

## MINIMAL VERTEX DEGREE SUM OF A 3-PATH IN PLANE MAPS

O.V. BORODIN\*

*Novosibirsk State University*  
*Novosibirsk, 630090, Russia*

### Abstract

Let  $w_k$  be the minimum degree sum of a path on  $k$  vertices in a graph. We prove for normal plane maps that: (1) if  $w_2 = 6$ , then  $w_3$  may be arbitrarily big, (2) if  $w_2 > 6$ , then either  $w_3 \leq 18$  or there is a  $\leq 15$ -vertex adjacent to two 3-vertices, and (3) if  $w_2 > 7$ , then  $w_3 \leq 17$ .

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Let  $d(v)$  be the degree of a vertex  $v$  in a 3-polytope, i.e. in a 3-connected planar graph. Franklin [5] proves that in each simplicial 3-polytope (whose all faces are triangles) of minimal degree 5 there is a path  $uvw$  such that  $d(u) \leq 6$ ,  $d(v) = 5$ , and  $d(w) \leq 6$ . Both 6's here are best possible. Kotzig [8] proves that each 3-polytope has an edge  $uv$  such that  $d(u) + d(v) \leq 13$ ; the bound is best possible. For a graph  $G$  having at least one path consisting of  $k$  vertices, called hereafter a  $k$ -path, we denote by  $w_k(G)$ , or sometimes  $w_k$ , the minimal vertex degree sum of a  $k$ -path in  $G$ . A plane map is *normal* if its every edge and face is incident with at least three edges. If a plane map is not normal, then as seen from  $K_{2,n}$ , not only  $w_3$ , but also  $w_2$  can be arbitrarily big. It is proved in [1] that each normal plane graph of minimal degree 5 has a face  $uvw$  such that  $d(u) + d(v) + d(w) \leq 17$ , which bound is best possible. If a 3-polytope is simplicial and no 4-vertex is adjacent to that of degree  $\leq 4$ , then, as proved in [2], there is a face  $uvw$  such that  $d(u) + d(v) + d(w) \leq 29$ , which bound is also sharp. Jendrol' [6] proves that each 3-polytope has a path  $uvw$  such that  $\max\{d(u), d(v), d(w)\} \leq 15$  (the bound is precise). Jendrol' [7] further shows that such a path must

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belong to one of ten classes, in which  $d(u) + d(v) + d(w)$  varies from 23 to 15. As reported by Enomoto and Ota in [3], Ando, Iwasaki and Kaneko [4] prove  $w_3 \leq 21$  for each 3-polytope, which is best possible due to Jendrol's construction [7].

It is natural to describe the classes of normal plane maps in which  $w_3$  is bounded above. Consider the following construction with  $w_2 = 6$  and  $w_3$  unbounded: join two vertices by  $n$  edges and place two adjacent 3-vertices inside each 2-face. It turns out that not all 3,3-edges are responsible for the unboundedness of  $w_3$ , but only those lying on 3-faces. More specifically, the purpose of this note is to prove the following

**Theorem 1.** *Each normal plane map without triangles incident with two 3-vertices has*

- (i) *either  $w_3 \leq 18$  or a vertex of degree  $\leq 15$  adjacent to two 3-vertices, and*
- (ii) *either  $w_3 \leq 17$  or  $w_2 = 7$ .*

**Corollary 2.** *Each normal plane map with  $w_2 > 6$  has  $w_3 \leq 21$ .*

In particular, Theorem 1 immediately implies that Franklin's bound  $w_3 \leq 17$  is valid for all normal plane maps of minimal degree  $\geq 4$ .

**Corollary 3.** *Each normal plane map without 3-vertices has  $w_3 \leq 17$ .*

The upper bound in the following statement is also immediate:

**Corollary 4.** *In each 3-polytope without 3-vertices there is a path  $uvw$  such that  $\max\{d(u), d(v), d(w)\} \leq 9$ .*

To attain the bound in Corollary 4, take the dual of the well-known (3,5,3,5)-Archimedean solid, and join every two 5-vertices lying in a common face by a path consisting of two 4-vertices. (The former 3-vertices now have degree 9, while the former 5-ones become 10-vertices.)

**Proof of Theorem 1.** Suppose that  $M'$  is a counterexample to (i) or (ii) of Theorem 1. In particular,  $M'$  has  $w_3 > 17$ . Let  $M$  be a counterexample on the same vertex set with the greatest number of edges.

**(A)**  *$M$  is a triangulation.*

Suppose there is a  $> 3$ -face  $f = abc \dots$ . Further suppose  $b$  is a vertex with the minimal degree among all vertices incident with  $f$ . Then  $M + ac$  is also a counterexample to the same statement (i) or (ii) as  $M$ . First observe that

if  $M$  has no 4-vertex adjacent to a 3-vertex or a  $\leq 15$ -vertex adjacent to two 3-vertices, then so does  $M + ac$ . Secondly, suppose  $w_3(M + ac) < w_3(M)$ . Then in  $M + ac$  there is a path  $zac$  or  $acz$ , say  $zac$ , such that  $z \neq b$  and  $d_M(z) + d_M(a) + 1 + d_M(c) + 1 < w_3(M)$ . But since  $d_M(b) \leq d_M(c)$ , we have  $w_3(M) \leq d_M(z) + d_M(a) + d_M(b) < d_M(z) + d_M(a) + 1 + d_M(c) + 1 < w_3(M)$ , which is a contradiction.

The next property follows immediately from (A):

**(B)** *No 3-vertex of  $M$  is adjacent to a 3-vertex.*

Throughout the paper, we denote the vertices adjacent to a vertex  $v$  in a cyclic order by  $v_1, \dots, v_{d(v)}$ .

Euler's formula  $|V| - |E| + |F| = 2$  for  $M$  may be written as

$$(1) \quad \sum_{v \in V} (d(v) - 6) = -12.$$

Every  $\leq 5$ -vertex contributes a negative *charge*  $\mu(v) = d(v) - 6$  to (1), while the charges of  $\geq 6$ -vertices are non-negative. Using the properties of  $M$  as a counterexample to (i) or (ii), we define a local redistribution of  $\mu$ 's, preserving their sum, such that the *new contribution*  $\mu'(v)$  is non-negative for all  $v \in V$ . This will contradict the fact that the sum of the new contributions is, by (1), equal to -12.

**Rule 1.** Suppose  $d(v) = 3$ . If each  $v_i$  has  $d(v_i) \geq 7$ , then each of them gives 1 to  $v$ . Suppose  $4 \leq d(v_1) \leq 6$ . Then each of  $v_2, v_3$  gives  $\frac{3}{2}$  to  $v$ .

Now  $\mu'(v)$  is completely determined for  $d(v) = 3$ , and clearly  $\mu'(v) = \mu(v) + 3 = 0$ .

**Rule 2.** Suppose  $d(v) = 4$ . If  $v$  is not adjacent to  $< 7$ -vertices, it receives  $1/2$  from each  $v_i$ . Suppose  $5 \leq d(v_1) \leq 6$ . Then  $v$  receives  $2/3$  from each of  $v_2, v_3, v_4$ . If  $d(v_1) = 4$ , then  $v_3$  gives  $4/5$  to  $v$ , and each of  $v_2, v_4$  gives  $3/5$ . Finally, if  $d(v_1) = 3$ , then  $v_3$  gives 1, while each of  $v_2, v_4$  gives  $1/2$ .

Clearly, each 4-vertex  $v$  has  $\mu'(v) = \mu(v) + 2 = 0$ .

**Rule 3.** Suppose  $d(v) = 5$ . If each of  $v_i$  has  $d(v_i) \geq 6$ , then four of  $v_i$  actually have  $d(v_i) \geq 7$ , and each  $\geq 7$ -neighbour gives  $1/4$  to  $v$ . Otherwise, if say  $3 \leq d(v_1) \leq 5$ , then  $v$  receives  $1/2$  from each of  $v_3, v_4$ .

Clearly,  $\mu'(v) \geq \mu(v) + 1 \geq 0$  if  $d(v) = 5$ .

Thus, if  $v$  is *minor*, i.e. has  $d(v) \leq 5$ , then  $\mu'(v) \geq 0$ . It remains to prove  $\mu'(v) \geq 0$  for  $d(v) \geq 7$ , because 6-vertices do not participate in discharging.

If  $7 \leq d(v) \leq 8$ , then either  $v$  is adjacent to at most one minor vertex, or  $v$  is an 8-vertex with all minor neighbours having degree 5. By Rules 1-3,  $v$  cannot give  $>1$  to any of its neighbours because  $3 + 6 + 8 < w_3$ , therefore  $\mu'(v) \geq 7 - 6 - 1 = 0$  in the first subcase above. In the second, observe that by Rule 3,  $v$  can give a positive charge to a 5-vertex  $y$  only if there are triangles  $vxy$  and  $vzy$ , where  $d(x) \geq 6$ ,  $d(z) \geq 6$ . We thus have  $\mu'(v) \geq 7 - 6 - 3 \times 1/4 > 0$  if  $d(v) = 7$ , and  $\mu'(v) \geq 8 - 6 - 4 \times 1/4 = 0$  if  $d(v) = 8$ .

Suppose  $d(v) = 9$ . If  $v$  has a 3-neighbour, then no other minor neighbour is possible because  $9 + 3 + 5 < w_3$ , therefore  $\mu'(v) \geq 9 - 6 - 3/2 > 0$ . Otherwise,  $v$  can do at most four transfers ( $9 + 4 + 4 < w_3$ ), each of which is not greater than  $2/3$ , which implies  $\mu'(v) \geq 9 - 6 - 4 \times 2/3 > 0$ .

Suppose  $d(v) = 10$ . If  $v$  has a 3-neighbour, then 4-neighbours are impossible. Since  $v$  then can do at most five transfers in total, we have  $\mu'(v) \geq 10 - 6 - 3/2 - 4 \times 1/2 > 0$ . Assume  $v$  has no 3-neighbours. By Rule 2,  $v$  gives  $4/5$  to single 4-neighbours, and  $3/5$  to each of 4-twins. To estimate the total expenditure of  $v$ , undertake the following averaging of transfers from  $v$  to its neighbours: If  $v_i$  receives  $4/5$ , then neither  $v_{i-1}$ , nor  $v_{i+1}$ , where indices are taken modulo  $d(v)$ , gets anything from  $v$  directly. We may imagine that  $v$  actually gives  $1/5$  to each of  $v_{i-1}$ ,  $v_{i+1}$ , and only  $2/5$  to  $v_i$ . Similarly, if  $v_i$  gets  $3/5$  from  $v$  by Rule 3, we may imagine that  $v_i$  actually gets only  $2/5$ , while the remaining  $1/5$  goes to that of  $v_{i-1}$ ,  $v_{i+1}$  which is not a 4-vertex. Observe that those vertices which did not receive anything from  $v$  directly, now receive at most  $2 \times 1/5 = 2/5$ . Thus,  $v$  gives on average  $\leq 2/5$  to each neighbour, that is  $\mu'(v) \geq 10 - 6 - 10 \times 2/5 = 0$ .

Next suppose  $d(v) = 11$ . Still, at most one 3-neighbour is possible. Besides, no 4-neighbour of  $v$  is adjacent to a 3-vertex by (i) and (ii), so that no 4-neighbour of  $v$  can receive 1 from  $v$ . If  $v_i$  receives  $3/2$  from  $v$ , then by Rule 1 we may assume that  $d(v_{i+1}) \leq 6$ . Observe that  $v_{i+2}$  does not receive anything from  $v$ , and therefore  $v$  may split its donation of  $3/2$  to  $v_i$  among  $v_{i-1}$ ,  $v_i$ ,  $v_{i+1}$  and  $v_{i+2}$  as follows:  $3/2 = 1/5 + 7/10 + 2/5 + 1/5$ . The argument used for  $d(v) = 10$  says that after averaging such a  $v_i$  receives  $< 1$  from  $v$ , while each of the other ten neighbours receives  $\leq 2/5$ , i.e.  $\mu'(v) \geq 11 - 6 - 1 - 10 \times 2/5 = 0$ . If there does exist a 3-neighbour receiving 1, or there is no 3-neighbours at all, then the argument used for  $d(v) = 10$  is still valid, because if  $v_i$  receives  $t$  from  $v$  where  $2/5 \leq t \leq 4/5$ , then  $v_{i-1}$  and  $v_{i+1}$  receive nothing from  $v$  (directly). Hereafter suppose  $d(v) \geq 12$ .

*Case 1.*  $M$  contradicts (ii).

It follows that if  $d(v_i) = 3$ , then  $d(v_{i+1}) > 4$ , whenever  $1 \leq i \leq d(v)$ . By Rule 3, if  $d(v_i) = 3$ , then neither  $v_{i+1}$ , nor  $v_{i-1}$  receives anything. We employ another averaging of the donations of  $v$  to its neighbours. If  $v_2$  receives  $3/2$ , then assume  $d(v_3) \leq 6$  and split this  $3/2$  amongst  $v_1, \dots, v_4$  as follows:  $1/4 + 1/2 + 1/2 + 1/4$ , respectively. Otherwise, whenever  $v_2$  receives  $> 1/2$  from  $v$ , it actually receives  $\leq 1$ . We then direct  $1/4$  to each of  $v_1, v_3$ , and  $\leq 1/2$  remains for  $v_2$ . As a result, each neighbour receives  $\leq 1/2$  from  $v$ , which implies  $\mu'(v) \geq d(v) - 6 - d(v)/2 = (d(v) - 12)/2 \geq 0$ . This completes the proof of (ii).

*Case 2.*  $M$  contradicts (i).

If  $d(v) \leq 15$ , then  $v$  cannot be adjacent to two 3-vertices, and the  $2/5$ -argument given for  $d(v) = 11$  is valid. Therefore assume  $d(v) \geq 16$ . We now employ yet another averaging. Whenever  $d(v_i) = 3$  and  $d(v_{i+1}) = 4$ , we redistribute what they receive from  $v$  among  $v_{i-1}, \dots, v_{i+2}$  as follows  $3/2 + 1/2 = 1/3 + 2/3 + 2/3 + 1/3$ , respectively.

If  $d(v_i) = 3$  and  $5 \leq d(v_{i+1}) \leq 6$ , i.e.  $v_i$  receives  $3/2$  by Rule 1, while  $v_{i+1}$  does nothing, then both  $v_{i-1}$  and  $v_{i+2}$  still receive nothing from  $v$  directly, and we split  $3/2 = 1/3 + 1/3 + 1/2 + 1/3$ , respectively. If  $v_i$  receives  $\leq 1$ , while each of  $v_{i-1}, v_{i+1}$  receives nought, then we instead send  $1/3$  to each of  $v_{i-1}, v_{i+1}$ , so that  $\leq 1/3$  remains for  $v_i$ . Observe that in the last two cases  $v_i$  saves for  $v$  at least  $1/3$  with respect to the normal level  $2/3$  of donations of  $v$  to its neighbours.

Clearly, after this averaging every neighbour of  $v$  indirectly receives from  $v$  at most  $2/3$ . It follows,  $\mu'(v) \geq d(v) - 6 - 2d(v)/3 = 2(d(v) - 18) \geq 0$ , i.e. we are done with  $d(v) \geq 18$ .

It remains to prove  $\mu'(v) \geq 0$  if  $16 \leq d(v) \leq 17$ . Observe that  $d(v) - 6 - 2d(v)/3$  is  $-1/3$  and  $-2/3$  for  $d(v) = 17$  and  $d(v) = 16$ , respectively. So, it suffices to find one or, respectively, two vertices receiving  $\leq 1/3$  after averaging to complete the proof in these two cases.

Consider a partition of cycle  $C_v = v_1 \dots v_{d(v)}$  into segments  $R_{i,j} = v_i \dots v_j$  where  $i < j$ , called *receivers*, such that neither  $v_i$  nor  $v_j$  receives by Rule 1-3 anything from  $v$ , whereas each  $v_q$  does whenever  $i < q < j$ . Clearly,  $j - i \leq 3$  because in  $M$  there are no three minor vertices in a row. If there is  $R_{i,j}$  such that  $j - i = 1$ , then each of  $v_i, v_j$  obviously has  $\leq 1/3$  after averaging, which implies  $\mu'(v) \geq 0$  as mentioned above. Assume each  $R_{i,j}$  is either *singular* ( $j - i = 2$ ) or *double* ( $j - i = 3$ ). Due to the residues of 17 and 16 modulo 3, there should be at least one singular  $R_{i,j}$  if  $d(v) = 17$ , and at

least two if  $d(v) = 16$ . But in each singular  $R_{i,j}$ , vertex  $v_{i+1}$  receives  $\leq 1/3$ , i.e. saves  $1/3$  for  $v$ . This completes the proof of  $\mu'(v) \geq 0$  if  $16 \leq d(v) \leq 17$ .

Thus we have proved  $\mu'(v) \geq 0$  for every  $v \in V$ , which contradicts (1):

$$0 \leq \sum_{v \in V} \mu'(v) = \sum_{v \in V} \mu'(v) = -12.$$

This completes the proof of Theorem 1. ■

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