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ON A CHARACTERIZATION OF GRAPHS BY AVERAGE LABELLINGS

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Abstract

The additive hereditary property of linear forests is characterized by the existence of average labellings.

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1. Introduction

There are many results in graph theory dealing with the connection between the existence of some labelling of vertices and the combinatorial structure of graphs. For example, the Four Colour Theorem can be formulated as the implication: If a given graph G is planar, then there exists a labelling of the vertex set of G with some properties. Also converse implications are known, for instance: If a digraph D has a Grundy function, then D has a kernel [1]. The most valuable are characterizations of the structural properties of graphs by means of the existence of some labelling of vertices of a graph, e.g.: a graph G is magic if and only if every edge of G belongs to a $(1-2)$-factor and every couple of edges is separated by a $(1-2)$-factor [7]. In this paper, a new approach to the characterization of some additive hereditary properties of graphs by using the so called average labellings will be presented.

2. Results

We will investigate finite simple graphs (i.e., undirected without loops or multiple edges). A class of graphs $\mathcal{C}$ is said to be a *hereditary* property if $\mathcal{C}$ is closed under taking subgraphs. $\mathcal{C}$ is called *additive* if it is closed under taking unions of its disjoint elements. The lattice of additive hereditary properties of graphs is described in [2], [3], [8], [9]. Every such class of graphs is determined by its maximal connected graphs or by the minimal graphs which do not belong to $\mathcal{C}$ [4], [5]. In the following, we present another way to characterize some additive hereditary properties of graphs.

Let $G = (V, E)$ be a graph with the vertex set V and with E as the set of edges. Let N denote the set of positive integers.

A mapping $f : V \to N$ is said to be an *average labelling* of G if for every $v_1, v_2, v_3 \in V$ fulfilling $(v_1, v_2), (v_2, v_3) \in E$ the equality

$$f(v_2) = \frac{f(v_1) + f(v_3)}{2}$$

holds (cf. [6]). We will say that an average labelling of G is *nontrivial* if every connected component of G with the exception of isolated vertices has at least two differently labelled vertices.

Denote by $\mathcal{LF}$ the class of linear forests. The maximal connected graphs of $\mathcal{LF}$ are exactly paths and the minimal forbidden graphs are cycles and $K_{1,3}$ (see [9]).

Theorem 1. *Let G be a given graph with $E \neq \emptyset$. Then $G \in \mathcal{LF}$ if and only if there exists a nontrivial average labelling of G.*

Proof. If $G \in \mathcal{LF}$, then it suffices to label every component of G (a path P_n) consecutively by $1, 2, ..., n$ starting from a vertex of the least degree. Now, let f be a nontrivial average labelling of a graph $G = (V, E)$. First, we will show that G does not contain $K_{1,3}$. On the contrary, suppose v_1, v_2, v_3, c are pairwise different vertices of G and let $(v_i, c) \in E$ for each $i \in \{1, 2, 3\}$. Then

$$f(c) = \frac{f(v_1) + f(v_2)}{2} = \frac{f(v_1) + f(v_3)}{2}$$

and we have $f(v_2) = f(v_3)$ and then $f(c) = f(v_i)$ for $i \in \{1, 2, 3\}$. By the induction, it is easy to see that for every vertex $v \in V$ which belongs to the component of G containing c it is $f(v) = f(c)$. Second, G does not contain a cycle C_n. Suppose $C_n = (\{u_1, u_2, \ldots, u_n\}, \{(u_1, u_2), (u_2, u_3), \ldots, (u_{n-1}, u_n), (u_n, u_1)\})$ is a subgraph of G. Since f is nontrivial there exist two adjacent vertices of C_n with different values. Without loss of generality let $f(u_1) < f(u_2)$. But then

$$f(u_2) < 2.f(u_2) - f(u_1) = f(u_3) < 2.f(u_3) - f(u_2) = f(u_4) < \dots$$

$$\dots < 2.f(u_{n-1} - f(u_{n-2}) = f(u_n) < 2.f(u_n) - f(u_{n-1}) = f(u_1),$$

a contradiction. Thus $G \in \mathcal{LF}$. ■

Denote by $\mathcal{O}$ the class of all totally disconnected graphs (without edges) and by $\mathcal{F}$ the class of forests (without cycles). The class $\mathcal{O}$ has K_2 as the only minimal forbidden graph; in the class $\mathcal{F}$, the minimal forbidden graphs are the cycles of all lengths. Both classes are characterizable by means of certain average labellings as follows.

Let $G = (V, E)$ be a graph and $v \in V$. Let us denote by $N(v)$ the set $\{w \in V : (v, w) \in E\} \cup \{v\}$ and by $|N(v)|$ the number of elements of $N(v)$. A mapping $f : V \to N$ is said to be a *neighbour average labelling* of G if for every $v \in V$ the equality

$$f(v) = \frac{\sum_{w \in N(v)} f(w)}{|N(v)|}$$

holds.

Theorem 2. *Let G be a given graph. Then $G \in \mathcal{O}$ if and only if there exists an injective neighbour average labelling of G.*

Proof. If $G = (V, E) \in \mathcal{O}$, then an arbitrary injective mapping from V to N is convenient. Conversely, let f be an injective neighbour average labelling of G. Let us observe an arbitrary connected component $H = (W, F)$ of G and suppose that w is a vertex of this component such that $f(w) \geq f(v)$ whenever $v \in W$. Then for every $u \in N(w)$ the equality $f(u) = f(w)$ must be true, because otherwise

$$f(w) > \frac{\sum_{u \in N(w)} f(u)}{|N(w)|}.$$

Hence $N(w) = \{w\}$ and H is an isolated vertex. ■

A mapping $f : V \to N$ is said to be a *cycle average labelling* of G if for every incident pair of edges $(v_1, v_2), (v_2, v_3)$ belonging to some cycle of G the equality

$$f(v_2) = \frac{f(v_1) + f(v_3)}{2}$$

is true.

Theorem 3. *Let G be a given graph. Then $G \in \mathcal{F}$ if and only if there exists an injective cycle average labelling of G.*

Proof. If $G = (V, E)$ is a forest, then an arbitrary injective valuation of V by positive integers is a cycle average labelling of G. Conversely, let f be an injective cycle average labelling of G and let $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)$ be the edges of a cycle of G. Without loss of generality, let $f(v_1) > f(v_2)$. Since

$$f(v_2) = \frac{f(v_3) + f(v_1)}{2} > \frac{f(v_3) + f(v_2)}{2}$$

we obtain $f(v_2) > f(v_3)$. Analogously $f(v_i) > f(v_{i+1})$ for every $i \in \{1, 2, \ldots, n\}$ modulo n. Thus $f(v_1) > f(v_n) > f(v_1)$, a contradiction. ■

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