GENERALIZED COLORINGS AND AVOIDABLE ORIENTATIONS*

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Abstract

Gallai and Roy proved that a graph is k-colorable if and only if it has an orientation without directed paths of length k. We initiate the study of analogous characterizations for the existence of generalized graph colorings, where each color class induces a subgraph satisfying a given (hereditary) property. It is shown that a graph is partitionable into at most k independent sets and one induced matching if and only if it admits an orientation containing no subdigraph from a family of $k + 3$ directed graphs.

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1. Introduction

In the late 1960s, Gallai [4] and Roy [7] proved that a graph is k-colorable (i.e., its vertex set can be partitioned into at most k independent sets) if and only if it admits an orientation of the edges such that every directed path has length at most $k - 1$. A characterization of similar flavor, in terms of avoidable types of orientations on cycles, was given by Minty [6]. As a common extension of those two classic results, the second author [8] proved that a graph is k-colorable ($k \geq 2$) if and only if it has an orientation such that, for all $\ell \equiv 1 \pmod{k}$, $\ell \geq 3$, every cycle of length ℓ contains more than ℓ/k oriented edges in each direction. For the *chromatic sum* (sometimes also called the *color cost*), a characterization with weights of directed paths was found by Caro [3], and his result was extended in [9] for a general coloring concept that includes, as particular cases, both the chromatic number and the chromatic sum.

   Motivated by those results, the following problem is raised among various questions in Section 3 of [9]:

Problem 1 [9]**.** *Let k be a positive integer. For which properties $\mathcal{P}$ does there exist a finite family $\mathcal{A}(\mathcal{P}, k)$ of directed graphs such that a graph can be (vertex) partitioned into at most k induced subgraphs of property $\mathcal{P}$ if and only if it admits an orientation containing no subdigraph isomorphic to any $\vec{A} \in \mathcal{A}(\mathcal{P}, k)$?*

More generally, we can put the analogous question where the combination of several properties is considered. Let $\mathcal{P}_1, \ldots, \mathcal{P}_k$ be graph properties such that each $\mathcal{P}_i$ is *hereditary* (if $G \in \mathcal{P}_i$ and $G' \subset G$, then $G' \in \mathcal{P}_i$ also holds) and *additive* ($G \in \mathcal{P}_i$ if and only if each connected component of G has property $\mathcal{P}_i$). A vertex partition $V_1 \cup \cdots \cup V_k = V$ of $G = (V, E)$ will be called a $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-*coloring* if the subgraph $G[V_i]$ of G induced by V_i satisfies property $\mathcal{P}_i$, for all $1 \leq i \leq k$. If such a partition exists, G is caled $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-*colorable*. This will also be written as $G \in \mathcal{P}_1 \circ \ldots \circ \mathcal{P}_k$; and if $\mathcal{P}_1 = \ldots = \mathcal{P}_k = \mathcal{P}$, then the shorthand $\mathcal{P}^k$ will be applied for $\mathcal{P}_1 \circ \ldots \circ \mathcal{P}_k$.

   A systematic study of hereditary and additive properties has been initiated by Mihók [5]. For an extensive account on the literature, we refer to the recent survey [1].

   For colorings with respect to non-identical properties, we now raise the following generalization of Problem 1.

Problem 2. *For which properties $\mathcal{P}_1, \ldots, \mathcal{P}_k$ does there exist a finite family $\mathcal{A} = \mathcal{A}(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ of directed graphs such that a graph has a $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-coloring if and only if it admits an orientation containing no subdigraph isomorphic to any $\vec{A} \in \mathcal{A}$?*

For short, let us call a graph G $\mathcal{A}$-*orientable* if admits an $\mathcal{A}$-*avoiding* orientation; i.e., an orientation without any subdigraph $\vec{A} \in \mathcal{A}$.

The aim of this paper is to make the first modest step towards the solution of Problem 2, investigating the following two properties:

$\mathcal{O}\quad :=\quad$ the graph has no edges,

$\mathcal{O}_1\quad :=\quad$ the graph has maximum degree 1.

Hence, $\mathcal{O}_1$ means that the graph consists of isolated edges and isolated vertices; and, with the notation introduced above, $G \in \mathcal{O}^k$ is equivalent to saying that the chromatic number of G is at most k.

The main result of this paper, Theorem 1, is a necessary and sufficient condition for the $(\mathcal{O}^k \circ \mathcal{O}_1)$-colorability of graphs. This characterization is formulated in the next section (where another — formally weaker but still equivalent — condition is also given), and will be proved in Section 3. Some related conjectures are presented in the concluding section.

2. AVOIDABLE ORIENTED SUBGRAPHS

In this section we describe the necessary and sufficient conditions for $G \in \mathcal{O}^k \circ \mathcal{O}_1$.

Consider the following collection $\mathcal{A}_k = \{\vec{A}_0, \vec{A}_1, \ldots, \vec{A}_{k+2}\}$ of $k + 3$ digraphs:

- $\vec{A}_0$ is the directed path $\vec{P}_{k+3}$ of length $k + 2$,
- for $1 \le i \le k + 1$, the digraph $\vec{A}_i$ is obtained from two directed paths $v_1 v_2 \cdots v_{k+2}$ and $v'_1 v'_2 \cdots v'_{k+2}$ by identifying v_j with v'_j for all $1 \le j < i$ and $j = k + 2$,
- $\vec{A}_{k+2}$ is the directed path $v_1 v_2 \cdots v_{k+1}$ with two pendant vertices v_{k+2} and v'_{k+2}, both dominated by v_{k+1}.

The digraph $\vec{A}_{k+2}$ may be veiwed as a construction similar to the graphs $\vec{A}_1, \ldots, \vec{A}_{k+1}$, obtained by the identification of v_j and v'_j for all $j < i = k+2$, with the only difference that v_{k+2} and v'_{k+2} are not identified in $\vec{A}_{k+2}$. The family $\mathcal{A}_2$ is exhibited in Figure 1.
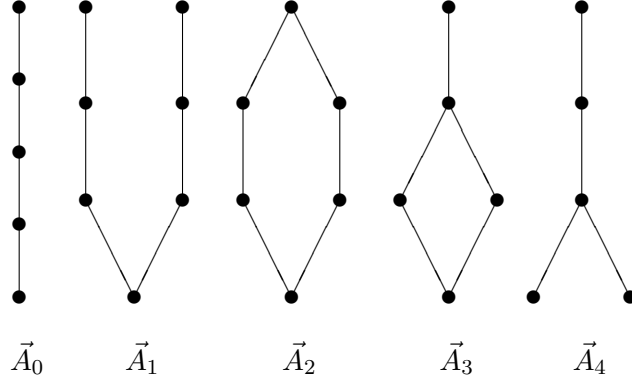
Figure 1. The family $\mathcal{A}_2$; avoidable subdigraphs for $\mathcal{O} \circ \mathcal{O} \circ \mathcal{O}_1$, orienting all edges downwards.

In the next section we shall prove the following result.

Theorem 1. *A graph is $(\mathcal{O}^k \circ \mathcal{O}_1)$-colorable if and only if it has an orientation containing no subdigraph $\vec{A} \in \mathcal{A}_k$.*

A similar but more restrictive and easier-to-prove necessary and sufficient condition can be obtained by considering the acyclic orientations only.

Proposition 1. *A graph is $(\mathcal{O}^k \circ \mathcal{O}_1)$-colorable if and only if it has an acyclic orientation containing no subdigraph $\vec{A} \in \mathcal{A}_k$.*

Proof of 'if'. The case of $k = 0$ (i.e., $\mathcal{O}_1$-colorability) is obvious, because excluding $\mathcal{A}_0$ (two adjacent edges with all the three possible orientations) yields a graph of maximum degree 1, even without the exclusion of directed circuits.

Let $k > 0$, and consider any graph $G = (V, E)$ with an $\mathcal{A}_k$-avoiding acyclic orientation $\vec{G}$. For $i = 1, \ldots, k$ we select the sets V_i recursively as follows. Setting $V'_0 := V$, and assuming that V'_{i-1} has already been defined (for some $i < k$), let $V_i \subseteq V'_{i-1}$ be the set of those vertices which have in-degree 0 in the induced subgraph $\vec{G}[V'_{i-1}]$. At the end, we define $V_{k+1} := V'_k$. Note that $V'_{i-1} \neq \emptyset$ implies $V_i \neq \emptyset$, because $\vec{G}$ is acyclic.

It is clear by definition that V_i is an independent set for all $1 \leq i \leq k$. Moreover, each $v \in V'_i$ of degree 0 in $\vec{G}[V'_i]$, $1 < i \leq k+1$ (and, in particular, each $v \in V_i$ for $1 < i \leq k$), is dominated by some $v' \in V_{i-1}$, because v has positive in-degree in $\vec{G}[V'_{i-1}]$.

We have to prove that $\vec{G}[V_{k+1}] \in \mathcal{O}_1$. Suppose on the contrary that V_{k+1} induces at least two adjacent edges, say a subgraph $\vec{A}^0 \in \mathcal{A}_0$ occurs. We can assume, without loss of generality, that the one or two vertices v_{k+1} of in-degree 0 in $\vec{A}^0$ have in-degree 0 also in $\vec{G}[V_{k+1}]$. (If this property is not valid in the $\vec{A}^0$ chosen originally, then $\vec{G}[V_{k+1}]$ always contains a suitable path of length 2.) By the observations above, for this (or, each of these two) v_{k+1}, we can choose a dominating vertex $v_k \in V_k$, then a vertex $v_{k-1} \in V_{k-1}$ dominating v_k, and so on, until the sequence of vertices $v_i \in V_i$ is chosen for all $1 \leq i \leq k$. After all, we obtain a subdigraph of $\vec{G}$ isomorphic to some $\vec{A} \in \mathcal{A}_k$, contradicting the assumption that the orientation is $\mathcal{A}_k$-avoiding. Consequently, G is indeed $(\mathcal{O}^k \circ \mathcal{O}_1)$-colorable.   ∎

The argument for the converse assertion ('only if') is identical to the one given for Theorem 1 in the next section.

Since the long directed paths are avoidable in either case, the acyclic characterization can be interpreted as one in terms of the avoidable family $\mathcal{A}'_k = \mathcal{A}_k \cup \{\vec{C}_i \mid 3 \leq i \leq k+3\}$, where $\vec{C}_i$ denotes the directed circuit of length i. In this way the condition of acyclicity is eliminated.

3. Proof for Unrestricted Orientations

In this section we prove Theorem 1.

Proof of avoidability. Suppose that $V_1 \cup \cdots \cup V_{k+1} = V$ is an $(\mathcal{O}^k \circ \mathcal{O}_1)$-coloring of the graph $G = (V, E)$, where the sets $V_1, \ldots, V_k$ are independent and V_{k+1} induces a subgraph of maximum degree 1. For $1 \leq i < j \leq k+1$, orient all edges joining V_i and V_j from the former to the latter; and, inside V_{k+1}, orient the edges arbitrarily. Denote by $\vec{G}$ the digraph obtained.

To see that this orientation is $\mathcal{A}$-avoiding, observe that in every directed path P, the set $V' := V_1 \cup \cdots \cup V_k$ meets P in a (possibly empty) subpath which is an initial segment of P, and V' and P share at most k vertices. As $G[V_{k+1}]$ contains no pair of consecutive edges, this implies $\vec{A}_0 \not\subset \vec{G}$.

Suppose next, for a contradiction, that $\vec{A}_\ell \subset \vec{G}$ for some $1 \leq \ell \leq k+2$. Since all edges are oriented from the partition classes V_i to V_j with a larger subscript, it follows that $v_{i+1}, v_{i+2}, v'_{i+1}, v'_{i+2} \in V_{i+1}$. But just one pair of those four vertices has been identified in $\vec{A}_\ell$, implying that $G[V_{k+1}]$ contains a vertex of in-degree at least 2 (for $\ell \in \{1, \ldots, k+1\}$) or out-degree at least 2 (for $\ell = k+2$). This contradiction proves that $\vec{G}$ is an $\mathcal{A}$-avoiding orientation of G.

It is immediately seen that the orientation defined above is acyclic, therefore the 'only if' part of Proposition 1 has been proved as well.

Proof of colorability. Suppose that $\vec{G}$ is an $\mathcal{A}$-avoiding orientation of the graph $G = (V, E)$. We define a vertex partition $V_1 \cup \cdots \cup V_{k+1}$ in the following way. First, consider all those oriented subforests $\vec{T} \subseteq \vec{G}$ (i.e., the subdigraphs whose edges form an acyclic subgraph T in G) in which every vertex has in-degree at most 1. In each such $\vec{T}$, for every vertex $v \in V$, denote by $h_T(v)$ the maximum number of vertices in a subpath of $\vec{T}$ that ends in v. (Actually, there is just one longest path ending in v, by the in-degree condition.) Assume $\vec{T}$ has been chosen such that

$$H_0 := \sum_{v \in V} h_T(v)$$

is as large as possible. Define now

$$v \in V_i \iff \begin{cases} 1 \le i \le k \ \wedge \ h_T(v) = i, \\ i = k+1 \ \wedge \ h_T(v) > k. \end{cases}$$

We are going to show that this partition is an $(\mathcal{O}^k \circ \mathcal{O}_1)$-coloring of G.

First, we prove that V_i is independent for all $1 \le i \le k$. Suppose on the contrary that $h_T(v) = h_T(v') \le k$ for some $vv' \in E$, and let the edge vv' be oriented from v to v' in $\vec{G}$. If v' has in-degree 0 in $\vec{T}$, then we add vv' to $\vec{T}$. We claim that $\vec{T} \cup vv'$ remains acyclic in G. Indeed, let $P = v_0 v_1 \cdots v_p \subset T$, $v_0 = v'$, $v_p = v$, be any undirected path. Since its first edge is oriented towards v_1, and $\vec{T}$ satisfies the in-degree condition, it follows by induction that P is in fact a directed path in $\vec{T}$. This would imply $h_T(v) = h_T(v') - p < h_T(v')$, however, contradicting our assumption. Thus, the edge vv' may be added to $\vec{T}$ without violating the conditions; but then the value of $h_T(v')$, and thus also of H_0, is increased, a contradiction to the choice of $\vec{T}$.

On the other hand, if v' has in-degree 1 in $\vec{T}$, then let v'' be the unique predecessor of v. We replace the edge $v''v'$ by vv' in $\vec{T}$. This transformation respects the in-degree condition, moreover it increases the value of $h_T(w)$ by 1 for every w to which there is a directed path from v' in $\vec{T}$ (including v' itself as well), because the first $h_T(v') - 1$ vertices of the longest path ending in w are replaced by the $h_T(v) = h_T(v')$ vertices of the longest path ending in v. Hence, a contradiction to the maximality of H_0 is obtained, implying that V_i is independent for all $i \le k$.

Consider now the subgraph G' induced by $V' := V_{k+1}$. Suppose, for a contradiction, that $G' \notin \mathcal{O}_1$; i.e., there exist three vertices $v, v', v'' \in V'$ such that one of the following cases holds:

1. $vv', v'v'' \in E(\vec{G})$
2. $vv', vv'' \in E(\vec{G})$,
3. $v'v, v''v \in E(\vec{G})$.

In the first two cases we may assume, without loss of generality, that $h_T(v) = k+1$; i.e., the predecessor of v in $\vec{T}$ belongs to V_k. Then v is the endpoint of a directed path P of length k, all of whose vertices different from v are contained in $V_1 \cup \cdots \cup V_k$. This P together with the corresponding two edges listed above would yield $\vec{A}_0$ (in Case 1) or $\vec{A}_{k+2}$ (in Case 2).

On the other hand, in Case 3, the predecessors of v' and v'' can be supposed to belong to V_k, otherwise V_{k+1} contains a directed path of length 2 and we are back to Case 1. Thus, there exist directed paths P' and P'' of length k that end in v' and v'', respectively, with all their vertices but v' and v'' being in $V_1 \cup \cdots \cup V_k$. Consider the subgraph $\vec{A}$ formed by the union of P', P'', the vertex v, and the two edges $v'v$ and $v''v$. If P' and P'' are vertex disjoint, then $\vec{A} \cong \vec{A}_1$; and if they share a vertex, say i is the largest subscript such that P' and P'' meet in V_i ($1 \leq i \leq k$), then $\vec{A}$ contains a subdigraph isomorphic to $\vec{A}_{i+1} \in \mathcal{A}_k$. This final contradiction completes the proof of Theorem 1.

4. Concluding Remarks and Open Problems

Concerning the characterization of generalized colorability in terms of finite families of avoidable digraphs, several natural questions arise. The one related most closely to the above results seems to be:

Problem 3. *Find a characterization for the property $\mathcal{O}^k \circ \mathcal{O}_1^\ell$ in terms of a finite set of avoidable digraphs, for $k \geq 0$, $\ell \geq 2$.*

We can show that $\mathcal{O}_1^2$ does have such a characterization, but we do not include it here because our avoidable family contains more than 20 digraphs, and perhaps there exists a more elegant description, also in the general case.

The comparison of Theorem 1 and Proposition 1 leads to the question whether there are properties $\mathcal{P}$ for which the unrestricted orientations behave differently from the acyclic ones; i.e., every avoidable family characterizing $\mathcal{P}$ is infinite, but there is a finite family $\mathcal{A}$ such that $G \in \mathcal{P}$ if and only if G admits an orientation containing no subdigraph $\vec{A} \in \mathcal{A} \cup \{\vec{C}_i \mid i \in \mathbb{N}\}$.

In the paper [10] it is shown that this is indeed the case: several graph classes demonstrate that a rather small avoidable family suffices for acyclic orientations, while the unrestricted case requires infinite avoidable families.

Generalizing the 'existence' aspect of Problem 3 concerning finite characterizations for the products of $\mathcal{O}$ and $\mathcal{O}_1$, we raise

Conjecture 1. *If both $\mathcal{P}$ and $\mathcal{Q}$ are characterizable by finite avoidable families, then so is $\mathcal{P} \circ \mathcal{Q}$ as well.*

The particular case $\mathcal{P} = \mathcal{Q}$ in this conjecture would also be of interest to settle.

Finally we note that, for any fixed family $\mathcal{A}$ of digraphs, the class of $\mathcal{A}$-orientable graphs is hereditary and additive. That is, if a graph G admits an $\mathcal{A}$-avoiding orientation, then so does every $G' \subset G$, and its connected components are $\mathcal{A}$-orientable if and only if so is the entire G. This simple remark motivates the following problem.

Conjecture 2. *If a family $\mathcal{A}(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ satisfying the requirements of Problem 2 exists, then $\mathcal{P}_i$ is hereditary and additive, for all $1 \le i \le k$.*

The validity of this conjecture would be implied immediately by an affirmative answer to the following more general problem, raised first by Mihók and later independently by the second author.

Problem 4. *Let $\mathcal{P}_1, \ldots, \mathcal{P}_k$ be graph properties such that $\mathcal{P}_1 \circ \ldots \circ \mathcal{P}_k$ is additive and hereditary. Is then $\mathcal{P}_i$ additive and hereditary for all $1 \le i \le k$.*

Clearly, if the answer is affirmative for $k = 2$, then the same holds for all k as well, by induction. The following related result has been proved by Bucko et al. [2]: for the so-called *degenerate* properties $\mathcal{P}$ (i.e., those for which there exist bipartite graphs not satisfying $\mathcal{P}$), the additivity of a property $\mathcal{Q}$ follows whenever both $\mathcal{P}$ and $\mathcal{P} \circ \mathcal{Q}$ are additive.

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