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LOCALIZATION OF JUMPS OF THE POINT-DISTINGUISHING CHROMATIC INDEX OF $K_{n,n}$

Mirko Horňák and Roman Soták

Department of Geometry and Algebra, P. J. Šafárik University
Jesenná 5, 041 54 Košice, Slovakia

e-mail: hornak@turing.upjs.sk
sotak@duro.upjs.sk

Abstract

The point-distinguishing chromatic index of a graph represents the minimum number of colours in its edge colouring such that each vertex is distinguished by the set of colours of edges incident with it. Asymptotic information on jumps of the point-distinguishing chromatic index of $K_{n,n}$ is found.

Keywords: Point-distinguishing chromatic index, colour set, complete equibipartite graph.

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1. Introduction

Recently, the following kind of colourings in the chromatic graph theory has appeared: One wants to colour elements of a graph G of the same dimension (0 for vertices and 1 for edges) in such a way that the remaining elements of G (of "complementary" dimension) have to be distinguished using sets of colours of their incident elements. As usually, the minimum number of colours is searched for; if vertices are coloured, we obtain the line-distinguishing chromatic number of G, $\chi_1(G)$ (Frank et al. [6]), and colourings of edges yield the point-distinguishing chromatic index of G, $\chi_0(G)$ (Harary and Plantholt [7]). It has been shown by Hopcroft and Krishnamoorthy [8] that, for a general graph G, to determine $\chi_1(G)$ is an NP-complete problem. However, the situation can be different for graphs with a simple structure: $\chi_1(P_n)$ and $\chi_1(C_n)$ are known – see Al-Wahabi et al. [1].

If colourings are required to be proper (i.e., adjacent elements receive different colours), we obtain new invariants of a graph G, the harmonious chromatic number $h(G)$, corresponding to $\chi_1(G)$ (Miller and Pritikin [13]), and the observability $\mathrm{obs}(G)$, corresponding to $\chi_0(G)$ (Černý et al. [4]). The observability has been introduced independently by Burris and Schelp [3]; recently, in Favaron and Schelp [5], it has been named strong colouring number. There are several results for upper bounds of $h(G)$, see e.g. Lee and Mitchem [12] and Beane et al. [2].

If the problems of determining $\chi_0(G)$ and $\mathrm{obs}(G)$ are compared, their complexity depends on the structure of G and none of them can be stated to be more difficult than the other. Thus, according to [7], $\chi_0(Q_n) = n+1$, while the observability of cubes is known only asymptotically: by Horňák and Soták [11] $\lim\limits_{n\to\infty} \frac{\mathrm{obs}(Q_n)}{n} = 1 + q^*$, where $q^* = 0.293815\ldots$ is the unique solution of the equation $(x+1)^{x+1} = 2x$ in the interval $(0,\infty)$; the exact value of $\mathrm{obs}(Q_n)$ is computed only for $n \le 5$. The value of observability is known, from Horňák and Soták [9], for complete multipartite graphs with equipotent parts; on the other hand, even for complete equibipartite graphs we only know, due to [7], that for any integer $n \ge 2$

$$\lceil \log_2 n \rceil + 1 \le \chi_0(K_{n,n}) \le \lceil \log_2 n \rceil + 2,$$

or, in other words,

$$2^{k-2} + 1 \le n \le 2^{k-1} \Rightarrow \chi_0(K_{n,n}) \in \{k, k+1\}.$$

Zagaglia Salvi [14] found $\chi_0(K_{n,n})$ for $n \le 45$. The same author in [15] claims to have obtained the complete solution of the problem of computing $\chi_0(K_{n,n})$. However, we shall see that her solution fails.

Now, to be more precise, the *point-distinguishing chromatic index* $\chi_0(G)$ of a graph G is the minimum integer k admitting a k-colouring of edges of G such that for each pair (x,y) of different vertices of G the *colour set* of x – the set of colours of edges incident with x – is different from the colour set of y. Evidently, $\chi_0(G)$ is well defined only if G has no component K_2 and at most one of its components is K_1.

For integers p, q set

$$[p,q] := \bigcup_{i=p}^{q} \{i\}, \qquad [p,\infty) := \bigcup_{i=p}^{\infty} \{i\}$$

and for $k \in [3,\infty)$, $n \in [2,\infty)$ let $\mathcal{M}_{k,n}$ be the set of all square matrices M of order n with entries from $[1,k]$ such that the sets of elements occurring in lines (rows or columns) of M are distinct. Then we have the following evident statement:

Proposition 1. *If $n \in [2, \infty)$, then $\chi_0(K_{n,n}) = \min\{k \in [3, \infty)\, \mathcal{M}_{k,n} \neq \emptyset\}$.*

For $M \in \mathcal{M}_{k,n}$ and $i \in [1, n]$ let $\mathcal{R}_i(M)$ be the set of all entries of i-th row of M and

$$\mathcal{R}(M) := \{\mathcal{R}_i(M)\; i \in [1, n]\};$$

analogous sets concerning columns of M will be denoted by $\mathcal{C}_j(M)$ for $j \in [1, n]$ and $\mathcal{C}(M)$. Evidently, $\mathcal{R}(M)$ and $\mathcal{C}(M)$ are disjoint sets of cardinality n and any member of $\mathcal{R}(M)$ has a non-empty intersection with any member of $\mathcal{C}(M)$. On the other hand, for disjoint sets $\mathcal{R}$, $\mathcal{C}$ of n non-empty subsets of $[1, k]$ such that $R \in \mathcal{R}$ and $C \in \mathcal{C}$ implies $R \cap C \neq \emptyset$ in general there does not exist $M \in \mathcal{M}_{k,n}$ such that $\mathcal{R} = \mathcal{R}(M)$ and $\mathcal{C} = \mathcal{C}(M)$; nevertheless, by [14] (Theorem 4.5) such a matrix does exist provided $n \in [2^{k-2} + 1, 2^{k-1}]$ is great enough:

Theorem 1. *If $k \in [3, \infty)$ and $n \in [\lceil 2^k/3 \rceil, 2^{k-1}]$, then $\chi_0(K_{n,n}) = k$ if and only if there exist disjoint n-element sets $\mathcal{R}$, $\mathcal{C}$ of non-empty subsets of the set $[1, k]$ such that $R \cap C \neq \emptyset$ for every $R \in \mathcal{R}$ and $C \in \mathcal{C}$.*

Using this result it is easy to see that $\chi_0(K_{n,n}) = k$ for some $n \in [\lceil 2^k/3 \rceil + 1, 2^{k-1}]$ implies $\chi_0(K_{n-1,n-1}) = k$. Moreover, by Corollary 3.5 of [14] $\chi_0(K_{n,n}) = k$ for $n \in [2^{k-2}, \lfloor 2^k/3 \rfloor]$ and we can conclude that $\chi_0(K_{n,n})$ is a non-decreasing integer function of n with jumps of value 1; put

$$n_k := \max\{n \in [2, \infty) : \chi_0(K_{n,n}) = k\},$$

$$r_k := \frac{2n_k}{2^k} = \frac{n_k}{2^{k-1}}.$$

By definition, $2n_k$ is the maximum number of subsets of $[1, k]$ able to distinguish – as colour sets – vertices of a complete equibipartite graph and r_k expresses which portion (of all subsets of $[1, k]$) those subsets represent. With respect to considerations above we have $n_k \in [\lfloor 2^k/3 \rfloor, 2^{k-1} - 1]$ and $r_k \in \langle \frac{1}{2}, 1)$. Furthermore, Theorem 2.4 of [14] yields $n_{k+1} \geq 2n_k$, hence the sequence $\{r_k\}_{k=3}^{\infty}$ is non-decreasing and convergent and $\lim\limits_{k\to\infty} r_k \leq 1$. Theorem 3.1 of [15] states that $n_{k+1} = 2n_k$ for odd $k \in [5, \infty)$ and $n_{k+1} = 2n_k + 1$ for even $k \in [6, \infty)$; this would lead to $\lim\limits_{k\to\infty} r_k = \frac{17}{24}$. However, the aim of the present paper is to show that

$$\lim_{k\to\infty} r_k \geq 3 - \sqrt{5} = 0.763932\ldots > 0.708\bar{3} = \frac{17}{24}.$$

Thus, the mentioned theorem of [15] is not valid (implicitly); an explicit contradiction to it has been found by Horňák and Soták [10]: $n_7 = 46$. (The starting terms of $\{n_k\}_{k=3}^{\infty}$ are $n_3 = 2$, $n_4 = 5$, $n_5 = 11$ and $n_6 = 22$, see [14], and nothing, besides trivial bounds, is known for $k \geq 8$.)

2. Asymptotic Behaviour of r_k

From $n_5 = 11 = \lceil 2^5/3 \rceil$ and $n_{k+1} \geq 2n_k$ we get $n_k \geq \lceil 2^k/3 \rceil$ for each $k \in [5, \infty)$, hence investigating n_k for $k \to \infty$ it is sufficient to study the existence of sets $\mathcal{R}, \mathcal{C}$ mentioned in Theorem 1.

Theorem 2. $\lim\limits_{k\to\infty} r_k \geq 3 - \sqrt{5}$.

Proof. Take $p, q \in [2, \infty)$ and put $A_i = [(i-1)p + 1, ip]$ for $i \in [1, q]$. As $\bigcup\limits_{i=1}^{q} A_i = [1, pq]$, any subset B of $[1, pq]$ can be presented as $\bigcup\limits_{i=1}^{q} B_i$ with $B_i = B \cap A_i \subseteq A_i$. Denote by $\mathcal{S}_{1-}$ ($\mathcal{S}_{1+}$) the set of all subsets of $[1, pq]$ with no (at least one) B_i equal to A_i and by $\mathcal{S}_{0-}$ ($\mathcal{S}_{0+}$) the set of all subsets of $[1, pq]$ with no (at least one) B_i being empty. Then evidently $\mathcal{S}_{1+} \cap (\mathcal{S}_{1-} \cap \mathcal{S}_{0-}) = \emptyset$ and any set of $\mathcal{S}_{1+}$ has a non-empty intersection with any set of $\mathcal{S}_{1-} \cap \mathcal{S}_{0-}$. If $|\mathcal{S}_{1+}| \geq |\mathcal{S}_{1-} \cap \mathcal{S}_{0-}| \geq \lceil 2^{pq}/3 \rceil$, we can set $\mathcal{R} = \mathcal{S}_{1-} \cap \mathcal{S}_{0-}$ and employ in the role of $\mathcal{C}$ any subset of $\mathcal{S}_{1+}$ equipotent with $\mathcal{S}_{1-} \cap \mathcal{S}_{0-}$; Theorem 1 then yields $n_{pq} \geq |\mathcal{S}_{1-} \cap \mathcal{S}_{0-}|$. Counting sets of $\mathcal{S}_{1+}$ with respect to j, the number of p- element sets B_i, we have

$$\begin{aligned} |\mathcal{S}_{1+}| &= \sum_{j=1}^{q} \binom{q}{j} (2^p - 1)^{q-j} \\ &= \sum_{j=0}^{q} \binom{q}{j} (2^p - 1)^{q-j} - (2^p - 1)^q = 2^{pq} - (2^p - 1)^q\,; \end{aligned}$$

on the other hand

$$|\mathcal{S}_{1-} \cap \mathcal{S}_{0-}| = (2^p - 2)^q\,.$$

Let $q(p)$ be the minimum $q \in [2, \infty)$ fulfilling

$$2^{pq} - (2^p - 1)^q \geq (2^p - 2)^q\,,$$

or, equivalently,

$$1 \geq \left(1 - \frac{1}{2^p}\right)^q + \left(1 - \frac{2}{2^p}\right)^q;$$

it is well defined since

$$\left(1-\frac{1}{2^p}\right)+\left(1-\frac{2}{2^p}\right)=2-\frac{3}{2^p}>1$$

and

$$\lim_{q\to\infty}\left(\left(1-\frac{1}{2^p}\right)^q+\left(1-\frac{2}{2^p}\right)^q\right)=0.$$

Defining inequalities for $q(p)$ are

$$1<\left(1-\frac{1}{2^p}\right)^{q(p)-1}+\left(1-\frac{2}{2^p}\right)^{q(p)-1}=:U(p),$$

$$1\geq\left(1-\frac{1}{2^p}\right)^{q(p)}+\left(1-\frac{2}{2^p}\right)^{q(p)}=:L(p).$$

Thus, we have

$$0<U(p)-L(p)=\left(1-\frac{1}{2^p}\right)^{q(p)-1}\cdot\frac{1}{2^p}+\left(1-\frac{2}{2^p}\right)^{q(p)-1}\cdot\frac{2}{2^p}<\frac{1}{2^p}+\frac{1}{2^{p-1}},$$

since

$$\left(1-\frac{1}{2^p}\right)^{q(p)-1},\left(1-\frac{2}{2^p}\right)^{q(p)-1}\in(0,1)\quad\text{for any } p\in[2,\infty).$$

Consequently, from $\lim\limits_{p\to\infty}\left(\frac{1}{2^p}+\frac{1}{2^{p-1}}\right)=0$, we obtain

$$\lim_{p\to\infty}(U(p)-L(p))=0.$$

Moreover,

$$0<U(p)-1\leq U(p)-L(p),$$

$$0\leq 1-L(p)<U(p)-L(p),$$

so that

$$\lim_{p\to\infty}U(p)=\lim_{p\to\infty}L(p)=1.$$

It is well known that the sequence $\left\{\left(1-\frac{a}{n}\right)^n\right\}_{n=\lceil a\rceil}^{\infty}$ is increasing and converges to e^{-a} for any $a\in(0,\infty)$. That is why,

$$\left(1-\frac{1}{2^p}\right)^{2^p}+\left(1-\frac{2}{2^p}\right)^{2^p}<e^{-1}+e^{-2}<1,$$

which yields $q(p) \le 2^p$ and

$$0 < \frac{q(p)}{2^p} \le 1.$$

Now we are going to prove that $\lim\limits_{p\to\infty} \frac{q(p)}{2^p}$ exists by showing that all convergent subsequences of $\{\frac{q(p)}{2^p}\}_{p=2}^{\infty}$ have the same limit. Let $\{p_i\}_{i=0}^{\infty}$ be an increasing sequence of integers, $p_i \in [2,\infty)$, such that the sequence $\left\{\frac{q(p_i)}{2^{p_i}}\right\}_{i=0}^{\infty}$ is convergent. Then, putting

$$t := \lim_{i\to\infty} \frac{q(p_i)}{2^{p_i}},$$

we obtain $t \in \langle 0,1\rangle$ and

$$1 = \lim_{i\to\infty} L(p_i) = \lim_{i\to\infty}\left(\left(\left(1-\frac{1}{2^{p_i}}\right)^{2^{p_i}}\right)^{q(p_i)/2^{p_i}} + \left(\left(1-\frac{2}{2^{p_i}}\right)^{2^{p_i}}\right)^{q(p_i)/2^{p_i}}\right)$$

$$= e^{-t} + e^{-2t};$$

the latter equality uses the fact that $x^{-ay} : (0,\infty) \times \mathbb{R} \to (0,\infty)$ is a continuous function for any parameter $a \in \mathbb{R}$. As the equation $1 = x + x^2$ has exactly one positive solution, we have

$$e^{-t} = \frac{\sqrt{5}-1}{2},$$

which is equivalent to

$$t = \ln 2 - \ln(\sqrt{5}-1).$$

It means that

$$\lim_{p\to\infty} \frac{q(p)}{2^p} = \ln 2 - \ln(\sqrt{5}-1).$$

Consider sets $\mathcal{S}_{0-}$ and $\mathcal{S}_{1-}$ corresponding to parameters p and $q = q(p)$. Then $|\mathcal{S}_{1-} \cap \mathcal{S}_{0-}| = (2^p-2)^{q(p)}$ and, for p large enough, $|S_{1-} \cap \mathcal{S}_{0-}| \ge \lceil 2^{pq(p)}/3\rceil$. To check this inequality, necessary for the application of Theorem 1, we need to show that

$$(2^p-2)^{q(p)} > \frac{2^{pq(p)}}{3},$$

or, in other words,

$$\left(1-\frac{2}{2^p}\right)^{q(p)} > \frac{1}{3};$$

this follows immediately from

$$\lim_{p\to\infty}\left(1-\frac{2}{2^p}\right)^{q(p)} = e^{-2t} = \frac{3-\sqrt{5}}{2} > \frac{1}{3}.$$

Thus, for sufficiently large p, we obtain $n_{pq(p)} \geq (2^p-2)^{q(p)}$ and, with respect to $\lim\limits_{p\to\infty} q(p) = \infty$,

$$\lim_{k\to\infty} r_k = \lim_{p\to\infty} r_{pq(p)} \geq \lim_{p\to\infty}\frac{(2^p-2)^{q(p)}}{2^{pq(p)-1}}$$
$$= 2\lim_{p\to\infty}\left(1-\frac{2}{2^p}\right)^{q(p)} = 2e^{-2t} = 3-\sqrt{5}.$$

■

3. Concluding Remarks

Provided that $|\mathcal{S}_{0-}| \geq |\mathcal{S}_{0+}\cap\mathcal{S}_{1+}| \geq \lceil 2^{pq}/3\rceil$, in the proof of Theorem 2 we can take $\mathcal{R} = \mathcal{S}_{0+}\cap\mathcal{S}_{1+}$ and $\mathcal{C}\subseteq\mathcal{S}_{0-}$. It is interesting that then the same inequality for $\lim\limits_{k\to\infty} r_k$ is reached as in Section 2.

The first difference $n_{k+1}-2n_k$ contradicting Theorem 3.1 of [15] is $n_7-2n_6 = 2$. Suppose that there are integers p, q such that $n_{k+1}-2n_k \geq p$ and $n_{k+1}-2n_k \leq q$ for any $k\in[7,\infty)$. Then $n_7 = 46$ leads to

$$n_{7+i} \geq 46\cdot 2^i + p\sum_{j=0}^{i-1} 2^j = (46+p)\cdot 2^i - p,$$
$$n_{7+i} \leq (46+q)\cdot 2^i - q \qquad \text{for any } \ i\in[1,\infty),$$

so that

$$\lim_{k\to\infty} r_k \geq \lim_{i\to\infty}\tfrac{(46+p)\cdot 2^i-p}{2^{6+i}} = \frac{46+p}{64},$$
$$\lim_{k\to\infty} r_k \leq \frac{46+q}{64}.$$

Then $p = 19$ would be in contradiction with $\lim\limits_{k\to\infty} r_k \leq 1$, since $\frac{46+19}{64} > 1$ and $q = 2$ would contradict Theorem 2, since $\frac{46+2}{64} = 0.75 < 3-\sqrt{5}$. That is why, there exist $l, m \in [7, \infty)$ such that $n_{l+1} - 2n_l \leq 18$ and $n_{m+1} - 2n_m \geq 3$. A natural question arises: Do there exist $j \in [7, \infty)$ and $q \in [3, \infty)$ such that $n_{k+1} - 2n_k \leq q$ for all $k \in [j, \infty)$?

To find a non-trivial upper bound for $\lim\limits_{k\to\infty} r_k$ seems to be a very difficult task. We conjecture that $\lim\limits_{k\to\infty} r_k = 3 - \sqrt{5}$, but we do not even know whether this limit is less than 1.

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