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LONG CYCLES AND NEIGHBORHOOD UNION IN 1-TOUGH GRAPHS WITH LARGE DEGREE SUMS

Vu Dinh Hoa

Wundtstr. 7/4L1
01217 Dresden, Germany

Abstract

For a 1-tough graph G we define $\sigma_3(G) = \min\{d(u) + d(v) + d(w) : \{u, v, w\}$ is an independent set of vertices$\}$ and $NC_{\sigma_3-n+5}(G) = \max\{\bigcup_{i=1}^{\sigma_3-n+5} N(v_i) : \{v_1, ..., v_{\sigma_3-n+5}\}$ is an independent set of vertices$\}$. We show that every 1-tough graph with $\sigma_3(G) \geq n$ contains a cycle of length at least $\min\{n, 2NC_{\sigma_3-n+5}(G) + 2\}$. This result implies some well-known results of Faßbender [2] and of Flandrin, Jung & Li [6]. The main result of this paper also implies that $c(G) \geq \min\{n, 2NC_2(G) + 2\}$ where $NC_2(G) = \min\{|N(u) \cup N(v)| : d(u, v) = 2\}$. This strengthens a result that $c(G) \geq \min\{n, 2NC_2(G)\}$ of Bauer, Fan and Veldman [3].

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Introduction

We consider only a finite undirected graph without loops and multiple edges. For undefined terms we refer to [3]. Let $\omega(G)$ denote the number of components of a graph G. A graph G is 1-*tough* if for every nonempty proper subset S of the vertex set V(G) of G we have $\omega(G - S) \leq |S|$. We use α to denote the cardinality of a maximum independent set of vertices of G. A cycle C in G is called a *dominating cycle* if the vertices of the graph $G - C$ are independent. The length ℓ(C) of a longest cycle C of G is denoted by $c(G)$. For $k \leq \alpha$ we denote by σ_k the minimum value of the degree sum of any k pairwise nonadjacent vertices and by $NC_k(G)$ the minimum cardinality of the neighborhood union of any k such vertices. For $k > \alpha$ we set $\sigma_k = k(n - \alpha(G))$ and $NC_k = n - \alpha(G)$. Instead of σ_1 and NC_1 we use the more common notation $\delta(G)$. If no ambiguity can arise, we sometimes write α instead of $\alpha(G)$, etc.

A number of results have been established concerning long cycles in graphs with large degree sums. For details we refer to a survey [4] and [7]. Since, clearly, $NC_t(G)$ is a non decreasing function of t and $NC_t(G) \geq \frac{1}{t}\sigma_t(G)$, analogous results in terms of NC_t would extend well-known previous results [5].

Let $d(u,v)$ denote the distance between u and v. Our main result in the present paper is Lemma 9 and its consequence.

Theorem 1. *If G is a 1-tough nonhamiltonian graph of order $n \geq 3$ with $\sigma_3 \geq n$, then there exists in G an independent set of $\sigma_3 - n + 5$ vertices $\{v_0, .., v_{\sigma_3 - n+4}\}$ such that $d(v_0, v_i) = 2$ $(i \geq 1)$ and $c(G) \geq 2|\bigcup_{i=0}^{\sigma_3-n+4} N(v_i)| + 2$.*

Clearly, Theorem 1 strengthens the result of Bauer et al. (Theorem 26 in [5]) that under the same hypothesis $c(G) \geq 2NC_2(G)$. Theorem 1 also implies the next result.

Theorem 2. *If G is a 1-tough graph of order $n \geq 3$ with $\sigma_3 \geq n$, then $c(G) \geq \min\{n, 2NC_{\sigma_3-n+5} + 2\}$.*

Theorem 1 and Theorem 2 are strongly related to other results of Broersma, Van den Heuvel & Veldman [7] and in Van den Heuvel [8].

Theorem 3 (Corollary 6 in [7]). *If G is a 1-tough graph of order $n \geq 3$ with $\sigma_3 \geq n$, then $c(G) \geq \min\{n, 2NC_{3\bar{\delta}-n+5}\}$, where $\bar{\delta} = \lceil \frac{\sigma_3}{3} \rceil$.*

Theorem 4 (Theorem 11 in [7]). *If G is a 1-tough graph of order $n \geq 3$ with $\sigma_3 \geq n + r \geq n$ and $n \geq 8t - 6r - 17$, then $c(G) \geq \min\{n, 2NC_t\}$.*

Theorem 5 (Corollary 7.12 in [8]). *If G is a 1-tough graph on $n \geq 3$ vertices, then $c(G) \geq \min\{n,\ 2NC_{\lfloor \frac{1}{2}(4\bar{\delta}-n+3)\rfloor}\}$.*

Theorem 2 is in a sense best possible. This can be seen from the construction by Bauer et al. [3] of a 1-tough graph G_n for odd $n \geq 15$. The graph G_n is obtained from $\bar{K}_{(n-1)/2} \cup K_3 \cup K_{(n-5)/2}$ by joining every vertex of $K_{(n-5)/2}$ to all vertices in $\bar{K}_{(n-1)/2} \cup K_3$ and by adding a matching between the vertices of K_3 and three vertices in $\bar{K}_{(n-1)/2}$. A variation of the graph G_n, with $K_{(n-5)/2}$ replaced by $\bar{K}_{(n-5)/2}$, has already appeared in [1].

But we do not know of the existence of 1-tough graphs G on $n \geq 3$ vertices with $\sigma_3 \geq n$ and $c(G) < n - 1$ for which Theorem 2 is best possible. Moreover, we cannot conclude Theorem 2 from Theorem 3, Theorem 4 and Theorem 5. Let $G_{(n,p)}$ denote the graph $(F_p \cup \bar{K}_{(n-1)/2-(2p+1)}) + K_{(n+1)/2-(2p+1)}$ for odd $n \geq 12p+3 \geq 27$, where F_p denotes the unique graph with a degree sequence ($d_1 = 1$, $d_2 = 1$, ..., $d_{2p+1} = 1$, $d_{2p+2} = 2p+1$, ..., $d_{4p+2} = 2p+1$). Then $G_{(n,p)}$ is a 1-tough graph on $n \geq 27$ vertices with $\sigma_3 \geq n$. By Theorem 2, $c(G_{(n,p)}) \geq n+1-4p$ which cannot be deduced from Theorem 3, Theorem 4 and Theorem 5.

Theorem 2 implies a recent result of Faßbender [2], conjectured in [3].

Corollary 6. *If G is a 1-tough graph of order $n \geq 13$ with $\sigma_3 \geq \frac{3n-14}{2}$, then G is hamiltonian.*

Proof. Clearly, $\sigma_3 \geq n$ for $n \geq 13$ and $\sigma_3 - n + 5 \geq \frac{n-4}{2}$ if $\sigma_3 \geq \frac{3n-14}{2}$. Since G is a 1-tough graph, $NC_{\lceil \frac{n-4}{2} \rceil} \geq \frac{n-2}{2}$. Hence, $2NC_{\sigma_3-n+5} + 2 \geq 2\frac{n-2}{2} + 2 = n$. By Theorem 2, $c(G) \geq \min\{n, 2NC_{\sigma_3-n+5}+2\} = n$. Thus, G is hamiltonian. ∎

Theorem 2 immediately implies a result of Flandrin, Jung & Li [6].

Corollary 7. *If G is a 2-connected graph of order n such that $d(u)+d(v)+d(w) \geq n + |N(u) \cap N(v) \cap N(w)|$ for every independent set $\{u, v, w\}$, then G is hamiltonian.*

Proof. Let G satisfy the stated conditions. Then G is 1-tough [4] and $n \leq 2NC_3$ [7]. The proof is completed by applying Theorem 2 (note that $NC_{\sigma_3-n+5} \geq NC_3$). ∎

PROOFS

Let C be a cycle in G with an assigned orientation. If x and y are two vertices of C then $x\overrightarrow{C}y$ denotes the path on C from x to y, inclusively x and y, following the assigned orientation. The same vertices in a reverse order are given by $y\overleftarrow{C}x$. We will consider $x\overrightarrow{C}y$ and $y\overleftarrow{C}x$ both as a path and as a vertex set. If c is a vertex on C, then c^+ and c^- are its successor and predecessor on C, respectively, according to the assigned orientation. If X is a set of vertices on C let $X^+ := \{x^+ : x \in X\}$ and $X^- := \{x^- : x \in X\}$. If $v \in V(G)$ and $H \subset V(G)$ then $N_H(v)$ is the set of all vertices in H adjacent to v. We denote $|N_H(v)|$ by $d_H(v)$. If G is a nonhamiltonian graph, we set $\mu(C) = \max\{d(v) : v \in V(G) - V(C)\}$ and $\mu(G) = \max\{\mu(C) : C$ is a longest cycle in $G\}$.

The following lemmas are already proved in [3].

Lemma 1 (Theorem 5 [3]). *Let G be a 1-tough graph with $\sigma_3 \geq n$. Then every longest cycle in G is a dominating cycle.*

Lemma 2 (see proof of Theorem 9 [3]). *Let G be a 1-tough graph with $\sigma_3 \geq n$. If G is nonhamiltonian, then G has a longest cycle C such that C avoids a vertex v_0 with $d(v_0) \geq \frac{\sigma_3}{3}$ in G.*

Lemma 3 (Lemma 8 [3]). *Let G be a 1-tough graph with $\sigma_3 \geq n$. Suppose C is a longest cycle in G. If $v_0 \in V(G) - V(C)$ and $A = N(v_0)$, then $(V(G) - V(C)) \cup A^+$ is an independent set of vertices.*

Assume G is nonhamiltonian. Let C be a cycle in G with an assigned orientation, $v \in V(G) - V(C)$ and v_1 ,..., v_k be the elements of $N(v)$, occurring on C in a consecutive order. For $i = 1$, 2, ..., k set $u_i = v_i^+$ and $w_i = v_{i+1}^-$ (indices modulo k). We set, for convenience, $\Im = \{$ i : there exists some $j \neq i$ such that $u_i w_j \in E(G)\}$.

The set $u_i \overrightarrow{\mathbf{C}} w_i$ will be called *a segment*; $u_i \overrightarrow{\mathbf{C}} w_i$ is a *p-segment* if $|u_i \overrightarrow{\mathbf{C}} w_i| = p$. Let S denote the set of 1-segments. The following lemma is observation (1) in the proof of Theorem 4 in Broersma et al. [7].

Lemma 4. $(V(G) - V(C)) \cup N(v)^+ \cup N(S)^+$ *is an independent set of vertices.*

If $d(v) = \mu(G)$ then $d(v) \geq n/3$ because of Lemma 2 and therefore $S \neq \emptyset$. Let u_{i_1}, u_{i_2}, ..., u_{i_s} be the vertices of the 1-segments and assume, without loss of generality, that $i_1 = 1$ and $d(u_1) \geq d(u_{i_2}) \geq ... \geq d(u_{i_s})$. Since $C' : vv_2 \overrightarrow{\mathbf{C}} v_1 v$ is a longest cycle, $\mu(G) \geq d(u_1)$.

Lemma 5. *If $\mu(G) = d(v) \leq \frac{\sigma_3+2}{3}$, then $d(v) = d(u_1)$.*

Proof. Suppose to the contrary that $d(u_1) \leq d(v)-1$. Let $t_C(v) = |V(C) - (N(v) \cup N(v)^+ \cup N(v)^-)|$. By $n-1 \geq \ell(C) = 3d(v) - s + t_C(v)$, $n - 1 + s - 3d(v) \geq t_C(v)$ (*). We distinguish 3 cases:

Case 1. $s = 1$.

By (*) and by Lemma 2, in fact, $\ell(C) = n - 1$, $d(v) = \frac{n}{3}$ and $t_C(v) = 0$. Since G is a 1-tough graph, $G - N(v)$ contains at most $d(v)$ components. Hence, there is $i_0 \neq j_0$ and some edge joining u_{i_0} with w_{j_0}. Now, $C' : vv_{j_0+1} \overrightarrow{\mathbf{C}} u_{i_0} w_{j_0} \overleftarrow{\mathbf{C}} v_{i_0+1} v$ is also a longest cycle which avoids w_{i_0}. Thus, $d(w_{i_0}) \leq d(v)$ by the maximality of $d(v)$, and therefore $d(u_1)+d(w_{i_0})+d(v) \leq 3d(v) - 1 = n - 1$, a contradiction.

Case 2. $s = 2$.
By (*), $\frac{(n+1)}{3} \geq d(v)$ and therefore $d(u_1) + d(u_{i_2}) + d(v) \leq 3d(v) - 2 \leq n - 1$, a contradiction.

Case 3. $s \geq 3$.
In this case we have $d(u_1) + d(u_{i_2}) + d(u_{i_3}) \leq 3d(v) - 3 \leq \sigma_3 - 1$, a contradiction. Thus, Lemma 5 is true. ■

Lemma 6. *If C contains only p-segments with $p \leq 3$, then $\Im \neq \emptyset$.*

Proof. Suppose to the contrary that $\Im = \emptyset$. We consider $G - (N(v) \cup \{u_i^+ : u_i \overrightarrow{\mathbf{C}} w_i$ is a 3-segment and $u_i w_i \notin E(G)\})$. Since G is a 1-tough graph there exists $i \neq j$ and some arc $\mathbf{B}$ joining a vertex p in $u_i \overrightarrow{\mathbf{C}} w_i$ with a vertex q in $u_j \overrightarrow{\mathbf{C}} w_j$. By Lemma 3 and since $\Im = \emptyset$, $p = u_i^+ = w_i^-$ or $q = u_j^+ = w_j^-$, say $p = u_i^+ = w_i^-$ and therefore $u_i w_i \in E(G)$. We distinguish two cases:

Case 1. $q = u_j$ (similar for the case $q = w_j$).
In this case $C' : vv_j \overleftarrow{\mathbf{C}} w_i u_i p \mathbf{B} u_j \overrightarrow{\mathbf{C}} v_i v$ would be a cycle longer than C, a contradiction.

Case 2. $q = u_j^+ = w_j^-$.
In this case $C' : vv_j \overleftarrow{\mathbf{C}} w_i u_i p \mathbf{B} q u_j w_j \overrightarrow{\mathbf{C}} v_i v$ would be a cycle longer than C, a contradiction. Thus Lemma 6 is true. ■

Lemma 7. *Suppose that $\Im \neq \emptyset$. Let $i_0 = \max \Im$ and $j_0 \neq i_0$ such that $u_{i_0} w_{j_0} \in E(G)$. Suppose that $v_{i_0} u_1 \in E(G)$ or $\{u_1 v_{j_0+1}, u_1 v_{j_0}\} \subset E(G)$. Then $d(u_{j_0}) + 2d(v) \leq \ell(C) + x$, where x is the number of vertices $u_i = w_i$ such that $v_{i_0} u_i \notin E(G)$ and $\{u_i v_{j_0+1}, u_i v_{j_0}\} \not\subseteq E(G)$.*

Proof. To prove this lemma we start with a trivial observation.

(*) If $u_i v_{i_0} \in E(G)$ or $u_i v_{j_0+1} \in E(G)$ then $u_i \in w_{j_0} \overrightarrow{\mathbf{C}} u_{i_0}$.

For $i = 1, 2, ..., $ k we set $L_i := u_i \overrightarrow{\mathbf{C}} v_{i+1}$. Then $d_{L_i}(u_{j_0}) \leq |L_i| - 1$ because of $u_i u_{j_0} \notin E(G)$ by Lemma 3. Since $d(u_{j_0}) = \sum_{i=1}^{k} d_{L_i}(u_{j_0})$ it suffices to show that $d_{L_i}(u_{j_0}) \leq |L_i| - 2$ (i.e. there exists on L_i some $z \neq u_i$ such that $z u_{j_0} \notin E(G)$) for $u_i \neq w_i$ and for $u_i = w_i$ with $v_{i_0} u_i \in E(G)$ or $\{u_i v_{j_0+1}, u_i v_{j_0}\} \subseteq E(G)$.

Note that $j_0 > i_0$ and $v_{j_0+1} \neq v_{i_0}$ by (*) (for $i = 1$). Thus $w_i u_{j_0} \notin E(G)$ if $w_i \neq u_i$ and $i \neq j_0$ because of the maximality of i_0. If $i = j_0$, then $v_{j_0+1} u_{j_0} \notin E(G)$ by (*). If $u_i = w_i$ with $v_{i_0} u_i \in E(G)$ or $\{u_i v_{j_0+1}, u_i v_{j_0}\} \subseteq E(G)$ then $u_i \in w_{j_0} \overrightarrow{\mathbf{C}} u_{i_0}$ by (*) and therefore $v_{i+1} u_{j_0} \notin E(G)$. Otherwise, $C' : v\ v_{j_0+1} \overrightarrow{\mathbf{C}} u_i\ v_{i_0} \overleftarrow{\mathbf{C}} v_{i+1}\ u_{j_0} \overrightarrow{\mathbf{C}} w_{j_0}\ u_{i_0} \overrightarrow{\mathbf{C}} v_{j_0} v$, when $u_i v_{i_0} \in E(G)$, and C' : $vv_i \overleftarrow{C} v_{j_0+1} u_i \; v_{j_0} \overleftarrow{C} u_{i_0} w_{j_0} \overleftarrow{C} u_{j_0} v_{i+1} \overrightarrow{C} v_{i_0} v$ when $\{u_i v_{j_0+1}, u_i v_{j_0}\} \subseteq E(G)$ would be a cycle longer than C, a contradiction. Thus Lemma 7 is true. ■

Theorem 1 is obviously established by the next two lemmas.

Lemma 8. *Let* $X = N(v) \bigcup \{N(u_i) : u_i \in S\}$. *Then* $\ell(C) \geq 2|X| + 2$.

Proof. Let x_1, ..., x_y be the vertices of X, occurring on C in a consecutive order. By Lemma 4, $X \cap X^+ = \emptyset$. Since G is a 1-tough graph, there exist some $i \neq j$ and some arc joining a vertex y on $x_i^+ \overrightarrow{C} x_{i+1}^-$ with a vertex z on $x_j^+ \overrightarrow{C} x_{j+1}^-$. Without loss of generality, assume that $|x_i^+ \overrightarrow{C} x_{j+1}^-| \leq |x_j^+ \overrightarrow{C} x_{j+1}^-|$. Then by Lemma 4, $z \notin \{x_j^+, x_{j+1}^-\}$ if $x_i^+ = x_{i+1}^-$. Thus, $\ell(C) \geq 2|X| + 2$. ■

Following Broersma et al. [7], we say that a property **P** holds by the longest cycle argument, denoted by **P**(C'), if the contrary implies the existence of a cycle C' longer than C.

Now, we give and prove a lower bound of so called 1-segments. Theorem 1 is established by the last lemma.

Lemma 9. *Let* G *be a* 1*-tough nonhamiltonian graph on* $n \geq 3$ *vertices with* $\sigma_3 \geq n$. *Then* G *contains a longest cycle* C *avoiding a vertex* v *with* $d(v) = \mu(G)$ *and* $s \geq \sigma_3 - n + 4$.

Proof. Assume to the contrary that $s \leq \sigma_3 - n + 3$ for any longest cycle C avoiding a vertex v with $d(v) = \mu(G)$. Let $t_C(v) = |V(C) - (N(v) \cup N(v)^+ \cup N(v)^-)|$.

Claim 1. *If* C *is a longest cycle in* G *avoiding a vertex* v *with* $d(v) = \mu(G)$, *then* $d(v) \leq \frac{\sigma_3+2}{3}$ *and* $t_C(v) \leq 2$ *with strict inequality if* $\mu(G) \neq \frac{\sigma_3}{3}$ *or* $\ell(C) \neq n - 1$.

Proof. Counting the vertices on C we get $n - 1 \geq \ell(C) = 3d(v) - s + t_C(v)$. Thus, $\sigma_3 + 2 - t_C(v) \geq 3d(v)$ and $\sigma_3 - 3d(v) + 2 \geq t_C(v)$, establishing Claim 1. ■

Claim 2. *If* C *is a longest cycle avoiding a vertex* v *with* $d(v) = \mu(G)$, *then* $\Im = \emptyset$.

Proof. Supposing that $\Im \neq \emptyset$, we determine $i_0 = \max \Im$ and $j_0 \neq i_0$ such that $u_{i_0} w_{j_0} \in E(G)$. First note that if $u_i = w_i$ and $d(u_i) = d(v)$, then by $P(C')$ $u_i u_{i_0}^+ \notin E(G)$ (C' : $vv_i \overleftarrow{C} u_{i_0}^+ \; u_i \overrightarrow{C} w_{j_0} u_{i_0} \overleftarrow{C} v_{j_0+1} v$ when $u_i \in u_{i_0} \overrightarrow{C} w_{j_0}$ and C' : $vv_{i+1} \overrightarrow{C} u_{i_0} w_{j_0} \overleftarrow{C} u_{i_0}^+ u_i \overleftarrow{C} \; v_{j_0+1} v$ when $u_i \in w_{j_0} \overleftarrow{C} u_{i_0}$). Similarly,

$u_i w_{j_0}^- \notin E(G)$. Consequently, $u_i v_{i_0} \in E(G)$ or $\{u_i v_{j_0+1}, u_i v_{j_0}\} \subset E(G)$ since, otherwise, $t_{C'}(u_i) \geq 3$ where $C' : v v_i \overleftarrow{C} v_{i+1} v$, which contradicts Claim 1. By Lemma 5 and by Claim 1, $d(u_1) = d(v)$. Now using Lemma 7 we have $d(u_1) + d(v) + d(u_{j_0}) = 2d(v) + d(u_{j_0}) \leq \ell(C) + x$, where x is the number of vertices $u_i = w_i$ such that $d(u_i) \leq d(v) - 1$. By $\sigma_3 \geq n$, $x \geq 1$. Hence, $d(v) + d(u_{i_s}) + d(u_{j_0}) \leq \ell(C) + x - 1$ and, by similar argument, $x \geq 2$. Note that by $\frac{\sigma_3+2}{3} \geq d(v)$, $x \leq 2$ by $d(u_{i_s}) + d(u_{i_{s-1}}) + d(u_{i_{s-2}}) \geq \sigma_3$ and, by $x \geq 2$, in fact, $x = 2$. Now we get $d(u_{i_{s-1}}) + d(u_{i_s}) + d(u_{j_0}) \leq \ell(C) < n$, a contradiction. ■

The next claim is obviously established by Lemma 6, Claim 2 and Claim 1.

Claim 3. *If C is a longest cycle and $v \in V(G) - V(C)$ such that $d(v) = \mu(G)$, then $t_C(v) = 2$ and C contains a 4-segment.*

By Claim 1, we get $\ell(C) = n - 1$ and $d(v) = \sigma_3/3$. Using the inequality $n - 1 \geq 3d(v) - s + t_C(v)$ and $t_C(v) = 2$ by Claim 3, we get $s \geq \sigma_3 - n + 3 \geq 3$. By $d(u_i) + d(u_1) + d(v) \geq \sigma_3$, we easily get:

Claim 4. *If C is a longest cycle avoiding a vertex v with $d(v) = \mu(G)$, then $d(u_i) \geq d(v) = \frac{\sigma_3}{3}$ and $d(w_i) \geq d(v)$ with equality if $u_i = w_i$.*

Claim 5. *If C is a longest cycle avoiding a vertex v with $d(v) = \mu(G)$, then $N(u_i) = N(v)$ for any $u_i = w_i$.*

Proof. Suppose that there exists some $u_i = w_i$ such that $N(u_i) \neq N(v)$. By Claim 4, either $u_t^+ u_i \in E(G)$ or $w_t^- u_i \in E(G)$, say $u_t^+ u_i \in E(G)$. Note that $u_t w_t \notin E(G)$ ($C' : v v_{i+1} \overrightarrow{C} u_t w_t \overleftarrow{C} u_t^+ u_i \overleftarrow{C} v_{t+1} v$) and $u_t w_t^- \notin E(G)$ ($C' : v v_i \overleftarrow{C} w_t^- u_t u_t^+ u_i \overrightarrow{C} v_t v$). Therefore there exits some j such that either $w_t^- w_j \in E(G)$ or $w_t^- u_j \in E(G)$ since $\omega(G - N(v) - \{u_t^+\}) \leq d(v) + 1$ by the toughness of G and by Claim 2. But $w_t^- u_j \notin E(G)$ ($C' : v v_{i+1} \overrightarrow{C} u_t^+ u_i \overleftarrow{C} u_j w_t^- \overrightarrow{C} v_j v$ when $u_j \in u_t^+ \overrightarrow{C} u_i$ and $C' : v v_j \overleftarrow{C} u_i u_t^+ \overleftarrow{C} u_j w_t^- \overrightarrow{C} v_i v$ when $u_j \notin u_t^+ \overrightarrow{C} u_i$) and therefore $w_t^- w_j \in E(G)$. Moreover, $w_j \in u_t \overrightarrow{C} u_i^-$ ($C' : v v_{j+1} \overrightarrow{C} u_t^+ u_i \overrightarrow{C} w_j w_t^- \overrightarrow{C} v_i v$). By Claim 2, $d(w_t) \leq d(v) - 1$ since $w_t v_{i+1} \notin E(G)$ ($C' : v v_{j+1} \overrightarrow{C} u_i u_t^+ \overleftarrow{C} v_{i+1} w_t w_t^- w_j \overleftarrow{C} v_{t+1} v$), $w_t v_{j+1} \notin E(G)$ ($C' : v v_{i+1} \overrightarrow{C} u_t^+ u_i \overleftarrow{C} v_{j+1} w_t w_t^- w_j \overleftarrow{C} v_{t+1} v$) and $w_t u_t^+ \notin E(G)$ ($C' : v v_{j+1} \overrightarrow{C} u_t^+ w_t w_t^- w_j \overleftarrow{C} v_{t+1} v$) (note that $u_t w_t \notin E(G)$), which contradicts Claim 4. Thus Claim 5 is true. ■

Now, a longest cycle C and a vertex $v_0 \in V(G) - V(C)$ with $d(v_0) = \mu(G)$ are fixed. Then there exists one t such that $|u_t \overrightarrow{C} w_t| = 4$ and $|u_i \overrightarrow{C} w_i| \leq 2$ for any $i \neq t$.

Since G is a 1-tough graph, $\omega(G - N(v_0)) \leq d(v_0)$ and therefore there exist $i \neq j$ and some $y \in u_i \overrightarrow{C} w_i$, $z \in u_j \overrightarrow{C} w_j$ such that $yz \in E(G)$. Since $\Im = \emptyset$ by Claim 2, either $i = t$ or $j = t$, say $j = t$ and assume, without loss of generality, that $y = u_i$. We distinguish two cases.

Case 1. $u_i u_t^+ \in E(G)$.
We consider the pair u_t and C':$v_0 v_i \overleftarrow{C} u_t^+ u_i \overrightarrow{C} v_t v_0$. By the maximality of $d(v_0)$, Claim 4 for v_0 and C yields $\mu(C') = d(u_t) = \mu(G)$. Now, Claim 3, 4 and 5 can be applied to u_t and C'. If $v_i u_t \notin E(G)$, then $v_{i-1}^+ \overrightarrow{C} v_i v_0$ is the 4-segment of u_t and C', consequently $u_{i-1} \neq w_{i-1}$. If $v_j u_t \notin E(G)$ for some $v_j \neq v_i$, v_t then $v_{j-1}^+ \overrightarrow{C} v_{j+1}^-$ is the 4-segment of u_t and C', therefore either $u_i \overrightarrow{C} w_i$ or $u_{i-1} \overrightarrow{C} w_{i-1}$ is a 2-segment of v_0 and C. It follows by $s \geq 3$ that some 1-segment of v_0 and C is also a 1-segment of u_t and C'. But this contradicts $N(u_t) \neq N(v)$ and Claim 5 (applied to both pairs v_0, C and u_t, C'). This rejects Case 1.

Case 2. $u_i w_t^- \in E(G)$.
In this Case $N(u_t^+) \cap N(v_0)^+ = \{u_t\}$ by Case 1. Since G is 1-tough and $u_t w_t \notin E(G)$ ($C' : v_0 v_t \overleftarrow{C} u_i w_t^- \overleftarrow{C} u_t w_t \overrightarrow{C} v_i v_0$) it follows that u_t^+ has a neighbor w_j. Clearly w_j is on $w_t \overrightarrow{C} v_i$ ($C' : v_0 v_{j+1} \overrightarrow{C} u_t^+ \ w_j \overleftarrow{C} u_i w_t^- \overrightarrow{C} v_i v_0$). Now consider the pair w_i and $C' : v_0 \ v_{i+1} \overrightarrow{C} u_t^+ \ w_j \overleftarrow{C} w_t^- \ u_i \overleftarrow{C} v_{j+1} \ v_0$ to obtain a contradiction as in Case 1. ■

Conjecture

The lower bound on the number of so called 1-segments on a longest cycle in Lemma 9 is best possible only for $c(G) = n - 1$.

Conjecture. *Let G be a 1-tough nonhamiltonian graph on $n \geq 3$ vertices with $\sigma_3 \geq n$. Then G contains a longest cycle C (with an assigned orientation) avoiding a vertex v with $d(v) = \mu(G)$ and $|N_C(v)^+ \cap N_C(v)^-| \geq \sigma_3 - n + 3\omega(G - C) + 1$.*

The graphs $G_{(n,p)}$ show that our Conjecture, if true, is best possible, also in case $c(G) < n - 1$.

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