Problems Column

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PARTITION PROBLEMS AND KERNELS OF GRAPHS

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1. Introduction

The graphs we consider are finite, simple and undirected. The number of vertices in a longest path in a graph G is denoted by $\tau(G)$. For positive integers k_1 and k_2 a graph G is (τ, k_1, k_2)-*partitionable* if there exists a partition $\{V_1, V_2\}$ of $V(G)$ such that $\tau(G[V_1]) \leq k_1$ and $\tau(G[V_2]) \leq k_2$. If this can be done for every pair of positive integers (k_1, k_2) satisfying $k_1 + k_2 = \tau(G)$, we say that G is τ-*partitionable.*

Let H_v denote the fact that the graph H is rooted at v. The set $S \subseteq V(G)$ is an H_v-*kernel* if

(i) there is no subgraph of $G[S]$ isomorphic to H and
(ii) for every $x \in V(G) - S$ there is a subgraph of $G[S \cup \{x\}]$ isomorphic to H_v with its root v at x.

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Similarly, a graph is H_v-*saturated* if it has a subset $S \subseteq V(G)$ such that

(i) H is not a subgraph of $G[S]$ and

(ii) for every $x \in V(G) - S$ which is adjacent to some vertex of S the graph H is a subgraph of $G[S \cup \{x\}]$ with its root v at x.

A graph G is called *decomposable* if it is the join of two graphs.

2. THE PROBLEMS

We start with a problem which is formulated as a conjecture in [3] and [1] (see also in [2]).

Conjecture 1. *Every graph is τ-partitionable.*

In [1] it is shown amongst others that every decomposable graph is τ-partitionable.

For a given (rooted) graph H_v, the question whether every graph G has an H_v-kernel is discussed in [2], [4] and [5]. It is shown amongst others that

(a) Every graph has an H_v-kernel if and only if every graph is H_v-saturated.

(b) Every graph has a P_v-kernel where P_v is a path of order at most six and v is an endvertex of P.

(c) Every graph has an S_v-kernel where S_v is a star and v is the center of the star or v is an endvertex of the star.

Clearly, if H_v is a vertex transitive graph, then every graph has an H_v-kernel (any maximal set of vertices inducing an H_v-free graph is an H_v-kernel). The fact that there are graphs H_v and G for which G has no H_v-kernel is illustrated in [2] and [4]. The general problem therefore is

Problem. *Describe the rooted graphs H_v for which every graph G has an H_v-kernel.*

Let the path P_v of order n be rooted at an endvertex. If every graph G has a P_v-kernel for every n then Conjecture 1 is true: If $\tau(G) = k_1 + k_2$, let V_1 be a Q_v-kernel where Q_v is a path (rooted at an endvertex) of order $k_1 + 1$ and let $V_2 = V(G) - S$. From (b) we immediately obtain that every graph is (τ, k_1, k_2)-partitionable if $\min\{k_1, k_2\} \leq 5$.

We are inclined to think that the following conjecture is also true for every path P_v rooted at an endvertex v.

Conjecture 2. *Every graph has a P_v-kernel.*

References

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