

EQUIVALENT CLASSES FOR K_3 -GLUINGS OF WHEELS

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Abstract

In this paper, the chromaticity of K_3 -gluings of two wheels is studied. For each even integer $n \geq 6$ and each odd integer $3 \leq q \leq [n/2]$ all K_3 -gluings of wheels W_{q+2} and W_{n-q+2} create an χ -equivalent class.

Keywords: chromatically equivalent graphs, chromatic polynomial, chromatically unique graphs, wheels.

1991 Mathematics Subject Classification: 05C15.

INTRODUCTION

The graphs which we consider here are finite, undirected, simple and loopless. Let G be a graph, $V(G)$ be its vertex set, $E(G)$ be its edge set, $\chi(G)$ be its chromatic number and $P(G, \lambda)$ be its chromatic polynomial. Two graphs G and H are said to be *chromatically equivalent*, or in short χ -equivalent, written $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is said to be *chromatically unique*, or in short χ -unique, if for any graph H satisfying $H \sim G$, we have $H \cong G$, i.e. H is isomorphic to G . A *wheel* W_n is a graph of order n , $n \geq 4$, obtained by the join of K_1 and a cycle C_{n-1} of order $n-1$. Let for a vertex x of G the symbol $N(x)$ denote a subgraph of G induced by the set of vertices adjacent to x .

A H -gluing of two graphs G and F is a graph obtained by identifying an induced subgraph of G isomorphic to H with such a subgraph of F in the disjoint union of G and F . Koh and Teo [5] gave a survey on several results on chromaticity of K_r -gluings of graphs for $r \geq 1$. One of more interesting results has been discovered by Koh and Goh [4]. They completely characterized χ -unique K_3 -gluings of complete graphs of order ≥ 3 and a K_4 -homeomorph.

In this paper, the χ -equivalent classes for K_3 -gluings of two wheels are studied. In computing chromatic polynomials, we make use of Whitney's reduction formula given in [8]. The formula is

$$(1) \quad P(G, \lambda) = P(G_{-e}, \lambda) - P(G/_e, \lambda)$$

or equivalently

$$(2) \quad P(G_{-e}, \lambda) = P(G, \lambda) + P(G/_e, \lambda)$$

where G_{-e} is the graph obtained from G by deleting an edge e and $G/_e$ is the graph obtained from G by contracting the edge e .

We also make use of the overlapping formula given in [8]. The formula is

$$(3) \quad P(G, \lambda) = P(H, \lambda)P(F, \lambda)/P(K_p, \lambda)$$

where G is a K_p -gluing of two disjoint graphs H and F , for $p \geq 1$.

PRELIMINARY RESULTS

We shall use the known results for χ -equivalent graphs presented in Lemma 1, where $I_G(F)$ denotes the number of induced subgraphs of G which are isomorphic to F .

Lemma 1 [6]. *Let G and H be two χ -equivalent graphs. Then*

- (i) $|V(G)| = |V(H)|$;
- (ii) $|E(G)| = |E(H)|$;
- (iii) $\chi(G) = \chi(H)$;
- (iv) $I_G(C_3) = I_H(C_3)$;
- (v) $I_G(C_4) - 2I_G(K_4) = I_H(C_4) - 2I_H(K_4)$;
- (vi) G is connected iff H is connected;
- (vii) G is 2-connected iff H is 2-connected.

The following simple immediate observation plays an important role in proving that graphs with triangles are χ -unique or χ -equivalent.

Lemma 2. *Let T be a tree with n vertices. Then there are $n - 1$ triangles in the join $T + K_1$.*

Lemma 3. *Let T be a tree with n vertices and let $v \notin V(T)$. Let H denote a graph obtained from T by adding the vertex v and m edges between v and vertices of T , ($m \leq n$). Then the number of triangles of H is $\leq m - 1$. Moreover, the equality holds if and only if the subgraph induced by the vertices adjacent to v is a tree.*

Lemma 4. *Let F be a unicyclic K_3 -free graph with n vertices and let $v \notin V(F)$. Let H denote a graph obtained from F by adding the vertex v and m edges between v and vertices of F , ($m \leq n$). Then the number of triangles of H is $\leq m$. Moreover, the equality holds if and only if the subgraph induced by the vertices adjacent to v is connected and it contains the cycle of F .*

Lemma 5. *Let F be a connected K_3 -free graph with n vertices and with only two fundamental cycles, and let $v \notin V(F)$. Let H be a graph obtained from F by adding the vertex v and $m \leq n$ edges between v and m vertices of F . Then the number of triangles of H is $\leq m + 1$. Moreover, the equality holds if and only if the subgraph induced by the vertices adjacent to v is connected and contains two fundamental cycles.*

Let us assume that $n \geq 6$ is an integer number. For an integer number q , $\frac{n}{2} \geq q \geq 3$, the graph W_{n+1}^q is obtained from W_{n+1} by adding exactly one new edge joining two vertices at distance q in the subgraph C_n of W_{n+1} . In other words, W_{n+1}^q is a K_3 -gluing of W_{n-q+2} and W_{q+2} identifying their central vertices.

Lemma 6. $(\lambda - 2)^2 \nmid P(W_{n+1}^q, \lambda)$. Moreover W_{n+1}^q is uniquely 3-colourable if n is even and q is odd, $\frac{n}{2} \geq q \geq 3$.

Proof. By using Whitney's reduction formula we have:

$$(4) \quad P(W_{n+1}^q, \lambda) = P(W_{n+1}, \lambda) - \frac{P(W_{n-q+1}, \lambda) \cdot P(W_{q+1}, \lambda)}{P(K_2, \lambda)}.$$

Evidently according to the known result for $P(C_n, \lambda)$ (see [1]), we get that

$$(5) \quad \begin{aligned} P(W_{n+1}, \lambda) &= \lambda\{(\lambda - 2)^n + (-1)^n(\lambda - 2)\} \\ &= \lambda(\lambda - 1)(\lambda - 2) \cdot P_s(W_{n+1}, \lambda), \end{aligned}$$

where

$$P_s(W_{n+1}, \lambda) = \begin{cases} (\lambda - 3) \sum_{i=0}^{(n-3)/2} (\lambda - 2)^{2i}, & \text{if } n \text{ is odd,} \\ \sum_{i=0}^{n-2} (-1)^i (\lambda - 2)^i, & \text{if } n \text{ is even.} \end{cases}$$

Note that

$$P_s(W_{n+1}, 2) = \begin{cases} -1, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even,} \end{cases}$$

and

$$P_s(W_{n+1}, 3) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

From (4) and (5) we get

$$\begin{aligned} P(W_{n+1}^q, \lambda) &= \lambda(\lambda - 1)(\lambda - 2) \cdot [P_s(W_{n+1}, \lambda) \\ &\quad - (\lambda - 2) \cdot P_s(W_{n-q+1}, \lambda) \cdot P_s(W_{q+1}, \lambda)]. \end{aligned}$$

Note that $(\lambda - 2) \mid P(W_{n+1}^q, \lambda)$. Let $P(W_{n+1}^q, \lambda) = (\lambda - 2)R(W_{n+1}^q, \lambda)$. Then $R(W_{n+1}^q, 2) = \pm 2$ and $P(W_{n+1}^q, \lambda)$ is not divisible by $(\lambda - 2)^2$. Since for an even n and an odd q we have $P(W_{n+1}^q, 3) = 6$, then W_{n+1}^q is uniquely 3-colourable. ■

Lemma 7 [2]. *Let G be a graph containing at least two triangles. If there is a vertex of a triangle having degree two in G , then $(\lambda - 2)^2 \mid P(G, \lambda)$.*

Lemma 8. *Let G be a graph obtained by K_2 -gluing of two graphs such that each of them has a triangle. Then $(\lambda - 2)^2 \mid P(G, \lambda)$.*

Proof. Directly from (3). ■

Lemma 9. *Let H and F be non-isomorphic χ -unique graphs. Then $K_1 + H \not\sim K_1 + F$.*

Proof. Evidently $P(G + K_1, \lambda) = \lambda \cdot P(G, \lambda - 1)$ for any graph G . Let H and F be non-isomorphic χ -unique graphs. Suppose that $P(H + K_1, \lambda) = P(F + K_1, \lambda)$ then $P(H, \lambda - 1) = P(F, \lambda - 1)$ and we get a contradiction. ■

MAIN RESULTS

We prove that each of χ -equivalent classes for some cases of W_{n+1}^q consists of two graphs.

Theorem 1. *For each even integer $n \geq 6$ and each odd integer $3 \leq q \leq [n/2]$ all K_3 -gluings of wheels W_{q+2} and W_{n-q+2} create a χ -equivalent class.*

Proof. Let n be even, ($n \geq 6$) and let $G \sim W_{n+1}^q$. Then $P(G, \lambda) = P(W_{n+1}^q, \lambda)$ and therefore, by Lemmas 1, 6 and 7 any candidate for G has the following properties: $|V(G)| = n+1$, $|E(G)| = 2n+1$, $I_G(C_3) = n+1$, G is a 2-connected unique 3-colourable graph and no vertex of any triangle of G has degree two in G .

Let V_1, V_2 and V_3 be colour classes of the uniquely 3-colouring of G and let $|V_i| = n_i$, $i = 1, 2, 3$. Evidently $n_1 + n_2 + n_3 = n+1$.

Let G_i be the subgraph of G induced by $V(G) - V_i$, where $i = 1, 2, 3$. Evidently, each of G_i , $i = 1, 2, 3$, is connected (see Theorem 12.16 in [3]). Therefore

$$(6) \quad \begin{aligned} 2n-1 &= (n_1 + n_2 - 1) + (n_1 + n_3 - 1) + (n_2 + n_3 - 1) \\ &\leq |E(G_3)| + |E(G_2)| + |E(G_1)| = 2n+1. \end{aligned}$$

Without loss of generality, we have two cases:

Case 1. Let G_3 and G_2 be trees and let G_1 be a connected graph with two fundamental cycles, say C, C' . Note that $|V(G_1)| = n_2 + n_3 = n+1 - n_1$ and $|E(G_1)| = n+2 - n_1$. Consequently, the number $m(V_1, V(G_1))$ of edges from V_1 to $V(G_1)$ satisfies the following equality

$$(7) \quad m(V_1, V(G_1)) = 2n+1 - (n+2 - n_1) = n + n_1 - 1.$$

Suppose that no vertex of V_1 is adjacent to all vertices of any cycle of G_1 . Then by Lemma 3 and formula (7)

$$n+1 = I_G(C_3) \leq \sum_{i=1}^{n_1} (\deg(v_i) - 1) = \sum_{i=1}^{n_1} \deg(v_i) - n_1 = n + n_1 - 1 - n_1 = n - 1,$$

and we get a contradiction. Therefore we can assume that some vertex $v \in V_1$ is adjacent to all vertices of a fundamental cycle of G_1 , say C , and since G_2 and G_3 are trees, then v is unique. Now if there exists no vertex of

V_1 adjacent to all vertices of the cycle C' of G_1 , where $C' \neq C$ then similarly, by Lemmas 3 and 4 we get that

$$(8) \quad n + 1 = I_G(C_3) \leq \sum_{i=1}^{n_1} (\deg(v_i) - 1) + 1 = n,$$

and it leads to a contradiction. Therefore according to the above argument there is exactly one vertex $v' \in V_1$ which is adjacent to all vertices of C' . Suppose that a subgraph of G_1 induced by the set of all vertices adjacent to a vertex of V_1 is disconnected. Looking at the tree structure of G_2 and G_3 and Lemmas 3-5 we obtain the inequality presented in formula (8), and it leads to a contradiction.

From the above it follows that

Lemma 10. *One of the vertices of V_1 , say v , is adjacent to all vertices of a connected subgraph of G_1 which contains C , and one of the vertices of V_1 , say v' , is adjacent to all vertices of a connected subgraph of G_1 which contains C' , and each of the other vertices of V_1 is adjacent to the vertices of a subtree of G_1 .*

Let us consider degrees of the vertices of G . Immediately by 2-connectivity of G and Lemmas 6, 7 and 10 we get that each vertex of V_1 has degree at least 3 in G . Similarly, each 1-degree vertex of G_1 has at least two neighbours in V_1 . Suppose that a 2-degree vertex x of G_1 has degree 2 in G . Then by Lemma 10 the vertex x does not belong to any cycle of G_1 and it is a cut vertex of G . It leads to a contradiction to 2-connectivity of G . It follows that

Lemma 11. $\deg(x) \geq 3$ for each $x \in V(G)$.

Suppose now that $V(N(x)) = V(G_1)$ for some $x \in V_1$. Then by Lemma 5 the vertex x belongs to $n_2 + n_3 + 1$ triangles of G , and each of $n + 1 - (n_2 + n_3 + 1) = n_1 - 1$ other triangles contains a vertex of $V_1 - \{x\}$. By formula (7) the number of edges from the set $V_1 - \{x\}$ to $V(G_1)$ is equal to $n + n_1 - 1 - (n_2 + n_3) = 2(n_1 - 1)$. So this fact and 2-connectivity of G imply that $\deg(y) = 2$ for each $y \in V_1 - \{x\}$. Therefore from Lemma 7, the set V_1 consists of exactly one vertex x and G_1 has not any vertex of degree one. Thus $\deg(x) = n$ and G is isomorphic to the join of K_1 and one of the three graphs presented in Figure 1.

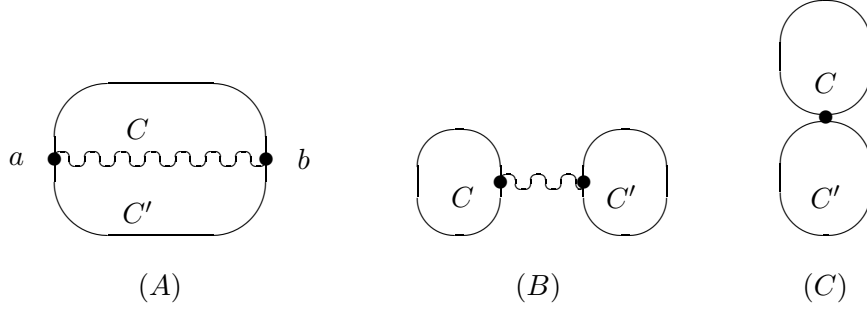


Figure 1

If G_1 is isomorphic to a graph of the structure (C) or (B), then Lemma 8 implies $(\lambda - 2)^2 \mid P(G, \lambda)$ and we get a contradiction to Lemma 6.

Therefore G_1 is isomorphic to a graph of the structure (A). Note that each of the three paths from the vertex a to b is odd length, since n is even and C, C' have even length. Since each generalized θ -graph is χ -unique [7], from Lemma 9 we get $G \cong W_{n+1}^q$.

We have to consider the case : $V(N(x)) \neq V(G_1)$ for each $x \in V_1$.

First suppose that the vertex $v \in V_1$ is adjacent to all vertices of C and C' , i.e., $v = v'$. The assumption of the case and Lemma 10 imply $V(G_1) - V(C \cup C') \neq \emptyset$. So there exists a vertex $u \in V(G_1) - V(N(v))$ such that $\deg_{G_1}(u) = 1$. Thus

$$(9) \quad n + 1 = I_G(C_3) \leq \sum_{i=1}^{n_1} (\deg(v_i) - 1) + 2 = n + 1.$$

Lemma 5 and $V(N(v)) \neq V(G_1)$ imply that v belongs to at most $n_2 + n_3$ triangles of G , and vertices of $V_1 - \{v\}$ belong to at least n_1 triangles. Moreover, the number of edges from $V_1 - \{v\}$ to $V(G_1)$ is at least $2(n_1 - 1) + 1$. Therefore $|V_1| \geq 2$.

Lemma 11 implies that the vertex u is adjacent to two different vertices $v_1, v_2 \in V_1 - \{v\}$. Let w be a neighbour of u in G_1 . From Lemmas 10, 11 we have that w is adjacent to v_1 and v_2 . Therefore we get either a cycle in the subgraph $N(w)$ or that G is a K_2 -gluing of two graphs with triangles. The first case contradicts acyclicity of G_2 and G_3 . By Lemma 8 the other case gives $(\lambda - 2)^2 \mid P(G, \lambda)$ and it contradicts Lemma 6.

Therefore suppose now that the vertex $v \in V_1$ is not adjacent to a vertex of C' . Thus $v \neq v'$. Applying the same arguments as before we get that

G_1 does not have any vertex of degree 1. Hence we can consider only the following three subcases: G_1 is a K_2 -gluing of two cycles of even order, a K_1 -gluing of two cycles of even order, or it consists of two cycles of even order and exactly one path connecting them.

Since n is even, then for the first case we get that $V_1 - \{v, v'\} \neq \emptyset$ and 2-connectivity of G , Lemma 10 and acyclicity of G_2 and G_3 imply $N(v_1) \cong K_2$ for each $v_1 \in V_1 - \{v, v'\}$ and this gives a contradiction to Lemma 11.

For two other cases Lemma 10 and acyclicity of G_2 and G_3 imply $|V(N(v_1)) \cap V(N(v_2))| \leq 2$, for each pair of different vertices $v_1, v_2 \in V_1$. Therefore by 2-connectivity of G we get that G is a K_2 -gluing of two graphs with triangles. Hence we get a contradiction to the Lemma 6.

Case 2. Let G_3 be a tree, and G_2, G_1 be unicyclic graphs with even cycles. Note that

$$\begin{aligned} |E(G_1)| &= |V(G_1)| = n + 1 - n_1, \\ |E(G_2)| &= |V(G_2)| = n_1 + n_3 = n + 1 - n_2. \end{aligned}$$

The number of edges from V_1 to $V(G_1)$ is equal to

$$(10) \quad 2n + 1 - (n + 1 - n_1) = n + n_1.$$

Similarly, the number of edges from V_2 to $V(G_2)$ is equal to

$$(11) \quad 2n + 1 - (n + 1 - n_2) = n + n_2.$$

Let C^1 be the cycle of G_1 , and C^2 be the cycle of G_2 .

Suppose that there is no vertex in V_1 adjacent to all of the vertices of C^1 . Then each vertex of V_1 is adjacent to a subforest in G_1 .

By Lemma 3 the number of triangles in G containing a vertex $v_i^1 \in V_1$ is at most $d(v_i^1) - 1$. So the number of triangles in G is at most

$$\begin{aligned} n + 1 &= I_G(C_3) \leq \sum_{i=1}^{n_1} (d(v_i^1) - 1) \\ (12) \quad &= \sum_{i=1}^{n_1} d(v_i^1) - n_1 = n + n_1 - n_1 = n, \end{aligned}$$

and we get a contradiction.

Therefore there exists at least one vertex $v^1 \in V_1$ adjacent to all of the vertices of C^1 . Suppose that there is another such vertex, i.e., let $w^1 \in V_1 - \{v^1\}$ and let w^1 be adjacent to all of the vertices of C^1 . Assume also without loss of generality that u_1, u_2, \dots, u_{2m} are consecutive vertices of C^1 , where $u_1, u_3, \dots, u_{2m-1} \in V_2$ and $u_2, u_4, \dots, u_{2m} \in V_3$. Note that the subgraph

induced by $\{u_1, v^1, u_3, w^1\}$ is a cycle in G_3 . This contradicts the fact that G_3 is a tree. Thus we have proved that there exists exactly one vertex v^1 in V_1 adjacent to all vertices in C^1 . Similarly, there exists exactly one vertex v^2 in V_2 adjacent to all vertices in C^2 . Suppose that a subgraph of G_1 induced by all vertices adjacent to a vertex of V_1 is disconnected. Hence by Lemmas 3-4 we get the formula (12), and it leads to a contradiction.

Thus we have the following observations.

Lemma 12. *One vertex, $v^1 \in V_1$, is adjacent to all of the vertices of a connected subgraph of G_1 which contains the even cycle. Each other vertex of V_1 is adjacent to the vertices of a subtree of G_1 .*

Similarly, by symmetry, the vertices of V_2 must satisfy the respective conditions of the following result.

Lemma 13. *One vertex, $v^2 \in V_2$, is adjacent to all of the vertices of a connected subgraph of G_2 which contains the even cycle. Each other vertex of V_2 is adjacent to the vertices of a subtree of G_2 .*

Lemma 12 and acyclicity of G_3 give the following lemma.

Lemma 14. $|V(N(v)) \cap V(N(v'))| \leq 3$ for $v, v' \in V_1$, $v \neq v'$.

Moreover, Lemma 11 presented in case 1 holds for G .

Subcase 2.1. Suppose that $N(v^1) = V(G_1)$. Then by Lemma 4 the vertex v^1 belongs to $n+1-n_1$ triangles in G , and each of other $n+1-(n+1-n_1) = n_1$ triangles contains a vertex of $V_1 - \{v^1\} \neq \emptyset$. Note that the number of edges from $V_1 - \{v^1\}$ to $V(G_1)$ is equal to $2n+1-2(n+1-n_1) = 2n_1-1 = 2(n_1-1)+1$. This and Lemma 11 lead to $|V_1| = 2$. Hence there exists exactly one vertex in V_1 different from v^1 , say w^1 , and its degree equals 3.

Therefore, from Lemma 7 and from the fact that n is even, the graph G_1 consists of C^1 and exactly one tree T rooted at a vertex of C^1 . Moreover, for each pair x, y of leaves of T we have that $\text{dist}_{G_1}(x, y) = 2$ and then T has only two leaves. Since n is even, T has an even number of vertices (including root vertex). Therefore $T \cong P_{2t}$ or T is a K_1 -gluing of P_{2t-1} and K_2 , where $t \geq 1$, and G_1 is one of the two graphs presented in Figure 2.

By Lemma 11 each leaf of the rooted tree T is adjacent to w^1 and v^1 . Lemmas 6, 8 imply that the graph G is not any K_2 -gluing of two graphs with triangles in each of them. Therefore G_1 is a unicyclic graph with one leaf and a cycle of length $n-2$.



Figure 2

If two of the vertices which are adjacent to w^1 have colour 2, then $\{x, w^1, y, v^1\}$ induces C_4 in G_3 , and we have a contradiction.

Therefore two of the vertices which are adjacent to w^1 have colour 3 and then $\{x, w^1, y, v^1\}$ induces C_4 in G_2 .

Hence G is K_3 -gluing of W_{n-1} and W_5 such that the centers of the wheels are not overlapped. Note that by Lemma 1(v) the graph G is isomorphic to W_{n+1}^q and this is possible only for $q = 3$.

Subcase 2.2. We can assume that $N(v^1) \neq V(G_1)$ and by symmetry $N(v^2) \neq V(G_2)$. Then by Lemmas 12, 13 each of the graphs G_1, G_2 is unicyclic with a vertex of degree one. Evidently by Lemma 11 each leave in G_1 is adjacent to at least two vertices of V_1 . Let v^1, v^2 be the vertices of Lemmas 12 and 13, respectively. Let x be a leave in G_1 which is not adjacent to v^1 , and let x^1 be the neighbour of x in G_1 .

Let x^2 be a neighbour of x^1 in G_1 such that $x^2 \neq x$ and $\deg(x^2) \geq 2$.

Lemmas 11, 12 imply that the vertex x has at least two neighbours in V_1 . Let us consider $N(x^1)$. Since G is not any K_2 -gluing of two graphs with triangles and G_3 has not any cycle, then Lemmas 6, 7, 11, 12 and 14 imply that $N(x^1)$ contains a cycle belonging to G_2 . Evidently, the cycle is unique. The same arguments give $x^1 \in V(C^1)$ and therefore G_1 has a unique rooted tree and it is isomorphic to a graph presented in Figure 3. Similarly, G_2 is isomorphic to a graph presented in Figure 3.

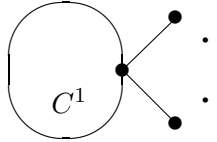


Figure 3

Let $a, b \in V(N(x^1)) \cap V(C^1)$, $\{w_1, \dots, w_t\} = V_1 - \{v^1\}$ and let $x = x_1, x_2, \dots, x_m$ be the leaves of G_1 . If neither a nor b is adjacent to a vertex w_j , $j = 1, \dots, t$, then G is a K_2 -gluing of two graphs with triangles, for K_2 induced by $\{v^1, x^1\}$ and we get a contradiction. Thus without loss of generality, we can assume that a is adjacent to w_1 . Then there exists an alternating sequence passing through all vertices of V_1 and all leaves of $V(G_1)$ and having one of the two forms

$$a, w_1, x_1, w_2, x_2, \dots, x_m, w_m, b, v^1$$

or

$$a, w_1, x_1, w_2, x_2, \dots, x_m, v^1.$$

The first case gives an odd cycle in G_2 and we get a contradiction. The other one gives a K_3 -gluing of two wheels which does not identify their central vertices. Since each generalized θ -graph is χ -unique [7], from Lemma 9 we get that these wheels must be isomorphic to W_{q+2} and W_{n-q+2} , respectively.

The proof is complete. \blacksquare

Since the wheels W_6, W_8 are not χ -unique graphs [2], [9] the χ -equivalent classes for other cases of n and q can contain more than two graphs. The graphs $G \simeq W_{n+1}^q$, for n odd or q even are not uniquely $\chi(G)$ -colourable. Thus, the proof techniques used in this paper cannot be used to characterize χ -equivalent classes for these graphs.

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Received 18 April 1997
Revised 28 August 1997