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PAIRED-DOMINATION

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Abstract

We are interested in dominating sets (of vertices) with the additional property that the vertices in the dominating set can be paired or matched via existing edges in the graph. This could model the situation of guards or police where each has a partner or backup. This paper will focus on those graphs in which the number of matched pairs of a minimum dominating set of this type equals the size of some maximal matching in the graph. In particular, we characterize the leafless graphs of girth seven or more of this type.

Keywords: domination, paired-domination, matching.

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1. Introduction

A *dominating set*, say S, of vertices in a graph G, is a set of vertices such that every vertex of G is either in S or adjacent to at least one member of S. A *paired-dominating set*, introduced by T. Haynes and P. Slater in [1], is a dominating set whose induced subgraph contains at least one perfect matching. As these authors suggest, this could model the situation in which the dominating set is a set of guards and each guard is assigned another adjacent one (they are designated as backups for each other). Our approach to the problem was motivated by a different application. We wish to find a set, say M, of independent edges whose end vertices serve as a dominating set in the graph. These edges could then be the set of "beats" that $|M|$ police could patrol. Although the set of police do not form a dominating set, we are trading constant surveillance for a saving of half the number of guards, perhaps not an unrealistic approach in the current economic climate.

As pointed out in [1], the size, denoted γ_p, of a minimum paired-dominating set is always bounded above by twice the size of any maximal matching (since the end vertices of a maximal matching form a paired-dominating set). In this paper we shall focus on those graphs which have a maximal matching whose end vertices actually form a minimum paired-dominating set (we shall use the term a γ_p-set, for brevity). Furthermore, we shall restrict our attention to connected graphs with no vertices of degree one. We will henceforth refer to vertices of degree one as *leaves*. Let $\mathcal{G}$ denote the leafless graphs of this type which have girth at least seven. In this situation where there are no cycles of length six or smaller, we completely characterize these graphs showing they must belong to an infinite family based on the nine cycle.

2. Preliminary Results

Let us consider a graph G in $\mathcal{G}$. That is, G has some maximal matching, say M, such that $V(M)$ forms a γ_p-set.

Lemma 1. *Let G be a graph in $\mathcal{G}$ and let M be a maximal matching such that $V(M)$ is a γ_p-set in G. Then no end vertex of an edge in M can be adjacent to an end vertex of another edge in M.*

Proof. Assume not. That is, let G and M satisfy the hypothesis but assume there are two edges, pq and rs say, in M such that q is adjacent to r. Since G is leafless and has girth at least six, p must have a neighbour, say v, which is not on the 4-path $pqrs$. Now v has a neighbour, say u, which is also not on the 4-path $pqrs$ due to the girth restriction and the leafless situation. Since M is a maximal matching either u or v must be incident with some edge in M. Hence, any neighbour of p, other than q, is adjacent to a vertex in $V(M)$ besides p. Similarly, any neighbour of s other than r is adjacent to a vertex in $V(M)$ besides s. Thus we could certainly interchange the two pairs represented by the edges pq and rs for the single pair q and r and have a paired-dominating set that is smaller than $2|M|$ which is a contradiction. ∎

Observe that the girth restriction in the lemma is sharp as illustrated by C_5 ($\gamma_p = 4$). The leafless property is also essential as shown by the graph in Figure 1 where $\{a, b, c, d\}$ is a γ_p-set.

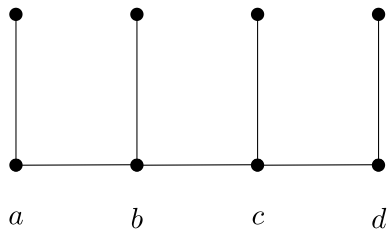

Figure 1. An example showing the necessity of the leafless property in Lemma 1

We now proceed to show the importance of 9-cycles in these graphs.

Lemma 2. *Let G be a graph in $\mathcal{G}$ and let M be a maximal matching such that $V(M)$ is a γ_p-set. If there is an 8-path, say $abcdefgh$, in which ab, de, and gh belong to M, then that 8-path must be part of a 9-cycle.*

Proof. Assume not. That is, let G be a graph and M be a maximal matching satisfying the hypothesis of the lemma, and let $abcdefgh$ be an 8-path where ab, de, and gh belong to M, but a and h do not share a common neighbour. Observe that since G is leafless and has girth at least seven (hence no 6-cycles), and since a and h do not share a common neighbour, then no pair of a, d, e, and h share a common neighbour. Now consider any vertex, say v, which is a neighbour of one of a, d, e, or h. Since v is not a leaf, it has another neighbour which is an end vertex of some edge in M and is not one of a, d, e, or h. This implies that we could interchange the three edges ab, de, and gh with the two edges bc and fg and obtain a paired-dominating set smaller than $2|M|$. This is a contradiction. Therefore, a and h must share a neighbour, implying that the 8-path is part of a 9-cycle. ■

Corollary 1. *If G belongs to $\mathcal{G}$, then the girth of G is at most nine.*

In fact, every edge of a graph in $\mathcal{G}$ lies on a cycle of length nine.

Lemma 3. *If G belongs to $\mathcal{G}$ then G does not contain a 7-cycle.*

Proof. Assume that G satisfies the above hypothesis but does contain a 7-cycle, say $C = abcdefg$. Let M be a maximal matching of G such that $V(M)$ is a γ_p-set. Then, by Lemma 1, C contains at most two edges of M.

Suppose C contains two edges of M. We may assume, without loss of generality, that the edges ab and de are in M. Then no edge of M is incident with f or g, by Lemma 1. But then fg is neither in M nor incident with any edge in M. This contradicts the fact that M is maximal. Hence, C contains at most one edge of M.

Suppose C contains exactly one edge, say ab, of M. By Lemma 1, neither c nor g is incident with an edge in M. Hence, d and f must each be incident with an edge in M, otherwise, cd and fg would neither be in M nor incident with an edge in M. Let xd and yf be the edges in M. We now have the path $yfgabcdx$ where yf, ab, and dx are in M. Therefore, by Lemma 2, the vertices x and y must have a common neighbour, say z. But this results in the 6-cycle $xzyfed$ which contradicts the girth restriction. Hence, C contains no edge of M. This too is impossible, however. Since M is maximal, at least one end vertex of each edge of C must be in $V(M)$. Because C is of odd length, this results in two edges of M having adjacent end vertices. We know this cannot be the case due to Lemma 1. Hence the girth of G is at least eight. ∎

Therefore, by Lemmas 2 and 3, any graph in $\mathcal{G}$ must have either girth eight or girth nine.

Lemma 4. *Let G be a graph in $\mathcal{G}$. If M is a maximal matching in G such that $V(M)$ is a γ_p-set, then no edge of M lies on an 8-cycle.*

Proof. Suppose G contains an 8-cycle, say $C = abcdefgh$. Let M be a matching satisfying the hypothesis. By Lemma 1, C contains at most two edges in M.

Suppose C contains two edges of M. By Lemma 1, we may assume, without loss of generality, they are either the pair ab and de or the pair ab and ef. If the former case holds then g must be incident with an edge in M. Let xg be the edge in M. But then $xghabcde$ is an 8-path with xg, ab, and de in M. Therefore, by Lemma 2, x and e must have a common neighbour, say y. Now we have the 5-cycle $xgfey$ which contradicts the girth restriction. Hence, the edges ab and ef must be in M. However, this results in the edges cd and gh being neither in M nor incident with any edge in M, by Lemma 1. This contradicts the fact that M is maximal. Hence, C contains at most one edge of M.

Suppose C contains exactly one edge of M, say ab. Then both d and g must be incident with edges in M which are not in C. But then, by Lemma 1, the edge ef is neither in M nor incident with any edge in M. This is impossible since M is maximal. Hence, C contains no edge of M. ∎

Lemma 5. *Suppose G is in $\mathcal{G}$. If M is a maximal matching such that $V(M)$ is a γ_p-set, then any vertex of G that has degree at least three must be incident with an edge of M.*

Proof. Let G and M satisfy the hypothesis, and let v be a vertex having degree at least three. Assume that v is not incident with any edge of M.

Suppose v has neighbours a, b and c. They each must be incident with some edge of M, since M is maximal. Let ax, by and cz be the edges in M. Since G has no leaves and girth at least seven, x must have a neighbour, say u, which is adjacent to neither y nor z. The vertex u itself must have another neighbour, say p. The vertex p cannot be adjacent to y nor z since this would create a 7-cycle in contradiction to Lemma 3. By Lemma 1, the vertex u is not incident with any edge in M. Hence, p must be incident with an edge in M, since M is maximal. Call this edge pq. By Lemma 1, q cannot be adjacent to y nor z. However, $qpuxavby$ is an 8-path with qp, xa and by in M. By Lemma 2, this path must be part of a 9-cycle. Hence, q and y share a common neighbour, say r. Similarly, q and z share a common neighbour, say s. The vertices r and s must be distinct due to girth restrictions. But now we have the 8-cycle $qrybvczs$ containing two edges which are in M. This is impossible by Lemma 4. Hence, we conclude that v must be incident with an edge in M. ■

3. The Characterization

We now wish to characterize all the graphs contained in the set $\mathcal{G}$. Define the infinite family of graphs, $\mathcal{F}$, to be the set of those graphs H which can be obtained from three nonempty sets of parallel edges, $\{u_r v_r : r = 1, \ldots, k\}$, $\{w_s x_s : s = 1, \ldots, l\}$ and $\{y_t z_t : t = 1, \ldots, m\}$, by connecting each of the pairs of vertices (v_r, w_s), (x_s, y_t) and (z_t, u_r) with a path of length two. Hence, for each such pair of vertices a new vertex is introduced which is a common neighbour of these vertices.

We will call the original set of $k + l + m$ parallel edges the *associated matching* of H.

The graphs in $\mathcal{F}$ are obviously leafless and have girth at least eight. Furthermore, the associated matching of any graph in $\mathcal{F}$ is a maximal matching. Note that if $k = l = m = 1$, then H is the 9-cycle and the end vertices of the associated matching form a γ_p-set. Hence, the 9-cycle has property P.

We are prepared to show that all the graphs in $\mathcal{F}$ are also in $\mathcal{G}$.

Theorem 1. *If G is a graph in $\mathcal{F}$, then G is also in $\mathcal{G}$.*

Proof. Let G be any graph in $\mathcal{F}$. Partition the edges in the associated matching into the sets $UV = \{u_r v_r : r = 1, \ldots, k\}$, $WX = \{w_s x_s : s =$

$1, 2, \ldots, l\}$, and $YZ = \{y_t z_t : t = 1, 2, \ldots, m\}$, as previously described. We wish to show that $\gamma_p \geq 2(k+l+m)$. That is, any matching (not necessarily maximal), say M, such that $V(M)$ is a paired-dominating set contains at least $k + l + m$ edges.

Let M be a matching such that $V(M)$ is a paired-dominating set. Since every u_r and z_t share a common neighbour of degree two, then either all the vertices $\{u_r : r = 1, 2, \ldots, k\}$ or all the vertices $\{z_t : t = 1, 2, \ldots, m\}$ must be incident with an edge in M. Similarly, either all of $\{v_r : r = 1, 2, \ldots, k\}$ or all of $\{w_s : s = 1, 2, \ldots, l\}$ must be incident with an edge in M, and either all of $\{x_s : s = 1, 2, \ldots, l\}$ or all of $\{y_t : t = 1, 2, \ldots, m\}$ must be incident with an edge in M. Without loss of generality assume that all the vertices $\{u_r : r = 1, 2, \ldots, k\}$ are met by the matching.

Case 1. Suppose that all the edges of UV are in M. The end vertices of these edges are not adjacent to any of the vertices $\{w_s : s = 1, 2, \ldots, l\}$ nor the vertices $\{z_t : t = 1, 2, \ldots, m\}$. Since no pair of vertices in $\{w_s, z_t : s = 1, \ldots, l, t = 1, 2, \ldots, m\}$ are adjacent or have a common neighbour, each vertex in this set requires a unique vertex to dominate it. Furthermore, no such set of dominating vertices contains an adjacent pair. Hence, another $l + m$ edges are required and $|M| \geq l + k + m$.

This also includes the case where $WX \subseteq M$ or $YZ \subseteq M$.

Case 2. Suppose that $UV \not\subseteq M$, $WX \not\subseteq M$ and $YZ \not\subseteq M$. By Lemma 1, no two edges of M can have adjacent end vertices. Therefore, both u_r and v_r are met by M only if $u_r v_r$ is in M. Since all of the vertices $\{u_r : r = 1, 2, \ldots, k\}$ are met by M and $UV \not\subseteq M$, then some v_r is not met by the matching. Therefore, all of $\{w_s : s = 1, 2, \ldots, l\}$ must be met by the matching. Since $WX \not\subseteq M$, then by Lemma 1 there is some x_s not met by the matching. Therefore, all of $\{y_t : t = 1, 2, \ldots, m\}$ must be met by the matching. This gives us a total of $k + l + m$ vertices all of which must be met by the matching, but none of which are adjacent. Therefore, at least $k + l + m$ edges are required and $|M| \geq l + k + m$.

So, at least $(l+k+m)$ edges are required in any matching, M, where $V(M)$ is a paired-dominating set. Therefore, $\gamma_p \geq 2(k + l + m)$. But we know the edge set $\{u_r v_r, w_s x_s, y_t z_t : r = 1, \ldots, k, s = 1, 2, \ldots, l, t = 1, \ldots, m\}$ is a maximal matching of size $k + l + m$. Hence, $\gamma_p = 2(k + l + m)$, and the vertex set of this maximal matching is a γ_p-set. Therefore, any graph in $\mathcal{F}$ must also be in $\mathcal{G}$. ■

It has been shown that for any graph in $\mathcal{F}$, the vertex set of the associated matching is a γ_p-set. In fact, for every graph in $\mathcal{F}$, other than the 9-cycle, the associated matching is the only maximal matching with this property. This can be easily verified using Lemma 1 and Lemma 4 from the previous section.

We now wish to show that the converse of Theorem 1 is true.

Theorem 2. *If G is a graph in $\mathcal{G}$, then G is also in $\mathcal{F}$.*

Proof. We proceed by induction on γ_p. Suppose G is a graph in $\mathcal{G}$ and $\gamma_p = 6$. Then G has a maximal matching, say M, such that $|M| = 3$ and $V(M)$ is a paired-dominating set. By Lemma 2, we know that G contains a 9-cycle, say $C = \{a, b, c, d, e, f, g, h, i\}$. If $G = C$ then we are done since C is obviously in $\mathcal{F}$. Suppose $G \neq C$. Since M is maximal and C has odd length, according to Lemma 1 there must be at least one edge of the matching which lies on C. Without loss of generality, let ab be that edge. Due to Lemma 1, the vertex c is not incident with any edge in M. Hence, d must be incident with an edge in M, since M is maximal. Similarly, h must be incident with an edge in M. If de is not in M, then the vertex f must be incident with an edge in M. This, however, is impossible since $|M| = 3$. Hence, de and, similarly, gh are in M. Hence $V(M) = \{a, b, d, e, g, h\}$ is a paired-dominating set in G. Since $G \neq C$ there is some vertex, say v, which is not on C. This vertex must be adjacent to at least one of $\{a, b, d, e, g, h\}$. Without loss of generality, we can assume that v is adjacent to a. Since G is leafless, v has another neighbour, say w, which must also be adjacent to one of $\{a, b, d, e, g, h\}$. This, however, results in a cycle of length at most seven. This is a contradiction due to the girth restriction together with Lemma 4. Hence, the only graph in $\mathcal{G}$ with $\gamma_p = 6$ is the 9-cycle.

Let M be a maximal matching in G such that $V(M)$ is a γ_p-set and $|V(M)| = 2n > 6$. Choose any edge, say uv, in M. Let $N(u) = \{v, u_1, u_2, \ldots, u_k\}$ and let $N(v) = \{u, v_1, v_2, \ldots, v_l\}$. We know from Lemma 1 that no u_i, $i = 1, \ldots, k$, or v_j, $j = 1, \ldots, l$, is incident with an edge in M. Hence, by Lemma 5, each u_i and v_j must have degree two. Let w_i be adjacent to u_i for each $i = 1, 2, \ldots, k$. Similarly, let x_j be adjacent to v_j for each $j = 1, 2, \ldots, l$. Note that each w_i and x_j must be incident with an edge in M, since M is maximal. Let $\{w_i y_i : i = 1, \ldots, k\}$, and $\{x_j z_j : j = 1, 2, \ldots, l\}$, be the edges in M. Now, by Lemma 2, it must be the case that y_i and z_j have a common neighbour, say q_{ij}, for all $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$. Let $S = \{u, v, u_i, w_i, y_i, v_j, x_j, z_j, q_{ij} : i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, l\}$. Note that due to girth restrictions and Lemma 4, all the vertices in S must be distinct.

Case 1. Suppose that at least one of the w_i's or x_j's has a neighbour not in S. Without loss of generality assume that w_1 has a neighbour, say a, not in S. By Lemma 1 and Lemma 5, a has degree two and is not incident with an edge in M. In addition, a is not adjacent to any vertex in S due to girth restrictions, Lemma 3 and Lemma 4. Hence, a has another neighbour, say b, where b is incident with an edge in M. Let bc be that edge.

For all $j = 1, 2, \dots, l$, we have the path $\{x_j, z_j, q_{1j}, y_1, w_1, a, b, c\}$ where $x_j z_j$, $y_1 w_1$ and bc are in M. Hence, by Lemma 2, x_j and c have a common neighbour, say r_j, for all $i = 1, 2, \dots, l$. Furthermore, for all $i = 1, 2, \dots, k$, we have the path $\{w_i, y_i, q_{i1}, z_1, x_1, r_1, c, b\}$. Hence, by Lemma 2, w_i and b have a common neighbour, say s_i, for all $i = 1, 2, \dots, k$. Note that each r_j and s_i has degree two, due to Lemma 1 and Lemma 5, and that $deg(u) \leq deg(b)$ and $deg(v) \leq deg(c)$. Now we wish to show that equality holds for both of these inequalities.

Suppose this is not the case, and that c has another neighbour, say d. Since M is maximal, d has a neighbour, say e, which is incident with an edge in M. Note that neither d nor e is in $S \cup \{a, b, c\} \cup \{s_i, r_j : i = 1, \dots, k, j = 1, 2, \dots, l\}$ due to girth restrictions. Now, let ef be the edge in M. By Lemma 2, the path $\{f, e, d, c, b, a, w_1, y_1\}$ implies that f and y_1 must have a common neighbour, say g. This gives us the path $\{e, f, g, y_1, w_1, u_1, u, v\}$ which, by Lemma 2, implies that e and v have a common neighbour. Hence, e is adjacent to v_j for some $j = 1, 2, \dots, l$. Without loss of generality, assume that e is adjacent to v_1. However, we now have the 6-cycle $\{e, d, c, r_1, x_1, v_1\}$, which is impossible due to the girth restriction. Hence $deg(v) = deg(c)$. It can be similarly shown that $deg(u) = deg(b)$.

Let us now consider the graph, $H = G \setminus (N(u) \cup N(v))$. This graph is connected, leafless, and has girth at least seven. Furthermore, the matching $M' = M \setminus \{uv\}$ is a maximal matching in H such that $V(M')$ is a γ_p-set in H. Hence, H is in $\mathcal{G}$ where $\gamma_p(H) = 2n - 2$. Then, by the induction hypothesis, H must be in $\mathcal{F}$ and M' is its associated matching. The edges of the associated matching of H can be partitioned into three sets of parallel edges, since H is in $\mathcal{F}$. The graph G is obtained from H by adding one more edge, uv, to the set of parallel edges containing bc. Therefore, G is also in $\mathcal{F}$.

Case 2. Suppose that all of the neighbours of the w_i's and x_j's lie in S. Hence, the only vertices which may be adjacent to some vertex not in S are the y_i's and z_j's. Suppose y_1 has a neighbour which is not in S. Let a be that neighbour. Since G is leafless a has another neighbour, say b. Note that b is neither in nor adjacent to any vertex in S due to Lemma 3, Lemma 4 and the girth restriction. Since M is maximal, b must be incident with some edge in M. Let bc be the edge in M. Now we have the path $\{v, u, u_1, w_1, y_1, a, b, c\}$ where vu, w_1y_1 and bc are in M. Hence, by Lemma 2, the vertices v and c have a common neighbour. Since $N(v) = \{u, v_1, v_2, \ldots, v_l\}$, we can assume, without loss of generality, that v_1 is adjacent to both v and c. This is impossible, however, since the only neighbours of v_1 are v and x_1 which are both in S. Hence, y_1 has no neighbours other than those in S. Similarly, no y_i or z_j has a neighbour outside of S. Hence, $V(G) = S$.

Obviously, if both u and v have degree 2 then $G = C_9$ and $\gamma_p = 6$. Therefore, at least one of u and v must have degree at least three. Without loss of generality, assume that u has degree at least three. Let $H = G \setminus \{u_1, w_1, y_1, q_{1j} : j = 1, \ldots, l\}$. Then H is connected, leafless, and in $\mathcal{G}$. Furthermore, the edge w_1y_1 is parallel to w_2y_2, which is contained in the associated matching of H. By the induction hypothesis, H is in $\mathcal{F}$, and, therefore, the graph G is also in $\mathcal{F}$. ■

Hence, Theorem 1 and Theorem 2 together tell us that the graphs in $\mathcal{G}$ are precisely those graphs in $\mathcal{F}$.

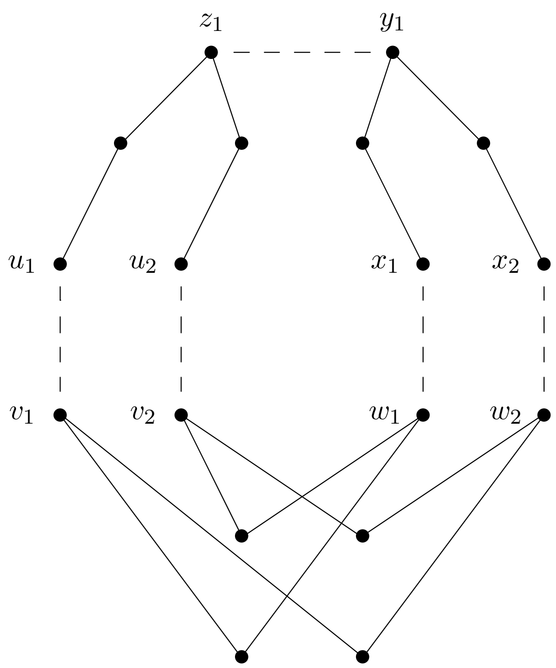

Figure 2. The graph P_{18}

Corollary 2. *If G belongs to $\mathcal{G}$ and contains an 8-cycle, then G contains the graph P_{18}, shown in Figure 2, as an induced subgraph.*

Proof. Suppose $G \in \mathcal{G}$ and P_{18} is not a subgraph of G. Then at least two of k, l and m are equal to one. Such graphs clearly do not contain any 8-cycles. ■

References

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