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KERNELS IN EDGE COLOURED LINE DIGRAPH

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Abstract

We call the digraph D an m-coloured digraph if the arcs of D are coloured with m colours. A directed path (or a directed cycle) is called monochromatic if all of its arcs are coloured alike. A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths if it satisfies the two following conditions (i) for every pair of different vertices $u, v \in N$ there is no monochromatic directed path between them and (ii) for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an xy-monochromatic directed path.

Let D be an m-coloured digraph and $L(D)$ its line digraph. The inner m-coloration of $L(D)$ is the edge coloration of $L(D)$ defined as follows: If h is an arc of D of colour c, then any arc of the form (x, h) in $L(D)$ also has colour c.

In this paper it is proved that if D is an m-coloured digraph without monochromatic directed cycles, then the number of kernels by monochromatic paths in D is equal to the number of kernels by monochromatic paths in the inner edge coloration of $L(D)$.

Keywords: kernel, kernel by monochromatic paths, line digraph, edge coloured digraph.

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1. Introduction

For general concepts we refer the reader to [1]. The existence of kernels by monochromatic paths in edge coloured digraphs was studied primarily by Sauer, Sands and Woodrow in [4]; they proved that any 2-coloured digraph has a kernel by monochromatic paths; sufficient conditions for the existence of kernels by monochromatic paths in m-coloured digraphs have been studied in [2], [3], [4], [5].

Definition 1.1. The line digraph of $D = (X, U)$ is the digraph $L(D) = (U, W)$ (we also denote $U = V(L(D)))$ and $W = A(L(D))$ with a set of vertices as the set of arcs of D, and for any $h, k \in U$ there is $(h, k) \in W$ if and only if the corresponding arcs h, k induce a directed path in D; i.e., the terminal endpoint of h is the initial endpoint of k.

In what follows, we denote the arc $h = (u, v) \in U$ and the vertex h in $L(D)$ by the same symbol.

If H is a subset of arcs in D it is also a subset of vertices of $L(D)$. When we want to emphasize our interest in H as a set of vertices of $L(D)$, we use the symbol H_L instead of H.

Definition 1.2. Let D be an m-coloured digraph and $L(D)$ its line digraph; the inner m-coloration of $L(D)$ is the edge coloration of $L(D)$ defined as follows: If h is an arc of D with colour c then any arc of the form (x, h) in $L(D)$ also has colour c.

Definition 1.3. A subset $N \subseteq V(D)$ is said to be independent by monochromatic paths if for every pair of different vertices $u, v \in N$ there is no uv-monochromatic directed path. The subset $N \subseteq V(D)$ is absorbant by monochromatic paths if for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an xy-monochromatic directed path. And a subset $N \subseteq V(D)$ is said to be a kernel by monochromatic paths if N is both independent and absorbant by monochromatic paths.

Definition 1.4. A sequence of vertices $x_1, x_2, \ldots, x_n$ such that $(x_i, x_{i+1}) \in U$ for $1 \leq i \leq n-1$ is called a *directed walk*; when $x_i \neq x_j$ for $i \neq j$, $1 \leq i, j \leq n$ will be called a *directed path.*

2. Kernels in Edge Coloured Line Digraph

Lemma 2.1. *Let D be an m-coloured digraph, $x_0, x_n \in V(D)$, $T = (x_0, x_1, \ldots, x_{n-1}, x_n)$ a monochromatic directed path in D and $a_0 = (x, x_0)$ be an arc of D whose terminal endpoint is x_0. There exists an $a_0 a_n$-monochromatic directed path in the inner m-coloration of $L(D)$, where $a_n = (x_{n-1}, x_n)$.*

Proof. Denote by $a_i = (x_{i-1}, x_i)$; for $i = 1, 2, \ldots, n$. Since T is a directed path in D, it follows from Definition 2.1 that $(a_1, a_2, \ldots, a_n)$ is a directed path in $L(D)$; in fact, the choice of a_0 and Definition 2.1 imply $(a_0, a_1, \ldots, a_n)$ is a directed path in $L(D)$.

Suppose without loss of generality that T is monochromatic of colour c. Since a_{i+1} has colour c for $0 \leq i \leq n-1$ it follows from Definition 1.2 that (a_i, a_{i+1}) has colour c for $0 \leq i \leq n-1$, hence $(a_0, a_1, \ldots, a_n)$ is a monochromatic directed path of colour c. ∎

Lemma 2.2. *Let D be an m-coloured digraph without monochromatic directed cycles, $a_0, a_n \in V(L(D))$. If there exists an a_0, a_n-monochromatic directed path in the inner m-coloration of $L(D)$, then the terminal endpoint of a_0 is different from the terminal endpoint of a_n and there exists a monochromatic directed path from the terminal endpoint of a_0 to the terminal endpoint of a_n in D.*

Proof. Let $(a_0, a_1, \ldots, a_n)$ be a monochromatic directed path of colour c in the inner m-coloration of $L(D)$ and $a_i = (x_i, x_{i+1}), 0 \leq i \leq n$. It follows from Definition 2.1 that $(x_1, \ldots x_{n+1})$ is a directed walk in D; since (a_i, a_{i+1}) has colour $c, 0 \leq i \leq n-1$ it follows from Definition 1.2 that a_{i+1} has colour c in D, $0 \leq i \leq n-1$. Hence $(x_1, x_2, \ldots, x_n, x_{n+1})$ is a monochromatic directed walk of colour c in D. Since D has no monochromatic directed cycles it follows that $x_i \neq x_j \ \ \forall\, i \neq j$, $1 \leq i \leq n+1$, $1 \leq j \leq n+1$; in particular $x_1 \neq x_{n+1}$ (Notice that any monochromatic closed directed walk contains a monochromatic directed cycle) and $(x_1, \ldots, x_{n+1})$ is a monochromatic directed path. ∎

Definition 2.1. Let $D = (X, U)$ be a digraph. We denote by $\mathcal{P}(\mathcal{X})$ the set of all the subsets of the set X and $f\colon \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{U})$ will denote the function defined as follows: for each $Z \subseteq X, f(Z) = \{(u, x) \in U \mid x \in Z\}$.

Lemma 2.3. *Let D be an m-coloured digraph without monochromatic directed cycles; if $Z \subseteq V(D)$ is independent by monochromatic paths in D, then $f(Z)_L$ is independent by monochromatic paths in the inner m-coloration of $L(D)$.*

Proof. We proceed by contradiction. Let D be an m-coloured digraph and $Z \subseteq V(D)$ independent by monochromatic paths. Suppose (by contradiction) that $f(Z)_L$ is not independent by monochromatic paths in the inner m-coloration of $L(D)$. Then there exists $h, k \in f(Z)_L$ and an hk-monochromatic directed path in the inner m-coloration of $L(D)$. It follows from Lemma 2.2 that the terminal endpoint of h is different from the terminal endpoint of k and there exists a monochromatic directed path from the terminal endpoint of h to the terminal endpoint of k. Since $h \in f(Z)_L$ (resp. $k \in f(Z)_L$) we have from Definition 2.1 that the terminal endpoint of h (resp. of k) is in Z; so we have a monochromatic directed path between two vertices of Z, a contradiction. ■

Theorem 2.1. *Let $D = (X, U)$ be an m-coloured digraph without monochromatic directed cycles. The number of kernels by monochromatic paths of D is equal to the number of kernels by monochromatic paths in the inner m-coloration of $L(D)$.*

Proof. Denote by $\mathcal{K}$ the set of all the kernels by monochromatic paths of D and by $\mathcal{K}^*$ the set of all the kernels by monochromatic paths in the inner m-coloration of $L(D)$.

(1) If $Z \in \mathcal{K}$, then $f(Z)_L \in \mathcal{K}^*$. Since $Z \in \mathcal{K}$, we have that Z is independent by monochromatic paths and Lemma 2.3 implies that $f(Z)_L$ is independent by monochromatic paths. Now we will prove that $f(Z)_L$ is absorbant by monochromatic pahts. Let $k = (u, v)$ be a vertex of $L(D)$ such that $k \in (V(L(D)) - f(Z)_L)$, it follows from Definition 2.1 that $v \in (V(D) - Z)$. Since Z is a kernel by monochromatic paths of D, it follows from Definition 1.3 that there exists $z \in Z$ and a monochromatic directed path from v to z in D, say $(v = x_0, x_1, \ldots, x_{n-1}, x_n = z)$. Then it follows from Lemma 2.1 that there exists an $(u, v)(x_{n-1}, x_n)$-monochromatic directed path in the inner m-coloration of $L(D)$ and since $z \in Z$, we have from Definition 2.1 that $(x_{n-1}, x_n = z) \in f(Z)_L$.

(2) The function $f' : \mathcal{K} \to \mathcal{K}^*$, where f' is the restriction of f to $\mathcal{K}$ is an injective function. Let $Z_1, Z_2 \in \mathcal{K}$ and $Z_1 \neq Z_2$. Let us suppose, *e.g.*, that $Z_1 - Z_2 \neq \emptyset$. Let $v \in (Z_1 - Z_2)$, since Z_2 is a kernel by monochromatic paths of D, it follows from Definition 1.3 that there exists $u \in Z_2$ and a vu-monochromatic directed path, let $h = (x_n, u)$ be the last arc of such a path. It follows from Definition 2.1 that $h \in f(Z_2)_L$. Finally, notice that since $v \in Z_1$, the subset Z_1 is independent by monochromatic paths and there exists a vu-monochromatic directed path, we have that $u \notin Z_1$ and then $h \notin f(Z_1)_L$. Hence $h \in (f(Z_2)_L - f(Z_1)_L)$ and so $f(Z_1)_L \neq f(Z_2)_L$.

Define a function $g : \mathcal{P}(U) \to \mathcal{P}(X)$ as follows:

If $H \subseteq U$, then $g(H) = C(H) \cup D(H)$, where $C(H) = \{x \in X \mid$ there exists $(z, x) \in H\}$ (the set of all the terminal endpoints of arcs of H).
$D(H) = \{x \in X \mid \delta_D^-(x) = 0$ and there is no monochromatic directed path from x to $C(H)\}$. (Where $\delta_D^-(x) = \{y \in V(D) \,|\, (y, x) \in U\}$).

(3) If $H_L \in \mathcal{K}^*$, then $g(H_L) \in \mathcal{K}$.

(3.1) If $H_L \in \mathcal{K}^*$, then $g(H_L)$ is independent by monochromatic paths.
Suppose that $H_L \in \mathcal{K}^*$, and let $u, v \in g(H_L)$, $u \neq v$; we will prove that there is no uv-monochromatic directed path in D. We will analyze several cases:

Case 1. $u, v \in C(H_L)$.
In this case we proceed by contradiction. Suppose (by contradition) that there exists an uv-monochromatic directed path $T = (u = x_0, x_1, \ldots, x_n = v)$ in D. Since $u, v \in C(H_L)$, u is the terminal endpoint of an arc $h \in H_L$ and v is the terminal endpoint of an arc $k \in H_L$.
When $k = (x_{n-1}, x_n = v)$ we have from Lemma 2.1 that there exists an hk-monochromatic directed path, a contradiction (because H_L is independent by monochromatic paths and $h, k \in H_L$).
Otherwise if $k \neq (x_{n-1}, x_n = v)$, we have $(x_{n-1}, x_n = v) \notin H_L$ (because if $(x_{n-1}, x_n = v) \in H_L$ we would have the monochromatic directed path $(h, a_0, a_1, \ldots, a_{n-1})$ where $a_i = (x_i, x_{i+1}), 0 \leq i \leq n-1$; from h to $(x_{n-1}, x_n = v) = a_{n-1}$ with $h, a_{n-1} \in H_L$, a contradiction). Since H_L is absorbant by monochromatic paths and $a_{n-1} = (x_{n-1}, x_n = v) \notin H_L$, there exists $b \in H_L$ and an $a_{n-1}b$-monochromatic directed path in the inner m-coloration of $L(D)$; let $(a_{n-1} = b_0, b_1, \ldots, b_m = b)$ be such a path. Since the terminal endpoint of k is v (the same as $a_{n-1} = b_0$) we have from Definitions 1.1 and 1.2 that also $(k, b_1, b_2, \ldots, b_m = b)$ is a monochromatic directed path in the inner m-coloration of $L(D)$ with $k, b \in H_L$, a contradiction.

Case 2. $u \in C(H_L), v \in D(H_L)$.
Since $v \in D(H_L)$, we have $\delta_D^-(v) = 0$, so there is no uv-monochromatic directed path in D.

Case 3. $u \in D(H_L), v \in C(H_L)$.
Since $u \in D(H_L)$, we have that there is no monochromatic directed path from u to $C(H_L)$, in particular there is no uv-monochromatic directed path.

Case 4. $u, v \in D(H_L)$.
Since $v \in D(H_L)$, we have $\delta_D^-(v) = 0$ and clearly, there is no uv-monochromatic directed path in D.

(3.2) If $H_L \in \mathcal{K}^*$, then $g(H_L)$ is absorbant by monochromatic paths.

Let $u \in X - g(H_L) = X - (C(H_L) \cup D(H_L))$. Since $u \notin (C(H_L) \cup D(H_L))$, we have that there is no arc in H whose terminal endpoint is u, and at least one of the two following conditions holds: $\delta_D^-(u) > 0$ or there exists a monochromatic directed path from u to $C(H_L)$.

We will analyze the two possible cases.

Case 1. There is no arc in H_L whose terminal endpoint is u and $\delta_D^-(u) > 0$. The hypothesis in this case implies that there exists an arc $(t, u) \in U - H_L$. Since $H_L \in \mathcal{K}^*$, we have that H_L is absorbant by monochromatic paths; hence there exists $p = (s, m) \in H_L$ and a monochromatic directed path from (t, u) to p. Now it follows from Lemma 2.2 that u is different from m and there exists a monochromatic directed path from u to m. Finally, notice that since $(s, m) \in H_L$, we have $m \in g(H_L)$. So there exists a monochromatic directed path from u to m with $m \in g(H_L)$.

Case 2. There is no arc in H_L whose terminal endpoint is u and there exists a monochromatic directed path from u to $C(H_L)$.
Clearly in this case we have a monochromatic directed path from u to $g(H_L) = C(H_L) \cup D(H_L)$.

(4) The function $g' : \mathcal{K}^* \to \mathcal{K}$, where g' is the restriction of g to $\mathcal{K}$ is an injective function. Let $N_L, P_L \in \mathcal{K}^*$, such that $N_L \neq P_L$. Let us suppose, e.g., that $N_L - P_L \neq \emptyset$. Let $h \in N_L - P_L$, and u the terminal endpoint of h. Since u is the terminal endpoint of an arc in N_L, we have that $u \in g(N_L)$.
Now we will prove that $u \notin g(P_L)$. Since P_L is absorbant by monochromatic paths and $h \notin P_L$, we have that there exists $k \in P_L$ and an hk-monochromatic directed path in the inner m-coloration of $L(D)$.
Let v be the terminal endpoint of k; hence $v \in g(P_L)$ and it follows from Lemma 2.2 that u is different from v and there exists an uv-monochromatic directed path in D. Since $g(P_L)$ is independent by monochromatic paths (This follows directly from (3) and Definition 1.3), we have that $u \notin g(P_L)$.
We conclude $u \in g(N_L) - g(P_L)$ and so $g(N_L) \neq g(P_L)$. Finally, notice that it follows from (2) and (4) that:
Card $\mathcal{K} \leq$ Card $\mathcal{K}^* \leq$ Card $\mathcal{K}$ and hence Card $\mathcal{K} =$ Card $\mathcal{K}^*$. ■

Note 2.1. Let D be an m-coloured digraph and $L(D)$ its line digraph; similarly as in Definition 1.2 we can define the outer m-coloration of $L(D)$ as follows: If h is arc of D with colour c, then any arc of the form (h, x) in $L(D)$ also has colour c. However, Theorem 2.1 does not hold if we change inner m-coloration of $L(D)$ by outer m-coloration of $L(D)$. In Figure 1, we show a digraph D without monochromatic directed cycles with one kernel by monochromatic paths such that the outer m-coloration of its line digraph (Figure 2) has no kernel by monochromatic paths.

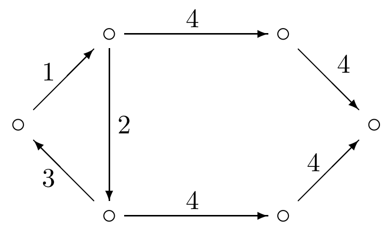

Figure 1

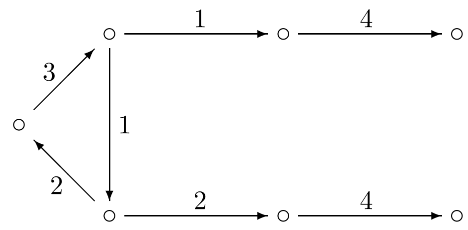

Figure 2

Note 2.2. Theorem 2.1 does not hold if we drop the hypothesis that D has no monochromatic directed cycles. In Figure 3, we show a digraph D with monochromatic directed cycles which has two kernels by monochromatic paths such that the inner m-coloration of its line digraph (Figure 4) has just one kernel by monochromatic paths. And in Figure 5, we show a digraph with monochromatic directed cycles without a kernel by monochromatic paths and its line digraph has two kernels by monochromatic paths (see Figure 6).

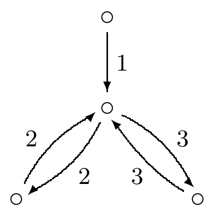

Figure 3

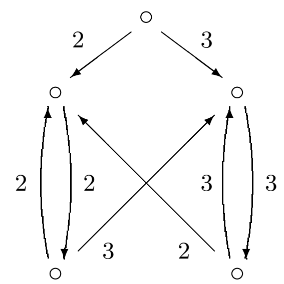

Figure 4

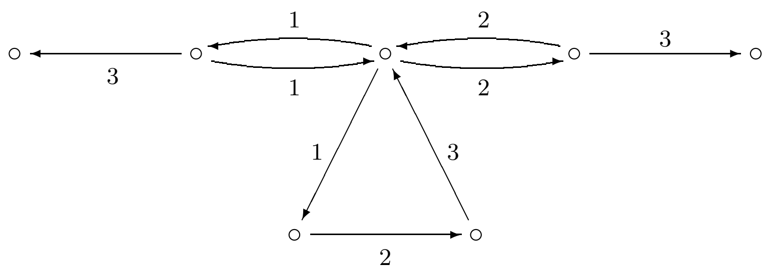

Figure 5

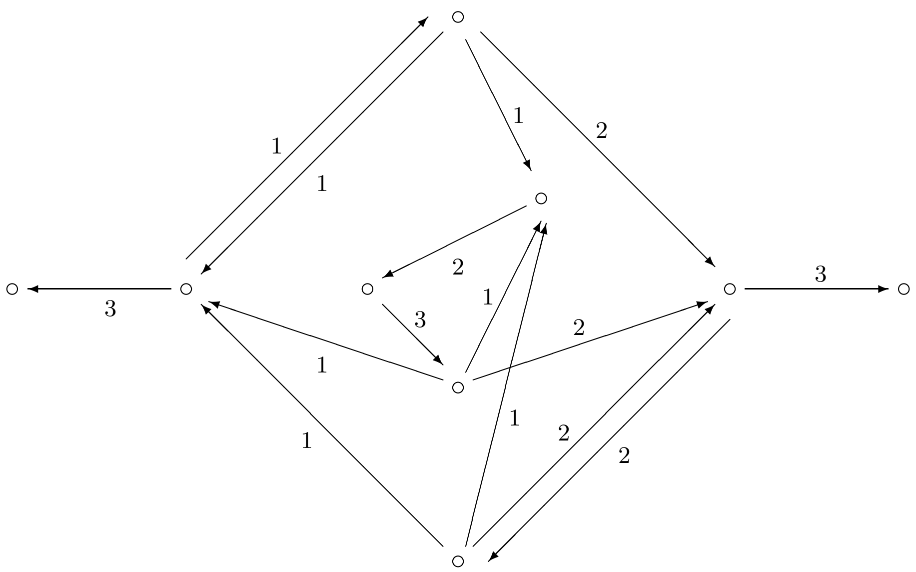

Figure 6

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