

## KERNELS IN EDGE COLOURED LINE DIGRAPH

H. GALEANA-SÁNCHEZ

*Instituto de Matemáticas, U.N.A.M., C.U.  
Circuito Exterior, D. F. 04510 México*

AND

L. PASTRANA RAMÍREZ

*Departamento de Matemáticas de la Facultad de Ciencias  
de la Universidad Nacional Autónoma de México  
México, D.F.*

### Abstract

We call the digraph  $D$  an  $m$ -coloured digraph if the arcs of  $D$  are coloured with  $m$  colours. A directed path (or a directed cycle) is called monochromatic if all of its arcs are coloured alike. A set  $N \subseteq V(D)$  is said to be a kernel by monochromatic paths if it satisfies the two following conditions (i) for every pair of different vertices  $u, v \in N$  there is no monochromatic directed path between them and (ii) for every vertex  $x \in V(D) - N$  there is a vertex  $y \in N$  such that there is an  $xy$ -monochromatic directed path.

Let  $D$  be an  $m$ -coloured digraph and  $L(D)$  its line digraph. The inner  $m$ -coloration of  $L(D)$  is the edge coloration of  $L(D)$  defined as follows: If  $h$  is an arc of  $D$  of colour  $c$ , then any arc of the form  $(x, h)$  in  $L(D)$  also has colour  $c$ .

In this paper it is proved that if  $D$  is an  $m$ -coloured digraph without monochromatic directed cycles, then the number of kernels by monochromatic paths in  $D$  is equal to the number of kernels by monochromatic paths in the inner edge coloration of  $L(D)$ .

**Keywords:** kernel, kernel by monochromatic paths, line digraph, edge coloured digraph.

**1991 Mathematics Subject Classification:** 05C20.

### 1. INTRODUCTION

For general concepts we refer the reader to [1]. The existence of kernels by monochromatic paths in edge coloured digraphs was studied primarily by Sauer, Sands and Woodrow in [4]; they proved that any 2-coloured digraph

has a kernel by monochromatic paths; sufficient conditions for the existence of kernels by monochromatic paths in  $m$ -coloured digraphs have been studied in [2], [3], [4], [5].

**Definition 1.1.** The line digraph of  $D = (X, U)$  is the digraph  $L(D) = (U, W)$  (we also denote  $U = V(L(D))$ ) and  $W = A(L(D))$  with a set of vertices as the set of arcs of  $D$ , and for any  $h, k \in U$  there is  $(h, k) \in W$  if and only if the corresponding arcs  $h, k$  induce a directed path in  $D$ ; i.e., the terminal endpoint of  $h$  is the initial endpoint of  $k$ .

In what follows, we denote the arc  $h = (u, v) \in U$  and the vertex  $h$  in  $L(D)$  by the same symbol.

If  $H$  is a subset of arcs in  $D$  it is also a subset of vertices of  $L(D)$ . When we want to emphasize our interest in  $H$  as a set of vertices of  $L(D)$ , we use the symbol  $H_L$  instead of  $H$ .

**Definition 1.2.** Let  $D$  be an  $m$ -coloured digraph and  $L(D)$  its line digraph; the inner  $m$ -coloration of  $L(D)$  is the edge coloration of  $L(D)$  defined as follows: If  $h$  is an arc of  $D$  with colour  $c$  then any arc of the form  $(x, h)$  in  $L(D)$  also has colour  $c$ .

**Definition 1.3.** A subset  $N \subseteq V(D)$  is said to be independent by monochromatic paths if for every pair of different vertices  $u, v \in N$  there is no  $uv$ -monochromatic directed path. The subset  $N \subseteq V(D)$  is absorbant by monochromatic paths if for every vertex  $x \in V(D) - N$  there is a vertex  $y \in N$  such that there is an  $xy$ -monochromatic directed path. And a subset  $N \subseteq V(D)$  is said to be a kernel by monochromatic paths if  $N$  is both independent and absorbant by monochromatic paths.

**Definition 1.4.** A sequence of vertices  $x_1, x_2, \dots, x_n$  such that  $(x_i, x_{i+1}) \in U$  for  $1 \leq i \leq n - 1$  is called a *directed walk*; when  $x_i \neq x_j$  for  $i \neq j$ ,  $1 \leq i, j \leq n$  will be called a *directed path*.

## 2. KERNELS IN EDGE COLOURED LINE DIGRAPH

**Lemma 2.1.** Let  $D$  be an  $m$ -coloured digraph,  $x_0, x_n \in V(D)$ ,  $T = (x_0, x_1, \dots, x_{n-1}, x_n)$  a monochromatic directed path in  $D$  and  $a_0 = (x, x_0)$  be an arc of  $D$  whose terminal endpoint is  $x_0$ . There exists an  $a_0 a_n$ -monochromatic directed path in the inner  $m$ -coloration of  $L(D)$ , where  $a_n = (x_{n-1}, x_n)$ .

**Proof.** Denote by  $a_i = (x_{i-1}, x_i)$ ; for  $i = 1, 2, \dots, n$ . Since  $T$  is a directed path in  $D$ , it follows from Definition 2.1 that  $(a_1, a_2, \dots, a_n)$  is a directed path in  $L(D)$ ; in fact, the choice of  $a_0$  and Definition 2.1 imply  $(a_0, a_1, \dots, a_n)$  is a directed path in  $L(D)$ .

Suppose without loss of generality that  $T$  is monochromatic of colour  $c$ . Since  $a_{i+1}$  has colour  $c$  for  $0 \leq i \leq n-1$  it follows from Definition 1.2 that  $(a_i, a_{i+1})$  has colour  $c$  for  $0 \leq i \leq n-1$ , hence  $(a_0, a_1, \dots, a_n)$  is a monochromatic directed path of colour  $c$ . ■

**Lemma 2.2.** *Let  $D$  be an  $m$ -coloured digraph without monochromatic directed cycles,  $a_0, a_n \in V(L(D))$ . If there exists an  $a_0, a_n$ -monochromatic directed path in the inner  $m$ -coloration of  $L(D)$ , then the terminal endpoint of  $a_0$  is different from the terminal endpoint of  $a_n$  and there exists a monochromatic directed path from the terminal endpoint of  $a_0$  to the terminal endpoint of  $a_n$  in  $D$ .*

**Proof.** Let  $(a_0, a_1, \dots, a_n)$  be a monochromatic directed path of colour  $c$  in the inner  $m$ -coloration of  $L(D)$  and  $a_i = (x_i, x_{i+1})$ ,  $0 \leq i \leq n$ . It follows from Definition 2.1 that  $(x_1, \dots, x_{n+1})$  is a directed walk in  $D$ ; since  $(a_i, a_{i+1})$  has colour  $c$ ,  $0 \leq i \leq n-1$  it follows from Definition 1.2 that  $a_{i+1}$  has colour  $c$  in  $D$ ,  $0 \leq i \leq n-1$ . Hence  $(x_1, x_2, \dots, x_n, x_{n+1})$  is a monochromatic directed walk of colour  $c$  in  $D$ . Since  $D$  has no monochromatic directed cycles it follows that  $x_i \neq x_j \ \forall i \neq j$ ,  $1 \leq i \leq n+1$ ,  $1 \leq j \leq n+1$ ; in particular  $x_1 \neq x_{n+1}$  (Notice that any monochromatic closed directed walk contains a monochromatic directed cycle) and  $(x_1, \dots, x_{n+1})$  is a monochromatic directed path. ■

**Definition 2.1.** Let  $D = (X, U)$  be a digraph. We denote by  $\mathcal{P}(\mathcal{X})$  the set of all the subsets of the set  $X$  and  $f: \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{U})$  will denote the function defined as follows: for each  $Z \subseteq X$ ,  $f(Z) = \{(u, x) \in U \mid x \in Z\}$ .

**Lemma 2.3.** *Let  $D$  be an  $m$ -coloured digraph without monochromatic directed cycles; if  $Z \subseteq V(D)$  is independent by monochromatic paths in  $D$ , then  $f(Z)_L$  is independent by monochromatic paths in the inner  $m$ -coloration of  $L(D)$ .*

**Proof.** We proceed by contradiction. Let  $D$  be an  $m$ -coloured digraph and  $Z \subseteq V(D)$  independent by monochromatic paths. Suppose (by contradiction) that  $f(Z)_L$  is not independent by monochromatic paths in the

inner  $m$ -coloration of  $L(D)$ . Then there exists  $h, k \in f(Z)_L$  and an  $hk$ -monochromatic directed path in the inner  $m$ -coloration of  $L(D)$ . It follows from Lemma 2.2 that the terminal endpoint of  $h$  is different from the terminal endpoint of  $k$  and there exists a monochromatic directed path from the terminal endpoint of  $h$  to the terminal endpoint of  $k$ . Since  $h \in f(Z)_L$  (resp.  $k \in f(Z)_L$ ) we have from Definition 2.1 that the terminal endpoint of  $h$  (resp. of  $k$ ) is in  $Z$ ; so we have a monochromatic directed path between two vertices of  $Z$ , a contradiction. ■

**Theorem 2.1.** *Let  $D = (X, U)$  be an  $m$ -coloured digraph without monochromatic directed cycles. The number of kernels by monochromatic paths of  $D$  is equal to the number of kernels by monochromatic paths in the inner  $m$ -coloration of  $L(D)$ .*

**Proof.** Denote by  $\mathcal{K}$  the set of all the kernels by monochromatic paths of  $D$  and by  $\mathcal{K}^*$  the set of all the kernels by monochromatic paths in the inner  $m$ -coloration of  $L(D)$ .

(1) If  $Z \in \mathcal{K}$ , then  $f(Z)_L \in \mathcal{K}^*$ . Since  $Z \in \mathcal{K}$ , we have that  $Z$  is independent by monochromatic paths and Lemma 2.3 implies that  $f(Z)_L$  is independent by monochromatic paths. Now we will prove that  $f(Z)_L$  is absorbant by monochromatic paths. Let  $k = (u, v)$  be a vertex of  $L(D)$  such that  $k \in (V(L(D)) - f(Z)_L)$ , it follows from Definition 2.1 that  $v \in (V(D) - Z)$ . Since  $Z$  is a kernel by monochromatic paths of  $D$ , it follows from Definition 1.3 that there exists  $z \in Z$  and a monochromatic directed path from  $v$  to  $z$  in  $D$ , say  $(v = x_0, x_1, \dots, x_{n-1}, x_n = z)$ . Then it follows from Lemma 2.1 that there exists an  $(u, v)(x_{n-1}, x_n)$ -monochromatic directed path in the inner  $m$ -coloration of  $L(D)$  and since  $z \in Z$ , we have from Definition 2.1 that  $(x_{n-1}, x_n = z) \in f(Z)_L$ .

(2) The function  $f': \mathcal{K} \rightarrow \mathcal{K}^*$ , where  $f'$  is the restriction of  $f$  to  $\mathcal{K}$  is an injective function. Let  $Z_1, Z_2 \in \mathcal{K}$  and  $Z_1 \neq Z_2$ . Let us suppose, e.g., that  $Z_1 - Z_2 \neq \emptyset$ . Let  $v \in (Z_1 - Z_2)$ , since  $Z_2$  is a kernel by monochromatic paths of  $D$ , it follows from Definition 1.3 that there exists  $u \in Z_2$  and a  $vu$ -monochromatic directed path, let  $h = (x_n, u)$  be the last arc of such a path. It follows from Definition 2.1 that  $h \in f(Z_2)_L$ . Finally, notice that since  $v \in Z_1$ , the subset  $Z_1$  is independent by monochromatic paths and there exists a  $vu$ -monochromatic directed path, we have that  $u \notin Z_1$  and then  $h \notin f(Z_1)_L$ . Hence  $h \in (f(Z_2)_L - f(Z_1)_L)$  and so  $f(Z_1)_L \neq f(Z_2)_L$ .

Define a function  $g: \mathcal{P}(U) \rightarrow \mathcal{P}(X)$  as follows:

If  $H \subseteq U$ , then  $g(H) = C(H) \cup D(H)$ , where  $C(H) = \{x \in X \mid \text{there exists } (z, x) \in H\}$  (the set of all the terminal endpoints of arcs of  $H$ ).  
 $D(H) = \{x \in X \mid \delta_D^-(x) = 0 \text{ and there is no monochromatic directed path from } x \text{ to } C(H)\}$ . (Where  $\delta_D^-(x) = \{y \in V(D) \mid (y, x) \in U\}$ ).

(3) If  $H_L \in \mathcal{K}^*$ , then  $g(H_L) \in \mathcal{K}$ .

(3.1) If  $H_L \in \mathcal{K}^*$ , then  $g(H_L)$  is independent by monochromatic paths.

Suppose that  $H_L \in \mathcal{K}^*$ , and let  $u, v \in g(H_L)$ ,  $u \neq v$ ; we will prove that there is no  $uv$ -monochromatic directed path in  $D$ . We will analyze several cases:

*Case 1.*  $u, v \in C(H_L)$ .

In this case we proceed by contradiction. Suppose (by contradiction) that there exists an  $uv$ -monochromatic directed path  $T = (u = x_0, x_1, \dots, x_n = v)$  in  $D$ . Since  $u, v \in C(H_L)$ ,  $u$  is the terminal endpoint of an arc  $h \in H_L$  and  $v$  is the terminal endpoint of an arc  $k \in H_L$ .

When  $k = (x_{n-1}, x_n = v)$  we have from Lemma 2.1 that there exists an  $hk$ -monochromatic directed path, a contradiction (because  $H_L$  is independent by monochromatic paths and  $h, k \in H_L$ ).

Otherwise if  $k \neq (x_{n-1}, x_n = v)$ , we have  $(x_{n-1}, x_n = v) \notin H_L$  (because if  $(x_{n-1}, x_n = v) \in H_L$  we would have the monochromatic directed path  $(h, a_0, a_1, \dots, a_{n-1})$  where  $a_i = (x_i, x_{i+1})$ ,  $0 \leq i \leq n-1$ ; from  $h$  to  $(x_{n-1}, x_n = v) = a_{n-1}$  with  $h, a_{n-1} \in H_L$ , a contradiction). Since  $H_L$  is absorbant by monochromatic paths and  $a_{n-1} = (x_{n-1}, x_n = v) \notin H_L$ , there exists  $b \in H_L$  and an  $a_{n-1}b$ -monochromatic directed path in the inner  $m$ -coloration of  $L(D)$ ; let  $(a_{n-1} = b_0, b_1, \dots, b_m = b)$  be such a path. Since the terminal endpoint of  $k$  is  $v$  (the same as  $a_{n-1} = b_0$ ) we have from Definitions 1.1 and 1.2 that also  $(k, b_1, b_2, \dots, b_m = b)$  is a monochromatic directed path in the inner  $m$ -coloration of  $L(D)$  with  $k, b \in H_L$ , a contradiction.

*Case 2.*  $u \in C(H_L), v \in D(H_L)$ .

Since  $v \in D(H_L)$ , we have  $\delta_D^-(v) = 0$ , so there is no  $uv$ -monochromatic directed path in  $D$ .

*Case 3.*  $u \in D(H_L), v \in C(H_L)$ .

Since  $u \in D(H_L)$ , we have that there is no monochromatic directed path from  $u$  to  $C(H_L)$ , in particular there is no  $uv$ -monochromatic directed path.

*Case 4.*  $u, v \in D(H_L)$ .

Since  $v \in D(H_L)$ , we have  $\delta_D^-(v) = 0$  and clearly, there is no  $uv$ -monochromatic directed path in  $D$ .

(3.2) If  $H_L \in \mathcal{K}^*$ , then  $g(H_L)$  is absorbant by monochromatic paths.

Let  $u \in X - g(H_L) = X - (C(H_L) \cup D(H_L))$ . Since  $u \notin (C(H_L) \cup D(H_L))$ , we have that there is no arc in  $H$  whose terminal endpoint is  $u$ , and at least one of the two following conditions holds:  $\delta_D^-(u) > 0$  or there exists a monochromatic directed path from  $u$  to  $C(H_L)$ .

We will analyze the two possible cases.

*Case 1.* There is no arc in  $H_L$  whose terminal endpoint is  $u$  and  $\delta_D^-(u) > 0$ . The hypothesis in this case implies that there exists an arc  $(t, u) \in U - H_L$ . Since  $H_L \in \mathcal{K}^*$ , we have that  $H_L$  is absorbant by monochromatic paths; hence there exists  $p = (s, m) \in H_L$  and a monochromatic directed path from  $(t, u)$  to  $p$ . Now it follows from Lemma 2.2 that  $u$  is different from  $m$  and there exists a monochromatic directed path from  $u$  to  $m$ . Finally, notice that since  $(s, m) \in H_L$ , we have  $m \in g(H_L)$ . So there exists a monochromatic directed path from  $u$  to  $m$  with  $m \in g(H_L)$ .

*Case 2.* There is no arc in  $H_L$  whose terminal endpoint is  $u$  and there exists a monochromatic directed path from  $u$  to  $C(H_L)$ .

Clearly in this case we have a monochromatic directed path from  $u$  to  $g(H_L) = C(H_L) \cup D(H_L)$ .

(4) The function  $g': \mathcal{K}^* \rightarrow \mathcal{K}$ , where  $g'$  is the restriction of  $g$  to  $\mathcal{K}$  is an injective function. Let  $N_L, P_L \in \mathcal{K}^*$ , such that  $N_L \neq P_L$ . Let us suppose, e.g., that  $N_L - P_L \neq \emptyset$ . Let  $h \in N_L - P_L$ , and  $u$  the terminal endpoint of  $h$ . Since  $u$  is the terminal endpoint of an arc in  $N_L$ , we have that  $u \in g(N_L)$ . Now we will prove that  $u \notin g(P_L)$ . Since  $P_L$  is absorbant by monochromatic paths and  $h \notin P_L$ , we have that there exists  $k \in P_L$  and an  $hk$ -monochromatic directed path in the inner  $m$ -coloration of  $L(D)$ .

Let  $v$  be the terminal endpoint of  $k$ ; hence  $v \in g(P_L)$  and it follows from Lemma 2.2 that  $u$  is different from  $v$  and there exists an  $uv$ -monochromatic directed path in  $D$ . Since  $g(P_L)$  is independent by monochromatic paths (This follows directly from (3) and Definition 1.3), we have that  $u \notin g(P_L)$ . We conclude  $u \in g(N_L) - g(P_L)$  and so  $g(N_L) \neq g(P_L)$ . Finally, notice that it follows from (2) and (4) that:

$\text{Card } \mathcal{K} \leq \text{Card } \mathcal{K}^* \leq \text{Card } \mathcal{K}$  and hence  $\text{Card } \mathcal{K} = \text{Card } \mathcal{K}^*$ . ■

**Note 2.1.** Let  $D$  be an  $m$ -coloured digraph and  $L(D)$  its line digraph; similarly as in Definition 1.2 we can define the outer  $m$ -coloration of  $L(D)$  as follows: If  $h$  is arc of  $D$  with colour  $c$ , then any arc of the form  $(h, x)$  in  $L(D)$  also has colour  $c$ . However, Theorem 2.1 does not hold if we change inner  $m$ -coloration of  $L(D)$  by outer  $m$ -coloration of  $L(D)$ . In Figure 1, we show a digraph  $D$  without monochromatic directed cycles with one kernel

by monochromatic paths such that the outer  $m$ -coloration of its line digraph (Figure 2) has no kernel by monochromatic paths.

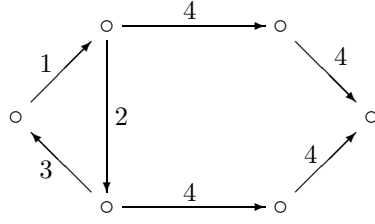


Figure 1

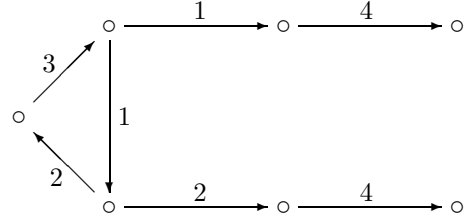


Figure 2

**Note 2.2.** Theorem 2.1 does not hold if we drop the hypothesis that  $D$  has no monochromatic directed cycles. In Figure 3, we show a digraph  $D$  with monochromatic directed cycles which has two kernels by monochromatic paths such that the inner  $m$ -coloration of its line digraph (Figure 4) has just one kernel by monochromatic paths. And in Figure 5, we show a digraph with monochromatic directed cycles without a kernel by monochromatic paths and its line digraph has two kernels by monochromatic paths (see Figure 6).

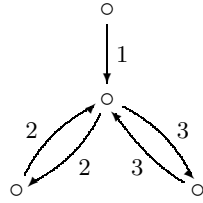


Figure 3

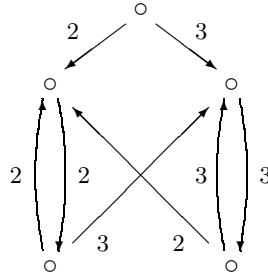


Figure 4

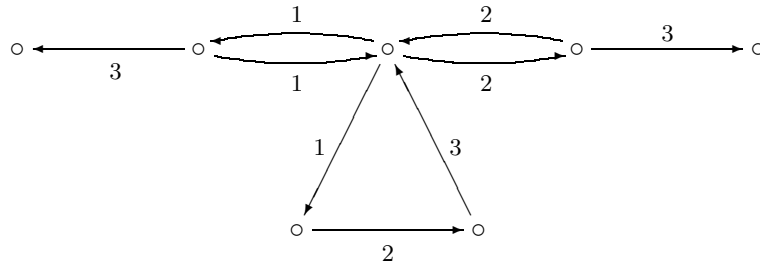


Figure 5



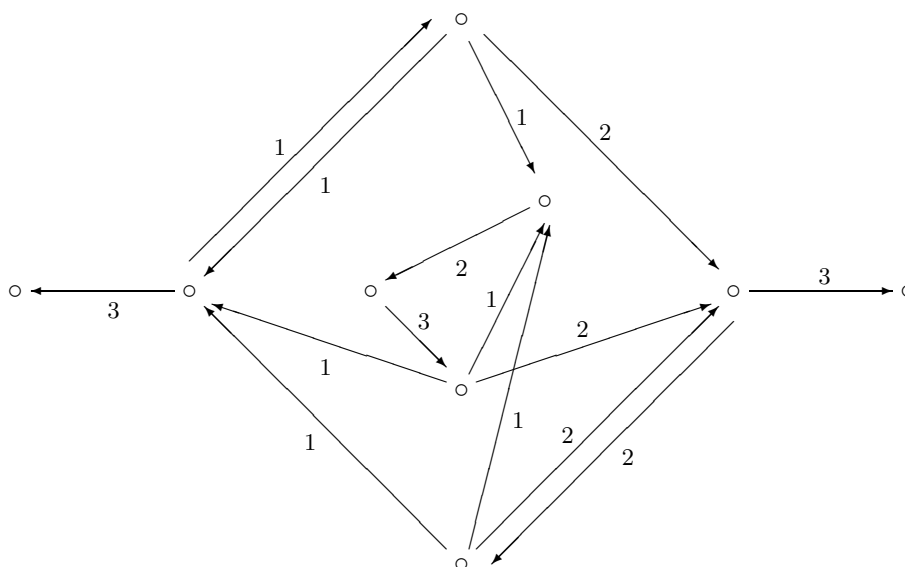


Figure 6

## REFERENCES

- [1] C. Berge, *Graphs* (North Holland, Amsterdam, New York, 1985).
- [2] H. Galeana-Sánchez, *On monochromatic paths and monochromatic cycles in edge coloured tournaments*, *Discrete Math.* **156** (1996) 103–112.
- [3] H. Galeana-Sánchez and J.J. García Ruvalcaba, *Kernels in  $\{C_3, T_3\}$ -free arc colorations of  $K_n - e$* , submitted.
- [4] B. Sands, N. Sauer and R. Woodrow, *On Monochromatic Paths in Edge Coloured Digraphs*, *J. Combin. Theory (B)* **33** (1982) 271–275.
- [5] Shen Minggang, *On Monochromatic Paths in  $m$ -Coloured Tournaments*, *J. Combin. Theory (B)* **45** (1988) 108–111.

Received 27 April 1997

Revised 22 September 1997