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DEGREE SEQUENCES OF DIGRAPHS WITH HIGHLY IRREGULAR PROPERTY

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Abstract

A digraph such that for each its vertex, vertices of the out-neighbourhood have different in-degrees and vertices of the in-neighbourhood have different out-degrees, will be called an HI-digraph. In this paper, we give a characterization of sequences of pairs of out- and in-degrees of HI-digraphs.

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The results presented here are the continuation of studies concerning degree sequences of graphs with highly irregular property understood in such a way that the vertices of the neighbourhood of each vertex have different degrees.

For directed graphs irregularity of this type can be defined in various ways, because each vertex v have out-degree $\deg^+(v)$, in-degree $\deg^-(v)$ and out-neighbourhood $N^+(v)$, in-neighbourhood $N^-(v)$. For example, in [1] authors consider digraphs G such that for every $v \in V(G)$ the following implication is true:

$$(u, w \in N^+(v) \text{ and } u \neq w) \Rightarrow \deg^+(u) \neq \deg^+(w).$$

In this paper, some problems are investigated concerning the existence of such graphs with special properties. In particular, the class of ditrees is examined.

We are interested in the characterization of degree sequences of digraphs which have the property of irregularity explained above. As far as this is concerned, we found digraphs described by below definition especially interesting.

Definition 1. A digraph G will be called an HI-*digraph*, if for each vertex $v \in V(G)$ the following condition holds:

$$(1) \qquad \begin{cases} (u, w \in N^+(v) \quad \text{and} \quad u \neq w) & \Rightarrow \quad \deg^-(u) \neq \deg^-(w), \\ (u, w \in N^-(v) \quad \text{and} \quad u \neq w) & \Rightarrow \quad \deg^+(u) \neq \deg^+(w). \end{cases}$$

We assume that the digraphs considered here have at least one arc.

The purpose of our considerations is to get necessary and sufficient conditions for a sequence of pairs of non-negative integers to be the sequence of pairs of semi-degrees of an HI-digraph.

Let $d^+ = (d_1^+, d_2^+, \ldots, d_n^+)$ and $d^- = (d_1^-, d_2^-, \ldots, d_n^-)$ be sequences of non-negative integers. We say that the sequence of pairs

$$(d_1^+, d_1^-), (d_2^+, d_2^-), \ldots, (d_n^+, d_n^-)$$

is HI-*digraphic* if there exists an HI-digraph G with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and such that $\deg^+(v_j) = d_j^+$, $\deg^-(v_j) = d_j^-$, for $j = 1, 2, \ldots, n$. Then G will be called an HI-*realization* of the sequence

$$(d_1^+, d_1^-), (d_2^+, d_2^-), \ldots, (d_n^+, d_n^-).$$

Proposition 1. *If a sequence (d_j^+, d_j^-), $j = 1, 2, \ldots, n$, is* HI-*digraphic, then for every permutation ϕ of the set $\{1, 2, \ldots, n\}$ the sequence $(d_j^+, d_{\phi(j)}^-)$, $j = 1, 2, \ldots, n$, is* HI-*digraphic.*

Proof. Let $G = (V, E)$, where $V = (v_1, v_2, \ldots, v_n)$, be an HI-realization of the sequence (d_j^+, d_j^-), j=1,2,...,n. We define a new digraph $G' = (V, E')$ such that

$$(v_i, v_j) \in E' \Leftrightarrow (v_i, v_{\phi(j)}) \in E.$$

Note that:

(a) $v_r \in N_{G'}^+(v_i) \Leftrightarrow v_{\phi(r)} \in N_G^+(v_i)$,
(b) $N_{G'}^-(v_j) = N_G^-(v_{\phi(j)})$.

It gives:

(c) $\deg_{G'}^+(v_j) = \deg_G^+(v_j) = d_j^+$,
(d) $\deg_{G'}^-(v_j) = \deg_G^-(v_{\phi(j)}) = d_{\phi(j)}^-$.

Now we show that G' is an HI-digraph. Let v_r and v_s be two vertices of out-neighbourhood of a vertex v_i in G'. Then, by (a), $v_{\phi(r)}$ and $v_{\phi(s)}$ are vertices of out-neighbourhood of a vertex v_i in G. Vertices $v_{\phi(r)}$ and $v_{\phi(s)}$ have different in-degrees in G. Then, by (d), v_r and v_s have different in-degrees in G'. Similarly, by (b) and (c), in the digraph G' vertices of in-neighbourhood have different out-degrees. ■

Remark 1. From Proposition 1 it follows that for characterization of HI-digraphic sequences of pairs (d_j^+, d_j^-), $j = 1, 2, \ldots, n$ it is sufficient to restrict our considerations to the case when both sequences

$$d^+ = (d_1^+, d_2^+, \ldots, d_n^+) \quad \text{and} \quad d^- = (d_1^-, d_2^-, \ldots, d_n^-)$$

are non-increasing. Then we say that the *pair* (d^+, d^-) *is* HI-*digraphic.*

If G is a digraph, then the maximum out-degree in G and the maximum in degree in G will be denoted by $\Delta^+(G)$ and $\Delta^-(G)$, respectively.

Proposition 2. *If G is an* HI-*digraph, then* $\Delta^+(G) = \Delta^-(G)$.

Proof. Let v and u be vertices of G for which $\deg^+(v) = \Delta^+(G)$ and $\deg^-(u) = \Delta^-(G)$. Note that:

(1) the set $\{\deg^-(w) : w \in N^+(v)\}$ has exactly $\deg^+(v)$ elements (by HI-graphicity of G),

(2) if $w \in N^+(v)$, then $\deg^-(w) \neq 0$,

(3) if some set Ω consists of ω positive integers, then $\omega \leq \max \Omega$.

By (1) we have

$$\Delta^+(G) = \deg^+(v) = \mid \{\deg^-(w) : w \in N^+(v)\} \mid,$$

and, by (2), (3),

$$\mid \{\deg^-(w) : w \in N^+(v)\} \mid \leq \max\{\deg^-(w); w \in N^+(v)\} \leq \Delta^-(G).$$

So,

$$\Delta^+(G) \leq \Delta^-(G).$$

Similarly, we obtain

$$\Delta^-(G) = \deg^-(u) \leq \max\{\deg^+(w) : w \in N^-(v)\} \leq \Delta^+(G). \qquad ■$$

Proposition 2 permits to put:

$$\Delta(G) = \Delta^+(G) = \Delta^-(G)$$

for an HI-digraph G.

Let

$$S^+(G) = \{s : s \neq 0 \text{ and } s = \deg^+(w) \text{ for some } w \in V(G)\},$$

$$S^-(G) = \{s : s \neq 0 \text{ and } s = \deg^-(w) \text{ for some } w \in V(G)\}.$$

Proposition 3. *If G is an* HI-*digraph, then* $\mid S^+(G) \mid = \Delta(G) = \mid S^-(G) \mid$.

Proof. Let v be a vertex of G such that $\deg^+(v) = \Delta^+(G)$. Note that

(1) $\Delta^+(G) = \mid \{\deg^-(w) : w \in N^+(v)\} \mid$,

(2) $\{\deg^-(w) : w \in N^+(v)\} \subseteq S^-(G)$ and

(3) $\mid S^-(G) \mid \leq \max S^-(G) \leq \Delta^-(G)$.

Aplying, in turn, (1), (2) and (3) we obtain

$$\Delta^+(G) \leq \mid S^-(G) \mid \leq \Delta^-(G).$$

Thus, by Proposition 2, we have $\mid S^-(G) \mid = \Delta(G)$. Similarly, we prove that $\Delta(G) = \mid S^+(G) \mid$. ∎

Theorem 1. *If a pair (d^+, d^-) of non-increasing sequences of non-negative integers has an* HI-*realization G with $\Delta(G) = m$, then*

$$(2)\begin{cases} d^+ = (\underbrace{m,\ldots,m}_{n_m}, \underbrace{m-1,\ldots,m-1}_{n_{m-1}}, \ldots, \underbrace{i,\ldots,i}_{n_i}, \ldots, \underbrace{1,\ldots,1}_{n_1}, \underbrace{0,\ldots,0}_{n_0}) \\ d^- = (\underbrace{m,\ldots,m}_{k_m}, \underbrace{m-1,\ldots,m-1}_{k_{m-1}}, \ldots, \underbrace{i,\ldots,i}_{k_i}, \ldots, \underbrace{1,\ldots,1}_{k_1}, \underbrace{0,\ldots,0}_{k_0}) \\ \text{where } n_0, k_0 \text{ are non-negative integers and } m, n_i, k_i \in N, \\ n_i \geq k_m, k_i \geq n_m \text{ for } i = 1, 2, \ldots, m, \\ \sum_{i=0}^m n_i = \sum_{j=0}^m k_j \text{ and } \sum_{i=1}^m i \cdot n_i = \sum_{j=1}^m j \cdot k_j. \end{cases}$$

Proof. Let n_i (k_i) denote the number of vertices of G with out-degree (in-degree) $i, i = 0, 1, \ldots, m$. By Proposition 3, the numbers n_i and k_i are positive for $i = 1, 2, \ldots, m$. From Definition 1 it follows that every vertex with out-degree m is joined with exactly one vertex of in-degree i for each $i = 1, 2, \ldots, m$ and every vertex with in-degree i is joined with at most one vertex of out-degree m. Then $k_i \geq n_m$ for $i = 1, 2, \ldots, m$. By the similar argumentation we obtain: $n_i \geq k_m$ for $i = 1, 2, \ldots, m$. The last equations of (2) hold by graphicity of (d^+, d^-). ∎

According to Theorem 1 says that the degree sequences of HI-digraphs with maximum degree m (out and in) have the analogous form as degree sequences of undirected HI-graphs (characterization of HI-graphic sequences has been presented in [4]). Namely, in each of them all integers of the set $\{1, 2, \ldots, m\}$ appear, moreover the number of elements equal to m in both sequences is the same. Observe that in (2) the number n_0 or k_0 can be equal to zero. By a remark below, we can restrict our consideration to one case when n_0 and k_0 are both positive.

Remark 2. The consideration of HI-graphicity of the pair (d^+, d^-) of the form (2) can be restricted to the case in which $n_0 \neq 0$ and $k_0 \neq 0$.

Proof. Let (d^+, d^-) be a pair of sequences of the form (2) and let (t^+, t^-) be the pair obtained by adding to both sequences d^+ and d^- one element equal to 0. Note that adding to HI-digraph G, being a realization of (d^+, d^-), a new isolated vertex we obtain an HI-realization T of (t^+, t^-). Conversely, deleting from an HI-digraph T, being a realization of (t^+, t^-), one isolated vertex we obtain an HI-realization G of (d^+, d^-). ∎

The case $n_0 \neq 0$ and $k_0 \neq 0$ everywhere extorts the assumption that HI-digraph has at least one isolated vertex, however in this case the notations are far uniform. The class of HI-digraphs with at least one isolated vertex we will denote by HI_0.

Let sequences $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_k)$ and a matrix $[c_{ij}]_{n \times k}$ have elements being non-negative integers. Let $\mathcal{M} = (X \cup Y, \mu)$, where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_k\}$ and $\mu : X \times Y \to N \cup \{0\}$ be a bipartite oriented multigraph. We will say that the multigraph $\mathcal{M}$ is a *realization of the pair* (a, b) *with bounds* c_{ij} if for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, k : \deg^+(x_i) = a_i$, $\deg^-(y_j) = b_j$ and $\mu(x_i, y_j) \leq c_{ij}$.

Lemma 1. *Let* (d^+, d^-) *be a pair of sequences of the form* (2)*, where* $n_0 \neq 0$ *and* $k_0 \neq 0$*, and let* $b^+ = (b_0^+, b_1^+, \ldots, b_m^+)$, $b^- = (b_0^-, b_1^-, \ldots, b_m^-)$ *be sequences* *such that* $b_i^+ = i \cdot n_i$, $b_j^- = j \cdot k_j$, $i, j = 0, 1, \ldots, m$. *If the pair* (d^+, d^-) *is* HI_0-*digraphic, then there exists a bipartite oriented mutigraph* $\mathcal{M} = (X \cup Y, \mu)$, *where* $\mu : X \times Y \to N \cup \{0\}$, *which realizes the pair* (b^+, b^-) *with bounds* $c_{ij} = min\{n_i, k_j\}$, $i, j = 0, 1, \ldots, m$.

Proof. Let $G = (V, E)$ be an HI_0-digraph which realizes the pair (d^+, d^-). We put:

$X_i = \{u \in V : \ \deg^+(u) = i\}$, $Y_j = \{u \in V; \ \deg^-(u) = j\}$,
$\mu(X_i, Y_j) = | \{(u, v) \in E : \ u \in X_i, v \in Y_j\} |$ for $i, j = 0, 1, \ldots, m$,
$X = \{X_0, X_1, \ldots, X_m\}$, $Y = \{Y_0, Y_1, \ldots, Y_m\}$.

It is not difficult to check that the multigraph $\mathcal{M} = (X \cup Y, \mu)$ has the required properties. ■

The multigraph $\mathcal{M}$ described in the proof of Lemma 1 will be denoted by $\mathcal{M}_G$.

Further we will prove the converse of Lemma 1. For this we use results of paper [3] adopting them for our aims.

Let $G = (V, E)$ be a digraph in which $V = \{v_1, v_2, \ldots, v_n\}$ and let $\{d_0^+, d_1^+, \ldots, d_r^+\}$, $\{d_0^-, d_1^-, \ldots, d_s^-\}$ be the sets of out- and in-degrees in G, respectively. We assume that $d_0^+ < d_1^+ < \ldots < d_r^+$ and $d_0^- < d_1^- < \ldots < d_s^-$. For $i = 0, 1, \ldots, r$ and $j = 0, 1, \ldots, s$ we put:

$$V_i^+ = \{v \in V : \ \deg_G^+(v) = d_i^+\},$$

$$V_j^- = \{v \in V : \ \deg_G^-(v) = d_j^-\},$$

Then for each vertex v of G we define column-vectors $\vec{v}^+$ and $\vec{v}^-$ in the following way:

$\vec{v}^+ = (t_0^+(v), t_1^+(v), \ldots, t_s^+(v))^T$, where $t_j^+(v)$ is equal to the number of arcs from the vertex v to vertices of the set V_j^-,

$\vec{v}^- = (t_0^-(v), t_1^-(v), \ldots, t_r^-(v))^T$, where $t_i^-(v)$ is equal to the number of arcs from vertices of the set V_i^+ to the vertex v.

In this way, with the digraph G, we can associate the pair (M_G^+, M_G^-) of matrices, where

$$M_G^+ = [\vec{v_1}^+, \vec{v_2}^+, \ldots, \vec{v_n}^+] \quad \text{and}$$

$$M_G^- = [\vec{v_1}^-, \vec{v_2}^-, \ldots, \vec{v_n}^-].$$

Note that M_G^+ is a $((s+1) \times n)$-matrix and M_G^- is a $((r+1) \times n)$-matrix (in [3] they have been called out- and in-distribution matrix, respectively).

By $\widetilde{M}_G^+$ we denote the matrix obtained from M_G^+ by a reordering of columns as follows:

$$\vec{v_r}^+ \text{ precedes } \vec{v_s}^+ \text{ in } \widetilde{M}_G^+$$
$$\Updownarrow$$
$$\text{(the sum of members of } \vec{v_r}^+ \text{ is less than the sum of members of } \vec{v_s}^+\text{) or}$$
$$\text{(these sums are equal and } r < s\text{).}$$

Similarly, we define the matrix $\widetilde{M}_G^-$. By $\widetilde{M}_i^+$, $i = 0, 1, \ldots, r$, we denote the submatrix of $\widetilde{M}_G^+$ which contains all columns with the sum of members equal to d_i^+. Analogously, we define $\widetilde{M}_j^-$, $j = 0, 1, \ldots, s$. Then we can identify the matrices $\widetilde{M}_G^+$ and $\widetilde{M}_G^-$ with the sequences

$$(\widetilde{M}_0^+, \widetilde{M}_1^+, \ldots, \widetilde{M}_r^+) \quad \text{and} \quad (\widetilde{M}_0^-, \widetilde{M}_1^- \ldots, \widetilde{M}_s^-),$$

respectively.

We say that a *pair* (A, B) *of sequences of matrices is* HI_0-*digraphic* if there exists a digraph G of the class HI_0 such that $A = \widetilde{M}_G^+$ and $B = \widetilde{M}_G^-$. Then the digraph G will be called an HI_0-*realization of* (A, B).

It is easy to check that the following proposition is true.

Proposition 4. *If G is HI_0-digraph with $\Delta(G) = m$, then $\widetilde{M}_G^+ = (A_0, A_1, \ldots, A_m)$ and $\widetilde{M}_G^- = (B_0, B_1, \ldots, B_m)$, where for $i = 0, 1, \ldots, m$ the following conditions hold:*

(i) *each of matrices A_i, B_i has $m + 1$ rows,*
(ii) *all matrices A_i, B_i have elements equal to 0 or 1, only,*
(iii) *each column of A_i (B_i) has exactly i elements equal to 1.*

In [3] the definition of a demi-bipartite graph is given. Namely, a *demi-bipartite graph* (*d.b-graph*) is a triple (X, Y, E), where X, Y are non-empty sets and $E \subseteq X \times Y$ (note that the sets X and Y can have an element in common).

A d.b-graph (X, Y, E), where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$ is called a *realization* of a pair $(a, b) = ((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_k))$ of sequences of non-negative integers, if for any $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, k$:

$$\deg^+(x_i) = a_i, \ \deg^-(y_j) = b_j, \ \text{ and}$$
$$\deg^-(x_i) = 0, \ \deg^+(y_j) = 0 \ \text{ for } \ x_i, y_j \notin X \cap Y.$$

From the criterion of d.b-graphicity of (a, b), (see [3] Lemma 4), the following follows immediately

Proposition 5. *If a and b are 0-1-sequences then (a,b) is realizable by a d.b-graph if and only if both sequences a and b have the same number of members equal to 1.*

Lemma 2. *Let m be a positive integer and let $A=(A_0,A_1,\ldots,A_m)$, $B=(B_0,B_1,\ldots,B_m)$ be sequences of matrices, where $A_i=[a^i_{pq}]$, $p=0,1,\ldots,m$ and $q=1,2,\ldots,n_i$; $B_j=[b^j_{rs}]$, $r=0,1,\ldots,m$ and $s=1,2,\ldots,k_j$. Moreover let $\sum_{i=0}^m n_i=\sum_{j=0}^m k_j$ and for A and B the conditions* (i) – (iii) *of Proposition 4 hold. Then (A,B) is* HI_0*-digrahic if and only if*

(iv) *for every $i,j=0,1,\ldots,m$ the j-th row of A_i and i-th row of B_j have the same number of members equal to 1.*

Proof. ($\Rightarrow$) Let $G=(V,E)$ be an HI_0-realization of the pair (A,B). We put: $V_i^+=\{v\in V:\deg^+(v)=i\}$, $V_j^-=\{v\in V:\deg^-(v)=j\}$, $E_{ij}=\{(u,v)\in E: u\in V_i^+ \text{ and } v\in V_j^-\}$. We consider the d.b-graphs $D_{ij}=(V_i^+,V_j^-,E_{ij})$. Note that each digraph D_{ij} realizes the pair of sequences

$$((a^i_{j1},a^i_{j2},\ldots,a^i_{jn_i}),\ \ (b^j_{i1},b^j_{i2},\ldots,b^j_{ik_j})).$$

Thus, by Proposition 5, we have (iv).

($\Leftarrow$) Let V be an arbitrary n-element set and let $\{X_i\}_{i\in\{0,1,\ldots,m\}}$, $\{Y_j\}_{j\in\{0,1,\ldots,m\}}$ be two partitions of the set V such that $\mid X_i\mid=n_i$ and $\mid Y_j\mid=k_j$ and $X_0\cap Y_0\neq\emptyset$. For $i,j=0,1,\ldots,m$ we put: $X_i=\{x^i_1,x^i_2,\ldots,x^i_{n_i}\}$, $Y_j=\{y^j_1,y^j_2,\ldots,y^j_{k_j}\}$. From Proposition 5 it follows that for every $i,j\in\{0,1,\ldots,m\}$ there exists a d.b-graph $H_{ij}=(X_i,Y_j,F_{ij}\}$ which is a realization of the pair

$$((a^i_{j1},a^i_{j2},\ldots,a^i_{jn_i}),\ \ (b^j_{i1},b^j_{i2},\ldots,b^j_{ik_j})).$$

This means that if $(x^i_q,y^j_s)\in F_{ij}$ then $a^i_{jq}=1$ and $b^j_{is}=1$. Moreover, if $i_1\neq i_2$ ($j_1\neq j_2$) then $F_{i_1j}\cap F_{i_2j}=\emptyset$ ($F_{ij_1}\cap F_{ij_2}=\emptyset$).

We consider the digraph $G=(V,F)$, where $F=\bigcup_{i,j\in\{0,1,\ldots,m\}}F_{ij}$. We will prove that $X_i=\{v\in V:\deg^+(v)=i\}$, $Y_j=\{v\in V:\deg^-(v)=j\}$, $G\in\mathrm{HI}_0$ and G realizes the pair (A,B).

Let $v\in X_i$. Then $v=x^i_q$ for some $q\in\{1,2,\ldots,n_i\}$. If $(x^i_q,y)\in F$, then $(x^i_q,y)\in F_{ij}$ for some $j\in\{0,1,\ldots,m\}$. Thus, $a^i_{jq}=1$. Since the sets F_{ij} are mutually disjoint, the number of arcs from x^i_q in F is equal to the number of elements eqaul to 1 in q-th column of the matrix A_i. By (iii), $\deg^+(v)=i$. In a similar way we obtain: if $v\in Y_j$ then $\deg^-(v)=j$.

Let $v \in V$ and let $u, w \in N^+(v)$, $u \neq w$. Then $(v, u) \in F_{ij_1}$ and $(v, w) \in F_{ij_2}$ for some $i, j_1, j_2 \in \{0, 1, \ldots, m\}$. Since d.b-graphs H_{ij_1} and H_{ij_2} realize the pairs of 0-1-sequences, so $j_1 \neq j_2$. Thus $\deg^-(u) \neq \deg^-(w)$. Obviously, vertices belonging to the set $X_0 \cap Y_0$ are isolated.

It is not difficult to check that $\widetilde{M}^+(G) = A$ and $\widetilde{M}^-(G) = B$. ■

Now we give the converse of Lemma 1.

Lemma 3. *Let (d^+, d^-) be a pair of sequences of the form* (2)*, where $n_0 \neq 0$ and $k_0 \neq 0$, and let*

$$b^+ = (b_0^+, b_1^+, \ldots, b_m^+), \;\; b^- = (b_0^-, b_1^-, \ldots, b_m^-)$$

be sequences such that $b_i^+ = i \cdot n_i$, $b_j^- = j \cdot k_j$, $i, j = 0, 1, \ldots, m$. If there exists a bipartite oriented mutigraph $\mathcal{M} = (X \cup Y, \mu)$, where $\mu : X \times Y \to N \cup \{0\}$, which realizes the pair (b^+, b^-) with bounds $c_{ij} = min\{n_i, k_j\}$, $i, j = 0, 1, \ldots, m$, then the pair (d^+, d^-) is HI_0*-digraphic.*

Proof. Let $\mathcal{M} = (X, Y, \mu)$, where

$$X = \{x_0, x_1, \ldots, x_m\} \;\text{ and }\; Y = \{y_0, y_1, \ldots, y_m\},$$

be a realization of (b^+, b^-) with the bounds c_{ij}. By Lemma 2 it is sufficient to prove that there exists a pair (A, B) of sequences of matrices for which conditions (i) – (iv) hold. We will give a method of construction of the sequences $A = (A_0, A_1, \ldots, A_m)$ and $B = (B_0, B_1, \ldots, B_m)$.

For each $i = 0, 1, \ldots, m$ we construct the matrix $A_i = [a_{pq}^i]_{(m+1) \times n_i}$ in the following way. As A_0 we put 0-matrix of the order $(m+1) \times n_0$. For $i \geq 1$, let $\alpha^{+i} = (\alpha_1^{+i}, \alpha_2^{+i}, \ldots, \alpha_h^{+i})$ be the sequence of the form:

$$\underbrace{1, \ldots, 1}_{\mu_{i1}}, \ldots, \underbrace{j, \ldots, j}_{\mu_{ij}}, \ldots, \underbrace{m, \ldots, m}_{\mu_{im}},$$

where

$$\mu_{ij} = \mu(x_i, y_j), \;\; j = 1, 2, \ldots, m$$

and let $\beta^{+i} = (\beta_1^{+i}, \beta_2^{+i}, \ldots, \beta_k^{+i})$ be the sequence which is formed by repetition, i times, of the sequence $(1, 2, \ldots, n_i)$. Note that the lengths h and k of sequences α^{+i} and β^{+i}, respectively, are equal because $h = \sum_{j=1}^m \mu_{ij} = b_i^+$ and $k = i \cdot n_i = b_i^+$. Let $I^{+i} = \{(\alpha_t^{+i}, \beta_t^{+i});\, t = 1, 2, \ldots, k\}$.

Note that $I^{+i} \subseteq \{(p,q): \ p = 1,2,\ldots,m, q = 1,2,\ldots,n_i\}$. Then we define the 0-1-matrix $A_i = [a^i_{pq}]$ puting: $a^i_{0q} = 0$ for every $q \in \{1,2,\ldots,n_i\}$ and $a^i_{pq} = 1$ if and only if $(p,q) \in I^{+i}$ for $p \geq 1$.

Similarly, for each $j = 0,1,\ldots,m$ we construct the matrix $B_j = [b^j_{rs}]_{(m+1)\times k_j}$. We define B_0 as the 0-matrix of the order $(m+1)\times k_0$. For $j \geq 1$ we consider the sequence $\alpha^{-j} = (\alpha_1^{-j}, \alpha_2^{-j}, \ldots, \alpha_h^{-j})$ of the form:

$$\underbrace{1,\ldots,1}_{\mu_{1j}},\ldots,\underbrace{i,\ldots,i}_{\mu_{ij}},\ldots,\underbrace{m,\ldots,m}_{\mu_{mj}},$$

where

$$\mu_{ij} = \mu(x_i, y_j), \ i = 1,2,\ldots,m$$

and the sequence $\beta^{-j} = (\beta_1^{-j}, \beta_2^{-j}, \ldots, \beta_k^{-j})$ formed by repetition, j times, of the sequence $(1,2,\ldots,k_j)$.

Obviously, the sequences α^{-j} and β^{-j} have the same number of elements. Let $I^{-j} = \{(\alpha_t^{-j}, \beta_t^{-j}) : t = 1,2,\ldots,k\}$. Then $B_j = [b^j_{rs}]$, where $b^j_{0s} = 0$ for every $s \in \{1,2,\ldots,k_j\}$ and $b^j_{rs} = 1$ if and only if $(r,s) \in I^{-j}$ for $r \geq 1$.

Obviously, for the matrices A_i and B_j conditions (i) and (ii) hold.

Let $q \in \{1,2,\ldots,n_i\}$ and $p \in \{1,2,\ldots,m\}$. Note that the number q appears exactly i times in the sequences $\beta^{+i}(q = \beta_t^{+i}$ for $t \in \{q, q+n_i, \ldots, q+(i-1)n_i\})$ however, the number p is repeated μ_{ip} times in the sequence α^{+i}, where $\mu_{ip} \leq c_{ip} = min\{n_i, k_p\} \leq n_i$. So, the set I^{+i} has exactly i pairs with second element equal to q. Then $\sum_{p=1}^{m+1} a^i_{pq} = i$. By similar argumentation we have $\sum_{r=1}^{m+1} b^j_{rs} = j$ for $s = 1,2,\ldots,k_j$. Thus the condition (iii) holds. To prove condition (iv) it is sufficient to note that:

- the number of elements equal to 1 in i-th row of the matrix B_j is equal to the number of elements equal to i in the sequence α^{-j} and
- the number of elements equal to 1 in j-th row of the matrix A_i is equal to the number of elements equal to j in the sequence α^{+i}.

By definition of sequences α^{-j} and α^{+i} these numbers are equal to μ_{ij}. ■

Example. An example which demostrates the method of constructing the HI_0-digraph with the help of results presented in this paper will be presented.

Let $d^+ = (3,3,2,2,2,2,1,1,1,0,0)$ and $d^- = (3,3,2,2,2,1,1,1,1,1,0)$. Accordingly to Lemma 3 we form the sequences $b^+ = (0,3,8,6)$ and

$b^- = (0, 5, 6, 6)$. The multigraph $\mathcal{M}$ in Figure 1 (the numbers about arcs denote the multiplicity of these arcs) realizes the pair (b^+, b^-) with bounds c_{ij}, $i, j = 0, 1, 2, 3$, given by the matrix

$$[c_{ij}] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 1 & 4 & 3 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix}.$$

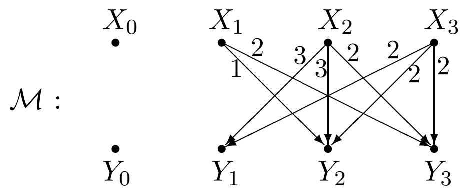

Figure 1

According to the procedure given in the proof of Lemma 3 the following auxiliary sequences α^{+i}, β^{+i} and α^{-i}, β^{-i} are constructed:

$$\begin{array}{llllll} \alpha^{+1} = & (2,3,3) & \alpha^{+2} = & (1,1,1,2,2,2,3,3) & \alpha^{+3} = & (1,1,2,2,3,3) \\ \beta^{+1} = & (1,2,3) & \beta^{+2} = & (1,2,3,4,1,2,3,4) & \beta^{+3} = & (1,2,1,2,1,2) \end{array}$$

$$\begin{array}{llllll} \alpha^{-1} = & (2,2,2,3,3) & \alpha^{-2} = & (1,2,2,2,3,3) & \alpha^{-3} = & (1,1,2,2,3,3) \\ \beta^{-1} = & (1,2,3,4,5) & \beta^{-2} = & (1,2,3,1,2,3) & \beta^{-3} = & (1,2,1,2,1,2) \end{array}.$$

Then we obtain the following pair (A, B) of sequences of matrices:

$$A = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

$$B = \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \right).$$

These matrices satisfy conditions (i) – (iv). To construct an HI-digraph which realizes the pair (A, B) we apply the method given in the proof of Theorem 3.

Let $V = \{v_1, v_2, \ldots, v_{11}\}$ and let

$X_0 = \{v_{11}, v_{10}\}$, $X_1 = \{v_9, v_8, v_7\}$, $X_2 = \{v_6, v_5, v_4, v_3\}$, $X_3 = \{v_2, v_1\}$, $Y_0 = \{v_{11}\}$, $Y_1 = \{v_{10}, v_9, v_8, v_7, v_6\}$, $Y_2 = \{v_5, v_4, v_3\}$, $Y_3 = \{v_1, v_2\}$.

All d.b-graphs H_{ij} for $i = 0$ or $j = 0$ and H_{11} have no arcs. The remaining d.b-graphs are presented in Figure 2.

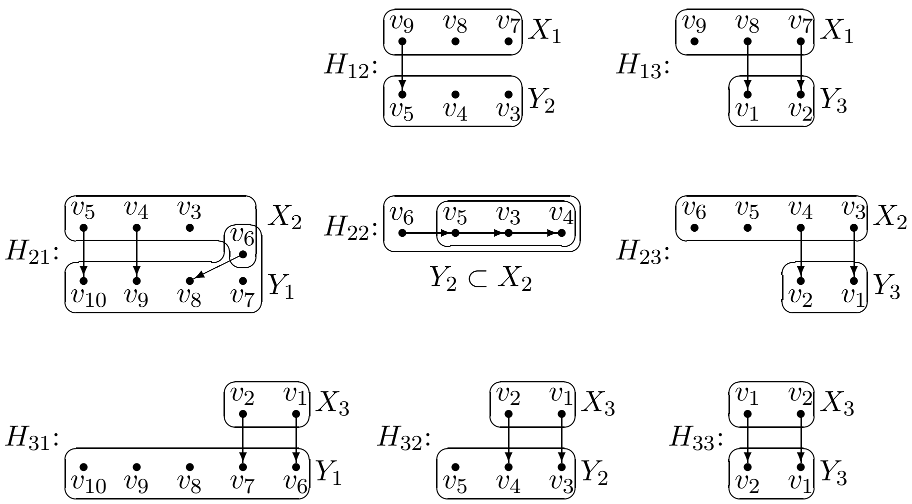

Figure 2

Then the digraph G (see Figure 3) which realizes the pair (d^+, d^-) is the union of the above d.b.-graphs.

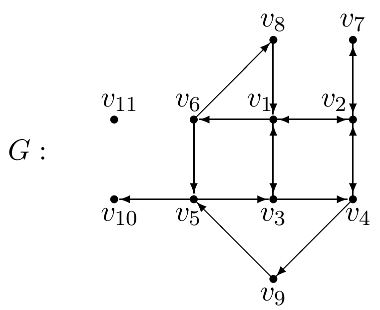

Figure 3

Finally, we give the main theorem of this paper which is a combinatorial characterization of the degree sequences of HI-digraphs. We use Hoffman's criterion ([2]) which is a characterization of the degree sequences of bipartite multigraphs with bounds for the multiplicity of edges. This criterion can be formulated as follows:

Let $\mathbf{d} = (d_1, d_2, \ldots, d_m)$ and $\tilde{\mathbf{d}} = (d_{m+1}, d_{m+2}, \ldots, d_{m+n})$ be sequences of non-negative integers such that $\sum_{i=1}^{m} d_i = \sum_{j=m+1}^{m+n} d_j = s$ and let $c_{ij} \geq 0$ for $1 \leq i \leq m$, $m+1 \leq j \leq m+n$ be given non-negative integers. There exists a bipartite multigraph which is a realization of the pair $(\mathbf{d}, \tilde{\mathbf{d}})$ with the bounds c_{ij} if and only if, for every $I \subseteq \{1, 2, \ldots, m\}$, $J \subseteq \{m+1, m+2, \ldots, m+n\}$ we have

$$\sum_{i \in I, j \in J} c_{ij} \geq \sum_{i \in I} d_i + \sum_{j \in J} d_j - s$$

From Lemma 1, Lemma 3 and above Hoffman's criterion, in the face of Remark 2, it immediately follows:

Theorem 2. *A pair (d^+, d^-) of sequences of the form* (2) *is* HI-*digraphic if and only if for every $I, J \subseteq \{0, 1, \ldots, m\}$ the following inequality holds:*

$$\sum_{i \in I, j \in J} min\{n_i, k_j\} \geq \sum_{i \in I} i \cdot n_i + \sum_{j \in J} j \cdot k_j - \sum_{i=0}^{m} i \cdot n_i. \tag{3}$$

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