &= (\lambda - 2)^2 (\lambda^2 - 5\lambda + 7)P(C_{n-4}, \lambda), \\ P(X_6, \lambda) &= (\lambda - 2)^2 ((\lambda - 2)P(C_{n-3}, \lambda) - (2\lambda - 5)P(C_{n-4}, \lambda)), \\ P(X_7, \lambda) &= (\lambda - 2)^2 ((\lambda - 3)P(C_{n-3}, \lambda) - (\lambda - 4)P(C_{n-4}, \lambda) + P(C_{n-5}, \lambda)), \\ P(X_8, \lambda) &= (\lambda - 2)^2 ((\lambda - 2)P(C_{n-3}, \lambda) - (2\lambda - 5)P(C_{n-4}, \lambda) - P(C_{n-5}, \lambda)), \end{aligned}$$

and the chromatic polynomials for other graphs G of the lemma are of the following form : $P(G, \lambda) = (\lambda - 1)(\lambda - 2)Q(G, \lambda)$, where the factor $Q(G, \lambda)$ is presented in Table 1 and $(\lambda - 2)^2 \nmid P(G, \lambda)$.

Table 1

G	$Q(G, \lambda)$
X_9	$[(\lambda - 2)^3 - (\lambda - 2)^2 + (\lambda - 2) - 1][(\lambda - 1)^{n-5} + (-1)^{n-4}] + (\lambda - 1)^{n-6} + (-1)^{n-5}$
X_{10}	$(\lambda - 2)[(\lambda - 1)^{n-3} - 2(\lambda - 1)^{n-4} - (\lambda - 4)(\lambda - 1)^{n-5} - (-1)^n(\lambda - 7)] - [(\lambda - 1)^{n-5} - (\lambda - 1)^{n-6} + 2(-1)^n]$
X_{11}	$(\lambda - 2)\{(\lambda - 1)^{n-3} - (\lambda - 1)^{n-4} + (\lambda - 1)^{n-5} + 3(-1)^{n-2} - 2(\lambda - 2)[(\lambda - 1)^{n-5} + (-1)^{n-4}]\} - [(\lambda - 1)^{n-5} - (\lambda - 1)^{n-6} + 2(-1)^{n-4}]$
X_{12}	$(\lambda - 2)[(\lambda - 1)^{n-3} + (-1)^{n-2}] - \{(\lambda - 3)[(\lambda - 1)^{n-4} + (-1)^{n-3}] + (\lambda^2 - 5\lambda + 7)[(\lambda - 1)^{n-5} + (-1)^{n-4}]\}$
X_{13}	$(\lambda^2 - 6\lambda + 9)[(\lambda - 1)^{n-4} + (-1)^{n-3}] + (2\lambda - 5)[(\lambda - 1)^{n-5} + (-1)^{n-4}]$
X_{14}, X_{15}	$(\lambda^2 - 5\lambda + 7)[(\lambda - 1)^{n-5}(\lambda - 2) - 2(-1)^n]$

Since $P(C_n, \lambda) = (\lambda-1)((\lambda-1)^{n-1} + (-1)^n)$, we get the following properties:

$$\begin{aligned} (\lambda-2)^5 & \mid P(X_0, \lambda); \\ (\lambda-2)^4 & \mid P(X_1, \lambda) \text{ and } (\lambda-2)^5 \nmid P(X_1, \lambda); \\ (\lambda-2)^3 & \mid P(X_i, \lambda) \text{ and } (\lambda-2)^4 \nmid P(X_i, \lambda) \text{ for } i = 2, 3; \\ (\lambda-2)^2 & \mid P(X_i, \lambda) \text{ and } (\lambda-2)^3 \nmid P(X_i, \lambda) \text{ for } 4 \leq i \leq 8; \\ (\lambda-2)^2 & \nmid P(X_i, \lambda) \text{ for } 9 \leq i \leq 15; \end{aligned}$$

Evidently graphs X_{14}, X_{15} are χ -equivalent. Looking at the above properties and checking the values of chromatic polynomials for $\lambda = 2, 3$ or 4 , we calculate that other pairs of the graphs are not χ -equivalent. This completes the proof. ■

Theorem 3. *For each $n \geq 8$, a 2-connected 3-chromatic graph of order n with five triangles and cyclomatic number six is χ -equivalent to one of the graphs $X_i(n), i = 0, \dots, 14$ presented in Figure 1.*

Proof. Let R be a 2-connected 3-chromatic graph of order $n \geq 8$ with five triangles and cyclomatic number six.

Suppose that there exists a graph $G \not\sim R$ and such that $G \sim R$.

Lemma 1 implies $|V(G)| = n, |E(G)| = n + 5, \chi(G) = 3, I_G(K_3) = 5, I_G(K_4) = 0$ and G is a 2-connected graph. Let H be a subgraph of G induced by the edges of the five triangles in G , and let $|V(H)| = h$. So

$$(1) \quad 6 \leq h$$

Now we define some parameters, which will be useful for the description of all possible candidates for H with h vertices and five triangles, and its supergraph G . Let

$$\begin{aligned} \alpha &= 2(|E(H)| - h), \\ \beta' &= |\{x \in V(H) \mid d_H(x) = 2 \text{ and } d_G(x) = 3\}|, \\ \beta'' &= |\{x \in V(H) \mid d_H(x) = 2 \text{ and } d_G(x) \geq 4\}|, \\ (2) \quad \gamma &= |\{x \in V(H) \mid d_H(x) \geq 3 \text{ and } d_G(x) = d_H(x) + 1\}|, \\ \gamma' &= |\{x \in V(H) \mid d_H(x) \geq 3 \text{ and } d_G(x) \geq d_H(x) + 2\}|, \\ \delta &= |\{x \in V(G) - V(H) \mid d_G(x) \geq 3\}|. \end{aligned}$$

We have

$$\begin{aligned} 2(n+5) &\geq \sum(d_H(x) \mid x \in V(H)) + \beta' + 2\beta'' + 2(n-h) + \gamma + 2\gamma' + \delta \\ &= \alpha + \beta' + 2\beta'' + \gamma + 2\gamma' + \delta + 2n. \end{aligned}$$

This implies that

$$(3) \quad \alpha + \beta' + 2\beta'' + \gamma + 2\gamma' + \delta \leq 10.$$

Let c be the number of connected components of the graph H . Evidently each connected component contains at least one triangle. Since G is a 2-connected graph and H has five triangles and it does not contain K_4 , then the cyclomatic number of H is equal to 5 if H is disconnected, and it is equal to 5 or 6 if H is connected. So by (2) we get

$$(4) \quad \alpha = 10 - 2c \text{ if } c > 1$$

and

$$(5) \quad \alpha \geq 8 \text{ if } c = 1.$$

The list of all possible candidates for H with h vertices and five triangles will be described by considering the following five cases. Three of them are very simple. For the cases 4 and 5 we use the known theorem of Erdős and Gallai on characterization of degree sequences (see [4], Theorem 6.2). All resulting graphs are presented in Figures 2–3 if $H \not\simeq G$ and in Figure 4 if $H \simeq G$.

Case 1. Suppose that $c = 5$, and let H_i , $i = 1, 2, 3, 4, 5$ be connected components of H . Evidently each of H_i is isomorphic to K_3 .

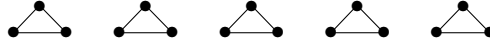
Now 2-connectivity of G and formulas (3)–(4) imply $\alpha = 0$, $\beta' = 10$, $\beta'' = 0$, $\gamma = 0$, $\gamma' = 0$, $\delta = 0$.

Case 2. Suppose that $c = 4$, and let H_i , $i = 1, 2, 3, 4$ be connected components of H . Now 2-connectivity of G and formulas (3)–(4) imply $\alpha = 2$, $\beta' + \gamma = 8$, $\beta'' = 0$, $\gamma' = 0$, $\delta = 0$. Evidently each of H_i , $i = 1, 2, 3$ is isomorphic to K_3 and H_4 is isomorphic to $2K_1 + K_2$ or $K_1 + 2K_2$.

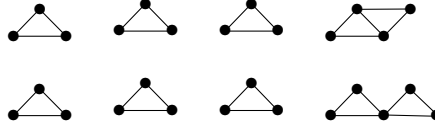
Case 3. Suppose that $c = 3$, and let H_i , $i = 1, 2, 3$ be connected components of H . Now 2-connectivity of G and formulas (3)–(4) imply $\alpha = 4$, $\beta' + \gamma = 6$, $\beta'' = 0$, $\gamma' = 0$, $\delta = 0$. Moreover, if a graph H_i has a cut vertex x , then the graph $H_i - x$ has exactly two connected components.

Thus if two graphs of H_i , $i = 1, 2, 3$ are isomorphic to K_3 then, the other one is isomorphic to the last graph presented in lines 1–5 of Figure 2 and if exactly one of H_i , $i = 1, 2, 3$ is isomorphic to K_3 , then the other two are isomorphic to a graph $2K_1 + K_2$ or $K_1 + 2K_2$ (see lines 6–8 of Figure 2).

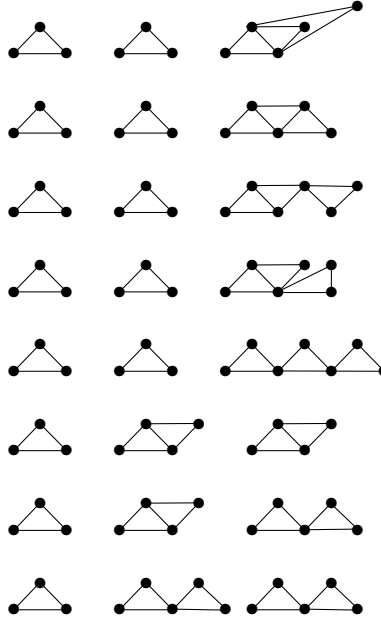
Case 4. Suppose that $c = 2$, and let H_1, H_2 be connected components of H . Then from (4) $\alpha \geq 6$ and from (3) $\beta' + \gamma = 4, \beta'' = 0, \gamma' = 0, \delta = 0$. So 2-connectivity of G implies that if a graph H_i has a cut vertex x , then the graph $H_i - x$ has two connected components. Thus we have to consider three cases for the second component H_2 . Namely, $H_2 \cong K_3, K_1 + 2K_2, 2K_1 + K_2$. Considering simple degree conditions we get all available candidates for H_1 presented in Figure 2 (continued).



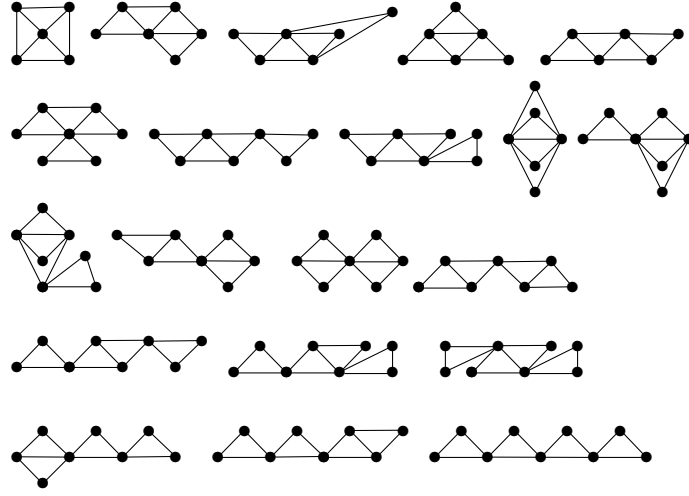
$c = 5$



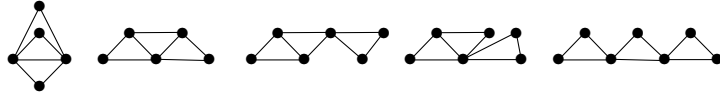
$c = 4$



$c = 3$



$$c = 2, H_2 = K_3$$



$$c = 2, H_2 = K_1 + 2K_2$$

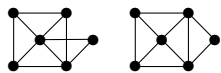
$$\text{or } H_2 = 2K_1 + K_2$$

Figure 2

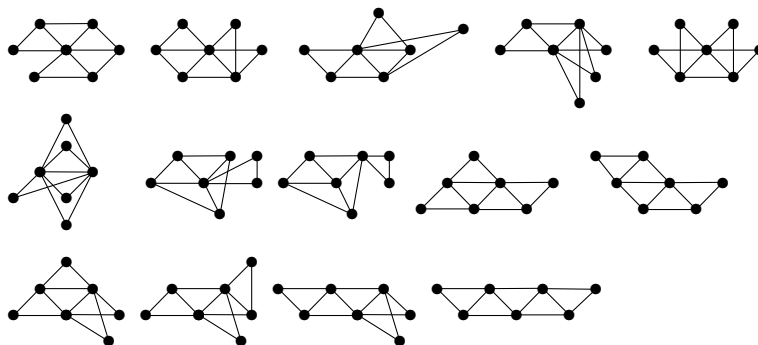
Case 5. Assume that $c = 1$. Then from (2) and (3) $\alpha \geq 8$ and $\beta' + \gamma \leq 2$. Moreover, $\beta'' + \gamma' + \delta = 0$. If $\beta' + \gamma = 0$ we get $H \simeq G$ and Figure 4 lists all such graphs (each edge belongs to a triangle). For the opposite case $\beta' + \gamma = 2$. This follows by 2-connectivity of G . Moreover, 2-connectivity of G implies that if a graph H has a cut vertex x , then the graph $H - x$ has two connected components. Since H has five triangles, we get $h \leq 11$. Considering $h = 6, \dots, 11$ and keeping the inequalities (2) and (3) we get all available candidates for H presented in Figure 3.

For each case of $c = 5, 4, 3, 2$ and 1 (if $H \not\simeq G$) each required 2-connected graph G is obtained from H by adding paths in such a way that exactly two vertices of each connected component of H are incident to an edge outside H . Looking at the graphs H and Lemma 2 we get all information on G presented in Table 1. The last column of Table 2 lists the graphs X_i that are χ -equivalent to respective graphs G . The column NB denotes a consecutive number of a graph H or H_1 for each respective group. This completes the proof. ■

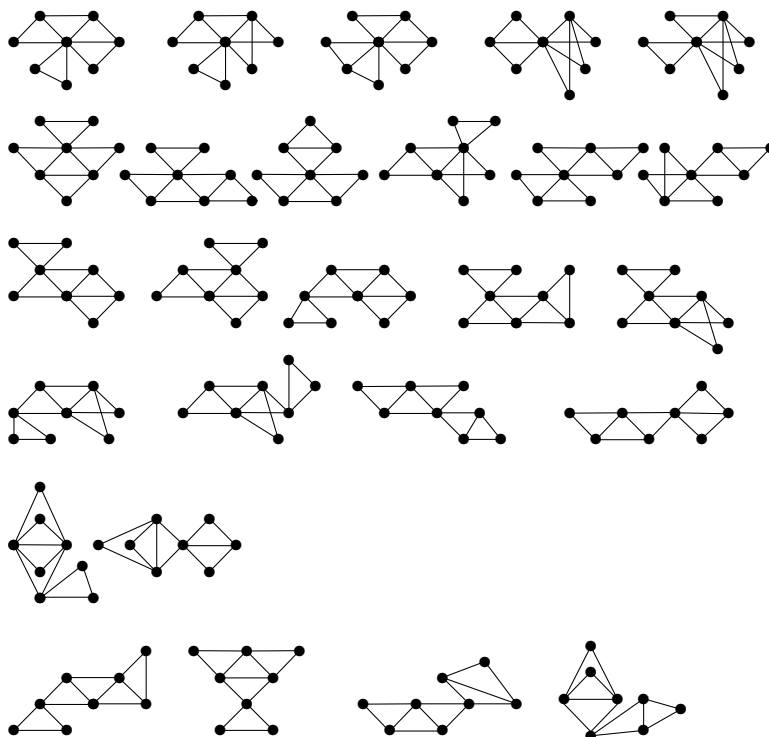
$$c = 1, h = 6$$



$$c = 1, h = 7$$



$$c = 1, h = 8$$



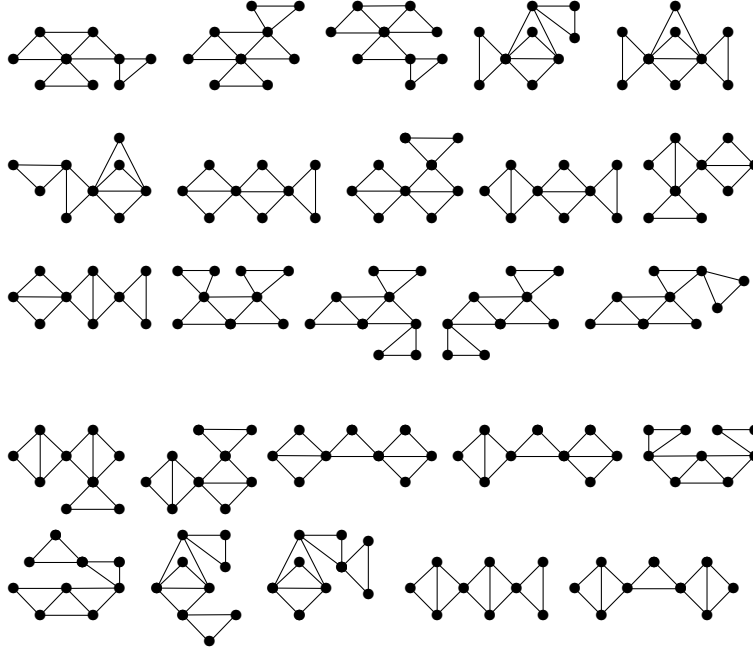
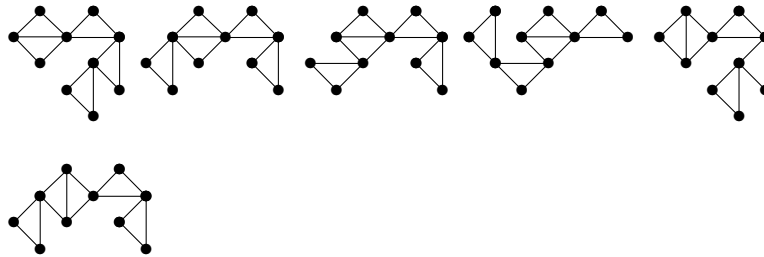
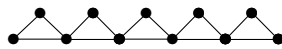
$c = 1, h = 9$

 $h = 10$

 $h = 11$


Figure 3

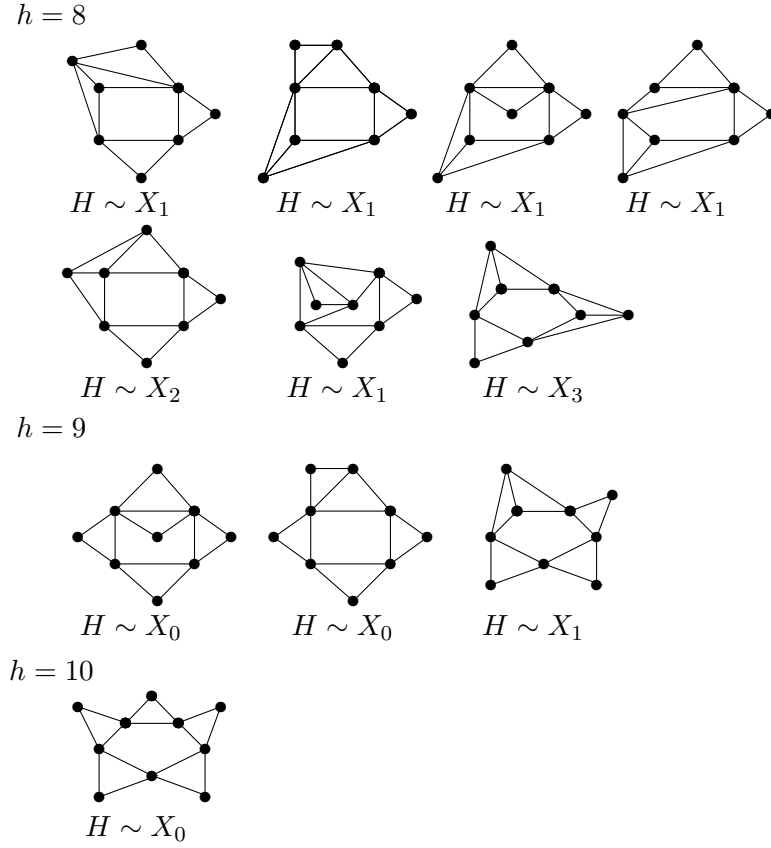


Figure 4

Immediately from the proof of Theorem 3 and Lemma 2 we get the following three results.

Corollary 4. *Each $X_i(n)$ for $9 \leq i \leq 13$ and $n \geq 8$ is a χ -unique graph of order n .*

The new chromatically unique graphs of order n are the graphs $X_i(n)$ for $i = 12, 13$ (see Figure 1) for each $n \geq 8$.

Corollary 5. *Each χ -equivalence class containing graphs $X_{14}(n), X_{15}(n)$ for $n \geq 8$ has exactly these two nonisomorphic elements.*

Corollary 6. *Each χ -equivalence class containing a graph $X_7(n)$, $n \geq 7$, consists of all graphs defined in the conjecture of Li and Whitehead [5].*

Table 2

$c =$	$h =$ or H_2	NB	$X_i, i =$
5	15	1	0
4	13	1	0, 1
	14	2	0
3	11	1	0, 1
	11	2	0, 1, 2
	12	3	0, 1
	12	4	0, 1
	13	5	0
	11	6	0, 1, 3
	12	7	0, 1
	13	8	0, 1
2	K_3	1	5, 6
		2	0, 1, 2, 4
		3	0, 1, 2
		4	0, 1, 2
		5	0, 1, 2, 8
		6	0
		7	0, 1, 2
		8	0, 1
		9	0, 1
		10	0
		11	0, 1
		12	0, 1
		13	0
		14	0, 1, 3
		15	1
		16	0
		17	0
		18	0
		19	0, 1
		20	0
	$K_1 + 2K_2$	1	0
		2	0, 1, 2
		3	0, 1
		4	0
		5	0
	$2K_1 + K_2$	1	0, 1
		2	0, 1, 2, 3, 7
		3	0, 1, 3
		4	0, 3
		5	0, 1
1	6	1	5, 6, 12, 14
	7	2	5, 6, 13, 14
		1	0, 1, 2, 4, 9
		2	0, 1, 2, 4
		3	0, 1, 2, 4
		4	0, 1, 2
		5	0, 1, 2
		6	0, 1
		7	5
		8	5, 6
		9	0, 1, 2, 4, 8
		10	0, 1, 2, 4, 8, 11
		11	0, 1, 2
		12	0, 1, 2, 8
		13	0, 2, 2, 8
		14	0, 1, 2, 8, 10

$c =$	$h =$ or H_2	NB	$X_i, i =$
1	8	1	0
		2	0
		3	0
		4	0
		5	0
		6	0, 1
		7	0, 1
		8	0, 1
		9	0, 1
		10	0, 1
		11	0, 1
		12	0, 1, 2
		13	0, 1
		14	0, 1, 2, 4
		15	0, 1, 2
		16	0, 1, 2
		17	0, 1, 2
		18	0, 1, 2
		19	0, 1, 3
		20	0, 1, 2
		21	0, 1
		22	0, 1
		23	0, 1, 2, 8
		24	0, 1, 2
		25	0, 1, 2, 3, 7
		26	0, 1, 3
	9	1	0
		2	0
		3	0
		4	0
		5	0
		6	0
		7	0
		8	0
		9	0, 1
		10	0
		11	1
		12	0
		13	0
		14	1
		15	0, 1
		16	0, 1
		17	0, 1
		18	0
		19	0
		20	2
		21	2
		22	1
		23	0, 1
		24	0, 1, 3
		25	0, 1, 3
	10	1	0
		2	0
		3	0
		4	0
		5	0, 1
		6	1
	11	1	0

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