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EQUIVALENT CLASSES FOR K_3-GLUINGS OF WHEELS

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Abstract

In this paper, the chromaticity of K_3-gluings of two wheels is studied. For each even integer $n \geq 6$ and each odd integer $3 \leq q \leq [n/2]$ all K_3-gluings of wheels W_{q+2} and W_{n-q+2} create an χ-equivalent class.

Keywords: chromatically equivalent graphs, chromatic polynomial, chromatically unique graphs, wheels.

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Introduction

The graphs which we consider here are finite, undirected, simple and loopless. Let G be a graph, $V(G)$ be its vertex set, $E(G)$ be its edge set, $\chi(G)$ be its chromatic number and $P(G, \lambda)$ be its chromatic polynomial. Two graphs G and H are said to be *chromatically equivalent*, or in short χ-equivalent, written $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is said to be *chromatically unique*, or in short χ-unique, if for any graph H satisfying $H \sim G$, we have $H \cong G$, i.e. H is isomorphic to G. A *wheel* W_n is a graph of order n, $n \geq 4$, obtained by the join of K_1 and a cycle C_{n-1} of order $n-1$. Let for a vertex x of G the symbol $N(x)$ denote a subgraph of G induced by the set of vertices adjacent to x.

A H-gluing of two graphs G and F is a graph obtained by identifying an induced subgraph of G isomorphic to H with such a subgraph of F in the disjoint union of G and F. Koh and Teo [5] gave a survey on several results on chromaticity of K_r-gluings of graphs for $r \geq 1$. One of more interesting results has been discovered by Koh and Goh [4]. They completely characterized χ-unique K_3-gluings of complete graphs of order ≥ 3 and a K_4-homeomorph.

In this paper, the χ-equivalent classes for K_3-gluings of two wheels are studied. In computing chromatic polynomials, we make use of Whitney's reduction formula given in [8]. The formula is

$$P(G,\lambda) = P(G_{-e},\lambda) - P(G/_e,\lambda) \tag{1}$$

or equivalently

$$P(G_{-e},\lambda) = P(G,\lambda) + P(G/_e,\lambda) \tag{2}$$

where G_{-e} is the graph obtained from G by deleting an edge e and $G/_e$ is the graph obtained from G by contracting the edge e.

We also make use of the overlaping formula given in [8]. The formula is

$$P(G,\lambda) = P(H,\lambda)P(F,\lambda)/P(K_p,\lambda) \tag{3}$$

where G is a K_p-gluing of two disjoint graphs H and F, for $p \geq 1$.

Preliminary Results

We shall use the known results for χ-equivalent graphs presented in Lemma 1, where $I_G(F)$ denotes the number of induced subgraphs of G which are isomorphic to F.

Lemma 1 [6]**.** *Let G and H be two χ-equivalent graphs. Then*

(i) $\mid V(G) \mid = \mid V(H) \mid$;

(ii) $\mid E(G) \mid = \mid E(H) \mid$;

(iii) $\chi(G) = \chi(H)$;

(iv) $I_G(C_3) = I_H(C_3)$;

(v) $I_G(C_4) - 2I_G(K_4) = I_H(C_4) - 2I_H(K_4)$;

(vi) *G is connected iff H is connected;*

(vii) *G is 2-connected iff H is 2-connected.*

The following simple immediate observation plays an important role in proving that graphs with triangles are χ-unique or χ-equivalent.

Lemma 2. *Let T be a tree with n vertices. Then there are $n-1$ triangles in the join $T + K_1$.*

Lemma 3. *Let T be a tree with n vertices and let $v \notin V(T)$. Let H denote a graph obtained from T by adding the vertex v and m edges between v and vertices of T, $(m \leq n)$. Then the number of triangles of H is $\leq m-1$. Moreover, the equality holds if and only if the subgraph induced by the vertices adjacent to v is a tree.*

Lemma 4. *Let F be a unicyclic K_3-free graph with n vertices and let $v \notin V(F)$. Let H denote a graph obtained from F by adding the vertex v and m edges between v and vertices of F, $(m \leq n)$. Then the number of triangles of H is $\leq m$. Moreover, the equality holds if and only if the subgraph induced by the vertices adjacent to v is connected and it contains the cycle of F.*

Lemma 5. *Let F be a connected K_3-free graph with n vertices and with only two fundamental cycles, and let $v \notin V(F)$. Let H be a graph obtained from F by adding the vertex v and $m \leq n$ edges between v and m vertices of F. Then the number of triangles of H is $\leq m+1$. Moreover, the equality holds if and only if the subgraph induced by the vertices adjacent to v is connected and contains two fundamental cycles.*

Let us assume that $n \geq 6$ is an integer number. For an integer number q, $\frac{n}{2} \geq q \geq 3$, the graph W_{n+1}^q is obtained from W_{n+1} by adding exactly one new edge joining two vertices at distance q in the subgraph C_n of W_{n+1}. In other words, W_{n+1}^q is a K_3-gluing of W_{n-q+2} and W_{q+2} identifying their central vertices.

Lemma 6. $(\lambda-2)^2 \nmid P(W_{n+1}^q, \lambda)$. *Moreover W_{n+1}^q is uniquely 3-colourable if n is even and q is odd , $\frac{n}{2} \geq q \geq 3$.*

Proof. By using Whitney's reduction formula we have:

$$P(W_{n+1}^q, \lambda) = P(W_{n+1}, \lambda) - \frac{P(W_{n-q+1}, \lambda) \cdot P(W_{q+1}, \lambda)}{P(K_2, \lambda)}. \tag{4}$$

Evidently according to the known result for $P(C_n, \lambda)$ (see [1]), we get that

$$\begin{aligned} P(W_{n+1}, \lambda) &= \lambda\{(\lambda-2)^n + (-1)^n(\lambda-2)\} \\ &= \lambda(\lambda-1)(\lambda-2) \cdot P_s(W_{n+1}, \lambda), \end{aligned} \tag{5}$$

where

$$P_s(W_{n+1},\lambda) = \begin{cases} (\lambda-3)\sum\limits_{i=0}^{(n-3)/2}(\lambda-2)^{2i}, & \text{if } n \text{ is odd,} \\ \sum\limits_{i=0}^{n-2}(-1)^i(\lambda-2)^i, & \text{if } n \text{ is even.} \end{cases}$$

Note that

$$P_s(W_{n+1},2) = \begin{cases} -1, & \text{if } n \text{ is odd ,} \\ 1, & \text{if } n \text{ is even,} \end{cases}$$

and

$$P_s(W_{n+1},3) = \begin{cases} 0, & \text{if } n \text{ is odd ,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

From (4) and (5) we get

$$\begin{aligned} P(W^q_{n+1},\lambda) &= \lambda(\lambda-1)(\lambda-2)\cdot[P_s(W_{n+1},\lambda) \\ &- (\lambda-2)\cdot P_s(W_{n-q+1},\lambda)\cdot P_s(W_{q+1},\lambda)]. \end{aligned}$$

Note that $(\lambda-2) \mid P(W^q_{n+1},\lambda)$. Let $P(W^q_{n+1},\lambda) = (\lambda-2)R(W^q_{n+1},\lambda)$. Then $R(W^q_{n+1},2) = \pm 2$ and $P(W^q_{n+1},\lambda)$ is not divisible by $(\lambda-2)^2$. Since for an even n and an odd q we have $P(W^q_{n+1},3) = 6$, then W^q_{n+1} is uniquely 3-colourable. ■

Lemma 7 [2]. *Let G be a graph containing at least two triangles. If there is a vertex of a triangle having degree two in G, then $(\lambda-2)^2 \mid P(G,\lambda)$.*

Lemma 8. *Let G be a graph obtained by K_2-gluing of two graphs such that each of them has a triangle. Then $(\lambda-2)^2 \mid P(G,\lambda)$.*

Proof. Directly from (3). ■

Lemma 9. *Let H and F be non-isomorphic χ-unique graphs. Then $K_1 + H \not\sim K_1 + F$.*

Proof. Evidently $P(G + K_1,\lambda) = \lambda \cdot P(G,\lambda-1)$ for any graph G. Let H and F be non-isomorphic χ-unique graphs. Suppose that $P(H + K_1,\lambda) = P(F + K_1,\lambda)$ then $P(H,\lambda-1) = P(F,\lambda-1)$ and we get a contradiction. ■

Main Results

We prove that each of χ-equivalent classes for some cases of W^q_{n+1} consists of two graphs.

Theorem 1. *For each even integer $n \geq 6$ and each odd integer $3 \leq q \leq [n/2]$ all K_3-gluings of wheels W_{q+2} and W_{n-q+2} create a χ-equivalent class.*

Proof. Let n be even, $(n \geq 6)$ and let $G \sim W^q_{n+1}$. Then $P(G,\lambda) = P(W^q_{n+1},\lambda)$ and therefore, by Lemmas 1, 6 and 7 any candidate for G has the following properties: $\mid V(G) \mid = n+1,\ \mid E(G) \mid = 2n+1,\ I_G(C_3) = n+1,$ G is a 2-connected unique 3-colourable graph and no vertex of any triangle of G has degree two in G.

Let V_1, V_2 and V_3 be colour classes of the uniquelly 3-colouring of G and let $\mid V_i \mid = n_i,\ i = 1,2,3$. Evidently $n_1 + n_2 + n_3 = n + 1$.

Let G_i be the subgraph of G induced by $V(G) - V_i$, where $i = 1,2,3$. Evidently, each of $G_i,\ i = 1,2,3,$ is connected (see Theorem 12.16 in [3]). Therefore

$$\begin{aligned} 2n - 1 &= (n_1 + n_2 - 1) + (n_1 + n_3 - 1) + (n_2 + n_3 - 1) \\ &\leq \mid E(G_3) \mid + \mid E(G_2) \mid + \mid E(G_1) \mid = 2n + 1. \end{aligned} \tag{6}$$

Without loss of generality, we have two cases:

Case 1. Let G_3 and G_2 be trees and let G_1 be a connected graph with two fundamental cycles, say $C,\ C'$. Note that $\mid V(G_1) \mid = n_2 + n_3 = n+1-n_1$ and $\mid E(G_1) \mid = n+2-n_1$. Consequently, the number $m(V_1, V(G_1))$ of edges from V_1 to $V(G_1)$ satisfies the following equality

$$m(V_1, V(G_1)) = 2n + 1 - (n + 2 - n_1) = n + n_1 - 1. \tag{7}$$

Suppose that no vertex of V_1 is adjacent to all vertices of any cycle of G_1. Then by Lemma 3 and formula (7)

$$n+1 = I_G(C_3) \leq \sum_{i=1}^{n_1}(\deg(v_i) - 1) = \sum_{i=1}^{n_1}\deg(v_i) - n_1 = n + n_1 - 1 - n_1 = n - 1,$$

and we get a contradiction. Therefore we can assume that some vertex $v \in V_1$ is adjacent to all vertices of a fundamental cycle of G_1, say C, and since G_2 and G_3 are trees, then v is unique. Now if there exists no vertex of

V_1 adjacent to all vertices of the cycle C' of G_1, where $C' \neq C$ then similarly, by Lemmas 3 and 4 we get that

$$n + 1 = I_G(C_3) \leq \sum_{i=1}^{n_1}(\deg(v_i) - 1) + 1 = n, \tag{8}$$

and it leads to a contradiction. Therefore according to the above argument there is exactly one vertex $v' \in V_1$ which is adjacent to all vertices of C'. Suppose that a subgraph of G_1 induced by the set of all vertices adjacent to a vertex of V_1 is disconnected. Looking at the tree structure of G_2 and G_3 and Lemmas 3-5 we obtain the inequality presented in formula (8), and it leads to a contradiction.

From the above it follows that

Lemma 10. *One of the vertices of V_1, say v, is adjacent to all vertices of a connected subgraph of G_1 which contains C, and one of the vertices of V_1, say v', is adjacent to all vertices of a connected subgraph of G_1 which contains C', and each of the other vertices of V_1 is adjacent to the vertices of a subtree of G_1.*

Let us consider degrees of the vertices of G. Immediately by 2-connectivity of G and Lemmas 6, 7 and 10 we get that each vertex of V_1 has degree at least 3 in G. Similarly, each 1-degree vertex of G_1 has at least two neighbours in V_1. Suppose that a 2-degree vertex x of G_1 has degree 2 in G. Then by Lemma 10 the vertex x does not belong to any cycle of G_1 and it is a cut vertex of G. It leads to a contradiction to 2-connectivity of G. It follows that

Lemma 11. $\deg(x) \geq 3$ *for each $x \in V(G)$.*

Suppose now that $V(N(x)) = V(G_1)$ for some $x \in V_1$. Then by Lemma 5 the vertex x belongs to $n_2 + n_3 + 1$ triangles of G, and each of $n + 1 - (n_2 + n_3 + 1) = n_1 - 1$ other triangles contains a vertex of $V_1 - \{x\}$. By formula (7) the number of edges from the set $V_1 - \{x\}$ to $V(G_1)$ is equal to $n + n_1 - 1 - (n_2 + n_3) = 2(n_1 - 1)$. So this fact and 2-connectivity of G imply that $\deg(y) = 2$ for each $y \in V_1 - \{x\}$. Therefore from Lemma 7, the set V_1 consists of exactly one vertex x and G_1 has not any vertex of degree one. Thus $\deg(x) = n$ and G is isomorphic to the join of K_1 and one of the three graphs presented in Figure 1.

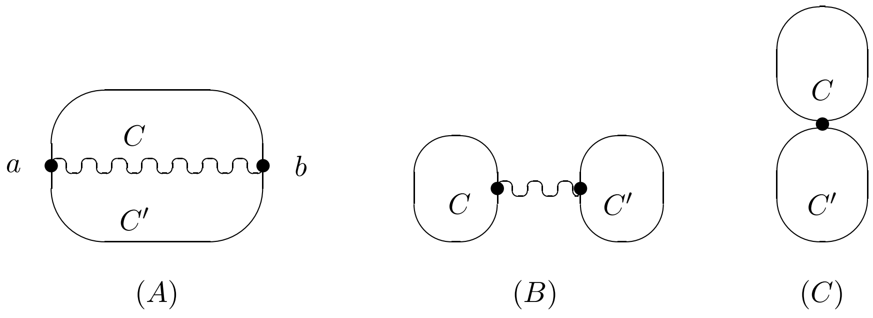

Figure 1

If G_1 is isomorphic to a graph of the structure (C) or (B), then Lemma 8 implies $(\lambda-2)^2|P(G,\lambda)$ and we get a contradiction to Lemma 6.

Therefore G_1 is isomorphic to a graph of the structure (A). Note that each of the three paths from the vertex a to b is odd length, since n is even and C, C' have even length. Since each generalized θ-graph is χ-unique [7], from Lemma 9 we get $G \cong W^q_{n+1}$.

We have to consider the case : $V(N(x)) \neq V(G_1)$ for each $x \in V_1$.

First suppose that the vertex $v \in V_1$ is adjacent to all vertices of C and C', i.e., $v = v'$. The assumption of the case and Lemma 10 imply $V(G_1) - V(C \cup C') \neq \emptyset$. So there exists a vertex $u \in V(G_1) - V(N(v))$ such that $\deg_{G_1}(u) = 1$. Thus

$$n+1 = I_G(C_3) \leq \sum_{i=1}^{n_1}(\deg(v_i)-1)+2 = n+1. \tag{9}$$

Lemma 5 and $V(N(v)) \neq V(G_1)$ imply that v belongs to at most n_2+n_3 triangles of G, and vertices of $V_1 - \{v\}$ belong to at least n_1 triangles. Moreover, the number of edges from $V_1-\{v\}$ to $V(G_1)$ is at least $2(n_1-1)+1$. Therefore $\mid V_1 \mid \geq 2$.

Lemma 11 implies that the vertex u is adjacent to two different vertices $v_1, v_2 \in V_1 - \{v\}$. Let w be a neighbour of u in G_1. From Lemmas 10, 11 we have that w is adjacent to v_1 and v_2. Therefore we get either a cycle in the subgraph $N(w)$ or that G is a K_2-gluing of two graphs with triangles. The first case contradicts acyclicity of G_2 and G_3. By Lemma 8 the other case gives $(\lambda-2)^2|P(G,\lambda)$ and it contradicts Lemma 6.

Therefore suppose now that the vertex $v \in V_1$ is not adjacent to a vertex of C'. Thus $v \neq v'$. Applying the same arguments as before we get that

G_1 does not have any vertex of degree 1. Hence we can consider only the following three subcases: G_1 is a K_2-gluing of two cycles of even order, a K_1-gluing of two cycles of even order, or it consists of two cycles of even order and exactly one path connecting them.

Since n is even, then for the first case we get that $V_1 - \{v, v'\} \neq \emptyset$ and 2-connectivity of G, Lemma 10 and acyclicity of G_2 and G_3 imply $N(v_1) \cong K_2$ for each $v_1 \in V_1 - \{v, v'\}$ and this gives a contradiction to Lemma 11.

For two other cases Lemma 10 and acyclicity of G_2 and G_3 imply $\mid V(N(v_1)) \cap V(N(v_2)) \mid \leq 2$, for each pair of different vertices $v_1, v_2 \in V_1$. Therefore by 2-connectivity of G we get that G is a K_2-gluing of two graphs with triangles. Hence we get a contradiction to the Lemma 6.

Case 2. Let G_3 be a tree, and G_2, G_1 be unicyclic graphs with even cycles. Note that

$\mid E(G_1) \mid = \mid V(G_1) \mid = n + 1 - n_1$,

$\mid E(G_2) \mid = \mid V(G_2) \mid = n_1 + n_3 = n + 1 - n_2$.

The number of edges from V_1 to $V(G_1)$ is equal to

$$2n + 1 - (n + 1 - n_1) = n + n_1. \tag{10}$$

Similarly, the number of edges from V_2 to $V(G_2)$ is equal to

$$2n + 1 - (n + 1 - n_2) = n + n_2. \tag{11}$$

Let C^1 be the cycle of G_1, and C^2 be the cycle of G_2.

Suppose that there is no vertex in V_1 adjacent to all of the vertices of C^1. Then each vertex of V_1 is adjacent to a subforest in G_1.

By Lemma 3 the number of triangles in G containing a vertex $v_i^1 \in V_1$ is at most $d(v_i^1) - 1$. So the number of triangles in G is at most

$$\begin{aligned} n + 1 &= I_G(C_3) \leq \sum_{i=1}^{n_1} (\deg(v_i^1) - 1) \\ &= \sum_{i=1}^{n_1} \deg(v_i^1) - n_1 = n + n_1 - n_1 = n, \end{aligned} \tag{12}$$

and we get a contradiction.

Therefore there exists at least one vertex $v^1 \in V_1$ adjacent to all of the vertices of C^1. Suppose that there is another such vertex, i.e., let $w^1 \in V_1 - \{v^1\}$ and let w^1 be adjacent to all of the vertices of C^1. Assume also without loss of generality that $u_1, u_2, ..., u_{2m}$ are consecutive vertices of C^1, where $u_1, u_3, ..., u_{2m-1} \in V_2$ and $u_2, u_4, ..., u_{2m} \in V_3$. Note that the subgraph induced by $\{u_1, v^1, u_3, w^1\}$ is a cycle in G_3. This contradicts the fact that G_3 is a tree. Thus we have proved that there exists exactly one vertex v^1 in V_1 adjacent to all vertices in C^1. Similarly, there exists exactly one vertex v^2 in V_2 adjacent to all vertices in C^2. Suppose that a subgraph of G_1 induced by all vertices adjacent to a vertex of V_1 is disconnected. Hence by Lemmas 3-4 we get the formula (12), and it leads to a contradiction.

Thus we have the following observations.

Lemma 12. *One vertex, $v^1 \in V_1$, is adjacent to all of the vertices of a connected subgraph of G_1 which contains the even cycle. Each other vertex of V_1 is adjacent to the vertices of a subtree of G_1.*

Similarly, by symmetry, the vertices of V_2 must satisfy the respective conditions of the following result.

Lemma 13. *One vertex, $v^2 \in V_2$, is adjacent to all of the vertices of a connected subgraph of G_2 which contains the even cycle. Each other vertex of V_2 is adjacent to the vertices of a subtree of G_2.*

Lemma 12 and acyclicity of G_3 give the following lemma.

Lemma 14. $|V(N(v)) \cap V(N(v')) \leq 3$ *for* $v, v' \in V_1,\ v \neq v'$.

Moreover, Lemma 11 presented in case 1 holds for G.

Subcase 2.1. Suppose that $N(v^1) = V(G_1)$. Then by Lemma 4 the vertex v^1 belongs to $n+1-n_1$ triangles in G, and each of other $n+1-(n+1-n_1) = n_1$ triangles contains a vertex of $V_1 - \{v^1\} \neq \emptyset$. Note that the number of edges from $V_1 - \{v^1\}$ to $V(G_1)$ is equal to $2n+1-2(n+1-n_1) = 2n_1 - 1 = 2(n_1-1)+1$. This and Lemma 11 lead to $|V_1| = 2$. Hence there exists exactly one vertex in V_1 different from v^1, say w^1, and its degree equals 3.

Therefore, from Lemma 7 and from the fact that n is even, the graph G_1 consists of C^1 and exactly one tree T rooted at a vertex of C^1. Moreover, for each pair x, y of leaves of T we have that $dist_{G_1}(x, y) = 2$ and then T has only two leaves. Since n is even, T has an even number of vertices (including root vertex). Therefore $T \cong P_{2t}$ or T is a K_1-gluing of P_{2t-1} and K_2, where $t \geq 1$, and G_1 is one of the two graphs presented in Figure 2.

By Lemma 11 each leaf of the rooted tree T is adjacent to w^1 and v^1. Lemmas 6, 8 imply that the graph G is not any K_2-gluing of two graphs with triangles in each of them. Therefore G_1 is a unicyclic graph with one leaf and a cycle of length $n-2$.

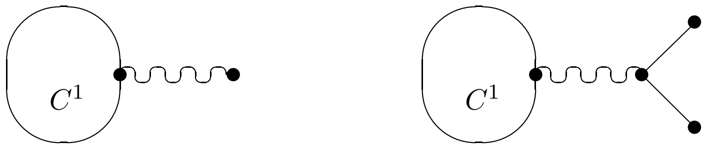

Figure 2

If two of the vertices which are adjacent to w^1 have colour 2, then $\{x, w^1, y, v^1\}$ induces C_4 in G_3, and we have a contradiction.

Therefore two of the vertices which are adjacent to w^1 have colour 3 and then $\{x, w^1, y, v^1\}$ induces C_4 in G_2.

Hence G is K_3-gluing of W_{n-1} and W_5 such that the centers of the wheels are not overlapped. Note that by Lemma 1(v) the graph G is isomorphic to W^q_{n+1} and this is possible only for $q = 3$.

Subcase 2.2. We can assume that $N(v^1) \neq V(G_1)$ and by symmetry $N(v^2) \neq V(G_2)$. Then by Lemmas 12, 13 each of the graphs G_1, G_2 is unicyclic with a vertex of degree one. Evidently by Lemma 11 each leave in G_1 is adjacent to at least two vertices of V_1. Let v^1, v^2 be the vertices of Lemmas 12 and 13, respectively. Let x be a leave in G_1 which is not adjacent to v^1, and let x^1 be the neighbour of x in G_1.

Let x^2 be a neighbour of x^1 in G_1 such that $x^2 \neq x$ and $\deg(x^2) \geq 2$.

Lemmas 11, 12 imply that the vertex x has at least two neighbours in V_1. Let us consider $N(x^1)$. Since G is not any K_2-gluing of two graphs with triangles and G_3 has not any cycle, then Lemmas 6, 7, 11, 12 and 14 imply that $N(x^1)$ contains a cycle belonging to G_2. Evidently, the cycle is unique. The same arguments give $x^1 \in V(C^1)$ and therefore G_1 has a unique rooted tree and it is isomorphic to a graph presented in Figure 3. Similarly, G_2 is isomorphic to a graph presented in Figure 3.

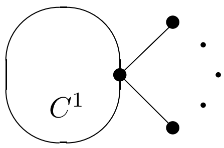

Figure 3

Let $a, b \in V(N(x^1)) \cap V(C^1)$, $\{w_1, ..., w_t\} = V_1 - \{v^1\}$ and let $x = x_1, x_2, ..., x_m$ be the leaves of G_1. If neither a nor b is adjacent to a vertex w_j, $j = 1, ..., t$, then G is a K_2-gluing of two graphs with triangles, for K_2 induced by $\{v^1, x^1\}$ and we get a contradiction. Thus without loss of generality, we can assume that a is adjacent to w_1. Then there exists an alternating sequence passing through all vertices of V_1 and all leaves of $V(G_1)$ and having one of the two forms

$a, w_1, x_1, w_2, x_2, ..., x_m, w_m, b, v^1$

or

$a, w_1, x_1, w_2, x_2, ..., x_m, v^1$.

The first case gives an odd cycle in G_2 and we get a contradiction. The other one gives a K_3-gluing of two wheels which does not identify their central vertices. Since each generalized θ-graph is χ-unique [7], from Lemma 9 we get that these wheels must be isomorphic to W_{q+2} and W_{n-q+2}, respectively.

The proof is complete. ■

Since the wheels W_6, W_8 are not χ-unique graphs [2], [9] the χ-equivalent classes for other cases of n and q can contain more than two graphs. The graphs $G \simeq W^q_{n+1}$, for n odd or q even are not uniquelly $\chi(G)$-colourable. Thus, the proof techniques used in this paper cannot be used to characterize χ-equivalent classes for these graphs.

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