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ISOMORPHIC COMPONENTS OF KRONECKER PRODUCT OF BIPARTITE GRAPHS

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Abstract

Weichsel (Proc. Amer. Math. Soc. 13 (1962) 47–52) proved that the Kronecker product of two connected bipartite graphs consists of two connected components. A condition on the factor graphs is presented which ensures that such components are isomorphic. It is demonstrated that several familiar and easily constructible graphs are amenable to that condition. A partial converse is proved for the above condition and it is conjectured that the converse is true in general.

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1. Introduction

The Kronecker product is one of the (four) most important graph products, and can be viewed as the product in the category of graphs, cf. Hell [4]. The product was used by Greenwell and Lovász [2] to demonstrate that for all $n \geq 3$, there is a uniquely n-colorable graph without odd cycles shorter than a given number s. Every graph is an induced subgraph of a Kronecker

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product of certain complete graphs, see Nešetřil [11]. Graph retracts, see for instance Pesch [12], form another area where the Kronecker product construction turned out to be useful. The product has several other applications, for instance in modelling concurrency in multiprocessor systems [10] and in automata theory. (For example, closure of regular sets, or closure of context-free sets with a regular set, under intersection may be proved by taking a Kronecker product of the respective machines.)

Weichsel [14] proved that the *Kronecker product* of two nontrivial graphs is connected if and only if both factors are connected and at least one of them possesses an odd cycle. If both factors are connected and bipartite, then their Kronecker product consists of two connected components. When are these components isomorphic? This natural question plays the central role in our paper.

The *Kronecker product* $G \times H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and edge set $\{\{(u,x),(v,y)\} \mid \{u,v\} \in E(G)$ and $\{x,y\} \in E(H)\}$. The terminology is justified by the fact that the adjacency matrix of a Kronecker graph product is given by the Kronecker matrix product of the adjacency matrices of the factor graphs; see [14] for details. However, this product is also known under several different names including *categorical product*, *tensor product*, *direct product*, *weak direct product*, *cardinal product* and *graph conjunction*.

The Kronecker product is commutative and associative in an obvious way. It is also distributive with respect to edge-disjoint union of graphs. In particular, if G and H are bipartite graphs which respectively appear as subgraphs of (not necessarily bipartite) graphs G_1 and H_1, then the two components of $G \times H$ appear as vertex-disjoint subgraphs of $G_1 \times H_1$. This lends some structure to the graph $G_1 \times H_1$. In fact, this view has been effectively used in Hamiltonian decompositions of certain Kronecker-product graphs [8].

The rest of the paper is organized as follows. In the next section, we give some more definitions, recall Weichsel's result and give an alternative proof of it. In Section 3, we prove that a certain condition on factor graphs is sufficient for the existence of an isomorphism between components of the Kronecker product of two bipartite graphs. It is also shown that several familiar and easily constructible graphs are amenable to that condition. A partial converse to the sufficient condition is established in the last section and it is conjectured that it holds in general. We conclude the paper with some remarks on the algorithmic aspects of related problems.

2. Preliminaries

By a graph is meant a finite, simple, undirected graph having at least two vertices. Unless indicated otherwise, graphs are also connected.

Another graph product which is of relevance here is the Cartesian product. The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $\{(u, x), (v, y)\} \in E(G \Box H)$ whenever $\{u, v\} \in E(G)$ and $x = y$, or $u = v$ and $\{x, y\} \in E(H)$. Note that the Cartesian product of two (or finitely many) graphs is connected if and only if every factor is. Observe also that $G \Box H$ is bipartite if and only if both G and H are bipartite [13] and that $G \times H$ is bipartite if and only if at least one of G and H is bipartite [3].

We have already mentioned the following result of Weichsel:

Theorem 2.1 [14]**.** *Let G and H be connected graphs. Then $G \times H$ is connected if and only if at least one of G and H contains an odd cycle.* ■

The next result refines a part of Theorem 2.1.

Lemma 2.2 [7]**.** *If $G = (V_0 \bigcup V_1, E)$ and $H = (W_0 \bigcup W_1, F)$ are bipartite graphs, then $(V_0 \times W_0) \bigcup (V_1 \times W_1)$ and $(V_0 \times W_1) \bigcup (V_1 \times W_0)$ are vertex sets of the two components of $G \times H$.* ■

While Lemma 2.2 is an alternative proof of the "only if" part of Theorem 2.1, the following argument (which is different from Weichsel's) takes care of the converse. Recall that $\times$-product is distributive with respect to the edge-disjoint union of graphs.

Let $G = (V, E)$ be a non-bipartite graph, and let $H = (W, F)$ be any graph. Choose spanning subgraphs G' and H' of G and H, respectively, and an edge $e = \{a, b\}$ of $G \setminus G'$ such that

(i) G' and H' are connected, bipartite graphs and

(ii) $G' + e$ is a non-bipartite graph.

G' and H' can always be constructed by successively removing edges from odd cycles. Let V_0 and V_1 (resp. W_0 and W_1) be the partite sets of G' (resp. H'). Note that $a, b \in V_0$ or $a, b \in V_1$. It suffices to show that $(G' + e) \times H'$ is a connected graph.

Without loss of generality, let $a, b \in V_0$. By Lemma 2.2, $G' \times H'$ consists of two connected components having vertex sets $(V_0 \times W_0) \bigcup (V_1 \times W_1)$ and $(V_0 \times W_1) \bigcup (V_1 \times W_0)$. Let $\{c, d\} \in E(H')$, where $c \in W_0$ and $d \in W_1$.

It is clear that $(a,c) \in V_0 \times W_0$, $(b,d) \in V_0 \times W_1$, and $\{(a,c),(b,d)\}$ is an edge of $(G'+e) \times H'$ which runs between $(V_0 \times W_0) \bigcup (V_1 \times W_1)$ and $(V_0 \times W_1) \bigcup (V_1 \times W_0)$. It immediately follows that the graph $(G'+e) \times H'$ is connected, and hence the result.

3. Property π

This is our main definition.

Definition 3.1. *A bipartite graph $G = (V_0 \bigcup V_1, E)$ is said to have a property π if G admits an automorphism φ such that $x \in V_0$ if and only if $\varphi(x) \in V_1$.* ■

Note that a bipartite graph which satisfies the foregoing definition possesses a certain symmetry, and has the same number of vertices in each partite set. Examples of graphs possessing the property π include even paths, even cycles, complete bipartite graphs with equal-sized partite sets, and the graph obtained from two copies of a star $K_{1,n}$ by introducing an edge between the center vertices of the two stars.

Theorem 3.2. *If G and H are bipartite graphs one of which has the property π, then the two components of $G \times H$ are isomorphic.*

Proof. Let $G = (V_0 \bigcup V_1, E)$ and $H = (W_0 \bigcup W_1, F)$ be bipartite graphs. Assume that the graph G possesses the property π with φ as an appropriate automorphism. The sets $(V_0 \times W_0) \bigcup (V_1 \times W_1)$ and $(V_0 \times W_1) \bigcup (V_1 \times W_0)$ correspond to vertex sets of the two components of the graph $G \times H$. Consider the mapping given by $(u,x) \mapsto (\varphi(u), x)$ between vertex sets of these components. It is straightforward to see that this is a well-defined bijection which corresponds to a desired isomorphism. ■

We next show that graphs with the property π enjoy certain interesting closure properties with respect to the Kronecker product and the Cartesian product.

Proposition 3.3. *If G and H are bipartite graphs having the property π, then so is each of the following:*

(i) *$\times$-product of G and a non-bipartite graph,*

(ii) *each component of the $\times$-product of G and H, and*

(iii) *$\square$-product of G and a bipartite graph.*

Proof. Let $G = (V_0 \bigcup V_1, E)$ and $H = (W_0 \bigcup W_1, F)$ be bipartite graphs which are respectively equipped with automorphisms φ and ψ in the sense of Definition 3.1.

(i) Let $K = (X, D)$ be a non-bipartite graph. The two partite sets of the connected, bipartite graph $G \times K$ are $V_0 \times X$ and $V_1 \times X$. The mapping given by $(v, x) \mapsto (\varphi(v), x)$ constitutes an automorphism of $G \times K$ corresponding to the property π.

(ii) Consider the component of the (bipartite) graph $G \times H$ on vertex set $(V_0 \times W_0) \bigcup (V_1 \times W_1)$. Note that $V_0 \times W_0$ and $V_1 \times W_1$ are the two partite sets of this component. The mapping given by $(v, w) \mapsto (\varphi(v), \psi(w))$ constitutes a desired automorphism of this component.

(iii) Let $K = (X_0 \bigcup X_1, D)$ be a bipartite graph. The two partite sets of the (connected) bipartite graph $G \Box K$ are $(V_0 \times X_0) \bigcup (V_1 \times X_1)$ and $(V_0 \times X_1) \bigcup (V_1 \times X_0)$. The mapping given by $(v, x) \mapsto (\varphi(v), x)$ constitutes a desired automorphism of this graph. ■

It follows from Proposition 3.3 that each of the following graphs has the property π: (i) the n-cube Q_n ($Q_1 = K_2$, $Q_n = K_2 \Box Q_{n-1}$, $n \geq 2$), (ii) the planar grid $P_m \Box P_n$, where m or n is even, (iii) the graph $C_{n_1} \Box \ldots \Box C_{n_r}$, where each n_i is even, and (iv) each component of the graph $C_{n_1} \times \ldots \times C_{n_r}$, where at least one n_i is even.

The following is a relevant result. It is a generalization of a similar result from [6] for the graph $C_m \times P_n$.

Proposition 3.4. *If m is even, $m/2$ is odd and G is a bipartite graph, then each component of the graph $C_m \times G$ is isomorphic to $C_{m/2} \times G$.*

Proof. Let m and $G = (V_0 \bigcup V_1, E)$ be as stated. Denote the consecutive vertices of C_m by 0, 1, ..., $m - 1$. Then vertex sets of the two components of the graph $C_m \times G$ are

$$(\{0, 2, \ldots, m-2\} \times V_0) \bigcup (\{1, 3, \ldots, m-1\} \times V_1)$$

and

$$(\{0, 2, \ldots, m-2\} \times V_1) \bigcup (\{1, 3, \ldots, m-1\} \times V_0).$$

By Theorem 3.2, these two components are isomorphic. Consider the mapping from the vertex set of one of these components to that of $C_{m/2} \times G$ given by $(i, x) \mapsto (i \bmod (m/2), x)$. That this is a well-defined bijection corresponding to a desired isomorphism is left to the reader. ■

4. A Conjecture

We conjecture that the converse of Theorem 3.2 holds as well:

Conjecture 4.1. *Let G and H be bipartite graphs. Then the components of $G \times H$ are isomorphic if and only if at least one of G and H has the property π.*

In the rest of the paper, let $G = (V_0 \bigcup V_1, E)$ and $H = (W_0 \bigcup W_1, F)$ be bipartite graphs and let X and Y be the two connected components of $G \times H$ induced by $(V_0 \times W_0) \bigcup (V_1 \times W_1)$ and $(V_0 \times W_1) \bigcup (V_1 \times W_0)$, respectively.

Note first that if X and Y are isomorphic, then at least one of $|V_0| = |V_1|$ or $|W_0| = |W_1|$ holds. Indeed, since $|X| = |Y|$ we have $|V_0||W_0| + |V_1||W_1| = |V_0||W_1| + |V_1||W_0|$. Therefore $|V_0|(|W_0| - |W_1|) = |V_1|(|W_0| - |W_1|)$ and thus the observation. However, much more is true, but first we need some more notation.

As usual, for $x \in V(G)$, let $N_G(x) = \{y \,|\, \{x, y\} \in E(G)\}$ and for $Q \subseteq V(G)$, let $N_G(Q) = \bigcup_{x \in Q} N_G(x)$. We will also write $N(Q)$ if the graph G will be clear from the context. For a graph G and $Q \subseteq V(G)$ let $\deg_G(Q)$ be the multiset $\{deg_G(v) \,|\, v \in Q\}$. Also, for a set $Q \subseteq V(G)$ let $N^i(Q)$, $i \geq 1$, be defined by $N^1(Q) = N(Q)$ and $N^i(Q) = N(N^{i-1}(Q))$, $i \geq 2$.

Theorem 4.2. *Let $\varphi : X \to Y$ be an isomorphism which maps $V_0' \times W_0'$ onto $V_1' \times W_0'$, where $V_0' \subseteq V_0$, $V_1' \subseteq V_1$ and $W_0' \subseteq W_0$. Then for all $i \geq 1$, $\deg_G(N^i(V_0')) = \deg_G(N^i(V_1'))$ hold.*

Proof. Since φ is an isomorphism, we have

$$\varphi(N_{G\times H}(V_0' \times W_0')) = N_{G\times H}(\varphi(V_0' \times W_0')) = N_{G\times H}(V_1' \times W_0')$$

and from the definition of the Kronecker product it follows

$$N_{G\times H}(V_1' \times W_0') = N_G(V_1') \times N_H(W_0').$$

Therefore, φ maps $N_{G\times H}(V_0' \times W_0') = N_G(V_0') \times N_H(W_0')$ onto $N_G(V_1') \times N_H(W_0')$.

Let v_0, v_1 and w be vertices of maximum degrees in $N_G(V_0')$, $N_G(V_1')$ and $N_H(W_0')$, respectively. Then $(v_0, w) \in N_G(V_0') \times N_H(W_0')$ and $(v_1, w) \in N_G(V_1') \times N_H(W_0')$ have the same (largest) degree and thus v_0 and v_1 must have the same degree in G. It follows that $\deg_{G\times H}(\{v_0\} \times N_H(W_0')) = \deg_{G\times H}(\{v_1\} \times N_H(W_0'))$, and therefore

$$\deg_{G\times H}((N(V_0') \setminus \{v_0\}) \times N(W_0')) = \deg_{G\times H}((N(V_1') \setminus \{v_1\}) \times N(W_0')).$$

Repeating the above argument, we conclude $\deg_G(N(V_0')) = \deg_G(N(V_1'))$. This proves the theorem for $i = 1$.

We have seen above that φ maps $N_G(V_0') \times N_H(W_0')$ isomorphically onto $N_G(V_1') \times N_H(W_0')$. But this is just the theorem assumption, hence as above we obtain $\deg_G(N^2(V_0')) = \deg_G(N^2(V_1'))$. Repeating the argument, we prove the equality for any $i \geq 1$. ■

Corollary 4.3. *Let A_0, A_1, B_0 and B_1 be the sets of vertices with maximum (minimum) degree in V_0, V_1, W_0 and W_1, respectively. Then if X and Y are isomorphic, at least one of $\deg_G(N^i(A_0)) = \deg_G(N^i(A_1))$ or $\deg_G(N^i(B_0)) = \deg_G(N^i(B_1))$ hold for all $i \geq 1$.*

Proof. Let $\varphi : X \to Y$ be an isomorphism. As X and Y are connected bipartite graphs, φ maps $V_0 \times W_0$ either onto $V_1 \times W_0$ or onto $V_0 \times W_1$. In the former case $\varphi(A_0 \times B_0) = A_1 \times B_0$ and by Theorem 4.2, $\deg_G(N^i(A_0)) = \deg_G(N^i(A_1))$. In the later case $\varphi(A_0 \times B_0) = A_0 \times B_1$ and by Theorem 4.2 and commutativity of the Kronecker product, $\deg_G(N^i(B_0)) = \deg_G(N^i(B_1))$. ■

Consider for example the graph G in Figure 1. G fulfils the conclusion of Corollary 4.3. On the other hand, the connected components of $G \times P_3$ are not isomorphic. Indeed, there is a vertex of degree 6 which is adjacent to vertices of degrees 2, 2, 3, 3, 4 and 4 in one of the components and there is no such vertex in the other component.

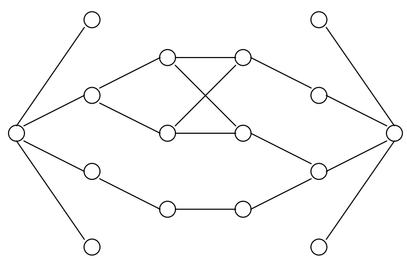

Figure 1. The graph G

Theorem 4.2 also implies that if X and Y are isomorphic then in at least one of the factors the partite sets have the same degree sequences.

Corollary 4.4. *If X and Y are isomorphic, then at least one of* $\deg_G(V_0) = \deg_G(V_1)$ *or* $\deg_H(W_0) = \deg_H(W_1)$ *holds.*

Proof. Let $\varphi : X \to Y$ be an isomorphism. As in Corollary 4.3 φ maps $V_0 \times W_0$ either onto $V_1 \times W_0$ or onto $V_0 \times W_1$. Assume the former case and let A_0 and A_1 be as in Corollary 4.3. Clearly, for any $i \geq 1$, $N^i(A_0) \subseteq N^{i+2}(A_0)$ and $N^i(A_1) \subseteq N^{i+2}(A_1)$. Furthermore, since G is connected, there must be some i such that $N^i(A_0) = V_0$ and $N^i(A_1) = V_1$. By Corollary 4.3, $\deg_G(V_0) = \deg_G(V_1)$.

In the case when φ maps $V_0 \times W_0$ onto $V_0 \times W_1$, we analogously obtain $\deg_H(W_0) = \deg_H(W_1)$. ■

Concluding Algorithmic Remarks

The Kronecker product seems to be interesting also from the algorithmic point of view. As we learned from Babai (at the Workshop on Cayley Graphs held in September 1996 in Montréal), Lalonde [9] proved that it is NP-complete to decide whether a given bipartite graph G has the property π provided in addition that the corresponding automorphism is of order 2. However, the problem of deciding whether G has the property π becomes isomorphism complete, cf. [1, page 1518].

Finally, we mention a recent very interesting algorithmic result of Imrich [5]. He namely proved that the unique prime factor decomposition of finite, nonbipartite, connected graphs with or without loops can be determined in polynomial time.

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