Discussiones Mathematicae
Graph Theory 17 (1997) 103–113

UNIQUELY PARTITIONABLE GRAPHS

Jozef Bucko

Department of Geometry and Algebra, P.J. Šafárik University
Jesenná 5, 041 54 Košice, Slovak Republic
e-mail: bucko@duro.upjs.sk

Marietjie Frick[1]

Department of Mathematics, Applied Mathematics and Astronomy
University of South Africa, P.O. Box 392, Pretoria, 0001 South Africa
e-mail: frickm@risc5.unisa.ac.za

Peter Mihók[2] and Roman Vasky[2]

Department of Geometry and Algebra, P.J. Šafárik University
Jesenná 5, 041 54 Košice, Slovak Republic
e-mail: mihok@kosice.upjs.sk
e-mail: vasky@duro.upjs.sk

Abstract

Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be properties of graphs. A $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of a graph G is a partition of the vertex set $V(G)$ into subsets $V_1, \ldots, V_n$ such that the subgraph $G[V_i]$ induced by V_i has property $\mathcal{P}_i$; $i = 1, \ldots, n$. A graph G is said to be uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable if G has exactly one $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition. A property $\mathcal{P}$ is called hereditary if every subgraph of every graph with property $\mathcal{P}$ also has property $\mathcal{P}$. If every graph that is a disjoint union of two graphs that have property $\mathcal{P}$ also has property $\mathcal{P}$, then we say that $\mathcal{P}$ is additive. A property $\mathcal{P}$ is called degenerate if there exists a bipartite graph that does not have property $\mathcal{P}$. In this paper, we prove that if $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are degenerate, additive, hereditary properties of graphs, then there exists a uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graph.

Keywords: hereditary property of graphs, additivity, reducibility, vertex partition.

1991 Mathematics Subject Classification: 05C15, 05C70.

---

[1]Research supported by the South African Foundation for Research Development.
[2]Research supported in part by the Slovak VEGA grant.

1. Notation and Background

All graphs considered in this paper are finite and simple. In general, we follow the notation and terminology of [15].

We denote the set of all mutually nonisomorphic graphs by $\mathcal{I}$. Each nonempty subset $\mathcal{P} \subseteq \mathcal{I}$ is also said to be a *property of graphs.* A property $\mathcal{P}$ is said to be *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$. A property $\mathcal{P}$ is *additive* if $G_1, G_2 \in \mathcal{P}$ implies that the disjoint union $G_1 \cup G_2$ is also in $\mathcal{P}$. We shall denote the set of all hereditary properties by $\mathbb{L}$, and the set of all additive, hereditary properties by $\mathbb{L}^a$. We list some additive, hereditary properties in Table 1. (We use the notation of [6] for most of them).

Table 1

The property	The graphs which have the property
$\mathcal{O}$	$G \in \mathcal{I}$; G is totally disconnected
$\mathcal{S}_k$	$G \in \mathcal{I}$; $\Delta(G) \leq k$
$\mathcal{W}_k$	$G \in \mathcal{I}$; the length of the longest path in G does not exceed k
$\mathcal{D}_k$	$G \in \mathcal{I}$; G is k-degenerate i.e., $\delta(H) \geq k$ for $H \subseteq G$
$\mathcal{T}_k$	$G \in \mathcal{I}$; G contains no subgraph homeomorphic to K_{k+2} or $K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}$
$\mathcal{I}_k$	$G \in \mathcal{I}$; G does not contain K_{k+2} as a subgraph

Any hereditary property $\mathcal{P}$ is uniquely determined by the set

$$\boldsymbol{F}(\mathcal{P}) = \{G \in \mathcal{I} |\ G \notin \mathcal{P} \text{ but each proper subgraph of } G \text{ belongs to } \mathcal{P}\}$$

of *minimal forbidden subgraphs* (see [6], [14], [16], [18]), or by the set of so-called $\mathcal{P}$-*maximal* graphs

$$\boldsymbol{M}(\mathcal{P}) = \{G \in \mathcal{P} | G + e \notin \mathcal{P} \text{ for every } e \in \overline{G}\},$$

(see [6], [24], [29]).

The *join* of two vertex disjoint graphs G_1 and G_2 is obtained by joining every vertex of G_1 to every vertex of G_2, and is denoted by $G_1 + G_2$.

A graph G is said to be $\mathcal{P}$-*strict* if $G \in \mathcal{P}$ and $G + K_1 \notin \mathcal{P}$.

Let $\mathcal{P}$ be a hereditary property, $\mathcal{P} \neq \mathcal{I}$. Then there is a nonnegative integer $c(\mathcal{P})$ such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \notin \mathcal{P}$, called the *completeness* of $\mathcal{P}$. Clearly, every $\mathcal{P}$-maximal graph G with $|V(G)| \geq c(\mathcal{P}) + 1$ is $\mathcal{P}$-strict.

For any property $\mathcal{P}$ we define the minimum degree of $\mathcal{P}$ as

$$\delta(\mathcal{P}) = \min\{\delta(G)|\ G \in \boldsymbol{F}(\mathcal{P})\},$$

and the chromatic number of $\mathcal{P}$ as

$$\chi(\mathcal{P}) = \min\{\chi(G)|\ G \in \boldsymbol{F}(\mathcal{P})\}.$$

Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be properties of graphs. A $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of a graph G is a partition $\{V_1, \ldots, V_n\}$ of $V(G)$ such that the subgraph $G[V_i]$ induced by V_i has property $\mathcal{P}_i$ for $i = 1, \ldots, n$. The property $\mathcal{R} = \mathcal{P}_1 \circ \ldots \circ \mathcal{P}_n$ is defined as the set of all graphs that have a $(\mathcal{P}_1, \ldots \mathcal{P}_n)$-partition.

If $\mathcal{P}_1 = \cdots = \mathcal{P}_n$, the property $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$ will be denoted by $\mathcal{P}^n$. For example, the class of all n-colourable graphs is denoted by $\mathcal{O}^n$.

If there exist properties $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$, then $\mathcal{R}$ is said to be a *reducible* property and $\mathcal{P}$, and $\mathcal{Q}$ are said to *divide* $\mathcal{R}$; otherwise $\mathcal{R}$ is called *irreducible* (see e.g., [6], [20], [23]). Different generalizations of regular colouring of the vertices of graphs (see e.g. [1], [2], [8], [9], [10], [11], [12], [20], [21], [22], [25], [26], [31]) can be expressed using the notion of reducible properties.

We shall need the following two lemmas concerning reducible properties.

Lemma 1. *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are (additive) hereditary properties of graphs, then the property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ is also (additive) hereditary.*

Lemma 2. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be hereditary properties of graphs and let G be a $\mathcal{P}_1 \circ \mathcal{P}_2$-maximal graph. If $\{V_1, V_2\}$ is any $(\mathcal{P}_1, \mathcal{P}_2)$-partition of G, then*

$$G = G[V_1] + G[V_2]$$

and the graph $G[V_i]$ are $\mathcal{P}_i$-maximal, $i = 1, 2$.

Proof. Suppose that there exists an edge $e = (x, y)$ such that $x \in V_1$ and $y \in V_2$ and $e \notin E(G)$. Then the graph $G + e \in \mathcal{P}_1 \circ \mathcal{P}_2$, contradicting our assumption that $G \in \boldsymbol{M}(\mathcal{P}_1 \circ \mathcal{P}_2)$. This proves that $G = G[V_1] + G[V_2]$.

Now suppose $G[V_1]$ is not $\mathcal{P}_1$-maximal. Then $G[V_1] + e \in \mathcal{P}_1$ for some $e \in E(\overline{G[V_1]})$. But then, again, $G + e \in \mathcal{P}_1 \circ \mathcal{P}_2$. This contradiction proves that $G[V_1]$ is $\mathcal{P}_1$-maximal. Likewise, $G[V_2]$ is $\mathcal{P}_2$-maximal. ∎

A graph $G \in \mathcal{P}_1\circ\ldots\circ\mathcal{P}_n$ is said to be *uniquely* $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$*-partitionable* if G has exactly one $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. The set of all uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graphs will be denoted by $\boldsymbol{U}(\mathcal{P}_1\circ\ldots\circ\mathcal{P}_n)$, e.g., $\boldsymbol{U}(\mathcal{O}^n)$ denotes the set of all uniquely n-colourable graphs (see [4], [19], [17]); $\boldsymbol{U}(\mathcal{S}_k^n)$ denotes the set of all uniquely $(m, k)^{\Delta}$-colourable graphs (see [12], [13], [32]); $\boldsymbol{U}(\mathcal{W}_k^n)$ has been studied in [12], [3] and $\boldsymbol{U}(\mathcal{D}_k^n)$ in [5], [27], and $\boldsymbol{U}(\mathcal{I}_k^n)$ in [7], [12] The basic properties of $\boldsymbol{U}(\mathcal{P}^n)$ have been investigated in [5], [23]. Another generalization of uniquely colourable graphs was introduced by X. Zhu in [33].

The notion of *degenerate* hereditary property appeared with regards to the famous Erdös-Simonovits formula

$$\text{ext}(n, \mathcal{P}) = \frac{\chi(\mathcal{P}) - 2}{\chi(\mathcal{P}) - 1}\binom{n}{2} + o(n^2),$$

where

$$\text{ext}(n, \mathcal{P}) = \max\{|E(G)| \mid G \in \mathcal{P} \text{ and } |V(G)| = n\},$$

A property $\mathcal{P} \in \mathbb{L}^a$ is said to be *degenerate* if $\chi(\mathcal{P}) = 2$, i.e., if $\boldsymbol{F}(\mathcal{P})$ contains some bipartite graph (see [29], [30]). Obviously, $\mathcal{O}, \mathcal{S}_k, \mathcal{Q}_k, \mathcal{O}_k, \mathcal{D}_k$ and $\mathcal{T}_k$ are degenerate properties of graphs, but the property $\mathcal{I}_k$ is not degenerate.

In [23] it is proved that if the property $\mathcal{P}$ is a reducible property of graphs, then $\boldsymbol{U}(\mathcal{P}^n) = \emptyset$ and we also proved that $\boldsymbol{U}(\mathcal{P}^n) \neq \emptyset$ for every degenerate property $\mathcal{P}$, which means that every degenerate property is irreducible. In Section 4 of this paper, we generalize this result by proving that $\boldsymbol{U}(\mathcal{P}_1\circ\ldots\circ\mathcal{P}_n) \neq \emptyset$ if $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are degenerate, additive, hereditary properties.

In Section 2 we present some basic properties of uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graphs, generalizing results known to hold for uniquely colourable graphs.

In Section 3 we provide a necessary and sufficient condition for one hereditary property to be divisible by another. This result is used to prove our main result, Theorem 3, which gives a necessary and sufficient condition for the existence of uniquely $\mathcal{P}\circ\mathcal{Q}$-partitionable graphs, when $\mathcal{P}$ and $\mathcal{Q}$ are additive, hereditary properties and $\mathcal{Q}$ is degenerate.

2. Basic Properties of Uniquely Partitionable Graphs

The results on uniquely $\mathcal{P}^n$-partitionable graphs obtained in [23] can be directly generalized to obtain the properties of uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graphs presented in the following two theorems.

Theorem 1. *Let $\mathcal{P}_1 \ldots, \mathcal{P}_n$ be hereditary properties of graphs, $n \geq 2$. If G is a uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graph and $\{V_1, \ldots, V_n\}$ is the unique $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of $V(G)$, then*

1. $G \notin \mathcal{P}_1 \circ \ldots \mathcal{P}_{j-1} \circ \mathcal{P}_{j+1} \circ \ldots \circ \mathcal{P}_n$ *for* $j = 1, \ldots, n$,
2. *the subgraphs* $G[V_i]$ *are* $\mathcal{P}_i$*-strict,* $i = 1, 2, \ldots, n$,
3. *if* $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, *then* $V_{i_1} \cup \ldots \cup V_{i_k}$ *induces a uniquely* $(\mathcal{P}_{i_1}, \ldots, \mathcal{P}_{i_n})$*-partitionable subgraph of* G,
4. $\delta(G) \geq \max\limits_{j} \sum\limits_{i=1,\, i \neq j}^{n} \delta(\mathcal{P}_i)$,
5. $|V(G)| \geq \sum\limits_{i=1}^{n} (c(\mathcal{P}_i) + 2) - 1$,
6. *the graph* $G = G[V_1] + \cdots + G[V_n]$ *is uniquely* $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$*-partitionable.*

Theorem 2. *Let $\mathcal{P}_1, \ldots \mathcal{P}_n$ be hereditary properties of graphs. If $G \in \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ and $\boldsymbol{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n) \neq \emptyset$, then G is an induced subgraph of some uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graph.*

3. Divisibility and Uniquely $(\mathcal{P}, \mathcal{Q})$-Partitionable Graphs

Lemma 3. *If $\mathcal{P}$ and $\mathcal{Q}$ are properties of graphs such that one of the following holds:*

1. $\mathcal{P}$ *divides* $\mathcal{Q}$
2. $\mathcal{Q}$ *divides* $\mathcal{P}$
3. *there exists a property* $\mathcal{S}$ *such that* $\mathcal{S}$ *divides both* $\mathcal{P}$ *and* $\mathcal{Q}$,

then $\boldsymbol{U}(\mathcal{P} \circ \mathcal{Q}) = \emptyset$.

Proof. 1. Suppose $\mathcal{Q} = \mathcal{P} \circ \mathcal{Q}^*$ for some property $\mathcal{Q}^*$. Let $G \in \mathcal{P} \circ \mathcal{Q}$ and let $\{V_1, V_2\}$ be a $(\mathcal{P}, \mathcal{Q})$-partition of G, with $V_1, V_2 \neq \emptyset$. Since $G[V_2] \in \mathcal{Q} = \mathcal{P} \circ \mathcal{Q}^*$, there exists a partition $\{V_{21}, V_{22}\}$ of $G[V_2]$, with $V_{21}, V_{22} \neq \emptyset$, such that $G[V_{21}] \in \mathcal{P}$ and $G[V_{22}] \in \mathcal{Q}^*$. But then $G[V_1 \cup V_{22}] \in \mathcal{P} \circ \mathcal{Q}^* = Q$, and thus $\{V_{21}, V_1 \cup V_{22}\}$ is a $(\mathcal{P}, \mathcal{Q})$-partition of G different from $\{V_1, V_2\}$, which implies that G is not uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable.

Cases (2) and (3) can be proved in an analogous way. ■

If $\mathcal{P}$ and $\mathcal{Q}$ are additive hereditary properties and $\mathcal{Q}$ is also degenerate, then converse of Lemma 1 also holds. In order to prove this, we introduce the concept of an extendible set.

Let $\mathcal{P}$ and $\mathcal{Q}$ be hereditary properties of graphs and let $G \in \mathcal{P}$. If S is a subset of $V(G)$ such that $G[S] \in \mathcal{Q}$ and for every graph $T \in \mathcal{Q}$ the graph $T + (G - S) \in \mathcal{P}$, then S is said to be a $(\mathcal{Q}, \mathcal{P})$-*extendible set* of G.
We shall need the following lemma.

Lemma 4. *Let $\mathcal{P}$ and $\mathcal{Q}$ be hereditary properties of graphs. If H is a graph with property $\mathcal{P}$ that has no $(\mathcal{Q}, \mathcal{P})$-extendible set, then there exists a $\mathcal{P}$-strict graph G such that G has no $(\mathcal{Q}, \mathcal{P})$-extendible set.*

Proof. Let H be a graph with property $\mathcal{P}$ such that H has no $(\mathcal{Q}, \mathcal{P})$-extendible set. Let G be a $\mathcal{P}$-strict graph such that $H \subseteq G$. Suppose, to the contrary, that G contains a $(\mathcal{Q}, \mathcal{P})$-extendible set S. Let $S' = S \cap V(H)$. Let T be any graph with property $\mathcal{Q}$. Then

$$T + (G - S) \in \mathcal{P}.$$

Since $T + (H - S') \subseteq T + (H - S)$ and $\mathcal{P}$ is hereditary, this implies that $T + (H - S') \in \mathcal{P}$, so that S' is an extendible set of H. ■

We have the following connection between divisibility and the existence of an extendible set.

Theorem 3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be hereditary properties of graphs. Then $\mathcal{Q}$ divides $\mathcal{P}$ if and only if every $\mathcal{P}$-maximal graph contains a $(\mathcal{Q}, \mathcal{P})$-extendible set.*

Proof. Suppose $\mathcal{Q}$ divides $\mathcal{P}$. Then there is a property $\mathcal{P}^*$ such that $\mathcal{P} = \mathcal{Q} \circ \mathcal{P}^*$. Let $G \in \mathcal{P}$ and let $\{V_1, V_2\}$ be a $(\mathcal{Q}, \mathcal{P}^*)$-partition of G. Let T be any graph with property $\mathcal{Q}$. Then $\{V(T), V_2\}$ is a $(\mathcal{Q}, \mathcal{P}^*)$-partition of $T + G[V_2]$, and hence $T + G[V_2] \in \mathcal{P}$. Since $G[V_2] = G - V_1$, this proves that V_1 is a $(\mathcal{Q}, \mathcal{P})$-extendible set of G.

To prove the converse, suppose every $\mathcal{P}$-maximal graph contains a $(\mathcal{Q}, \mathcal{P})$-extendible set. Let

$$\mathcal{S}(G) = \{S \subseteq V(G) | S \text{ is an extendible set of } G\}$$

and put

$$\mathcal{P}' = \{G - S | G \in \boldsymbol{M}(\mathcal{P}),\, S \in \mathcal{S}(G)\}.$$

Now let $\mathcal{P}^*$ be the property consisting of all subgraphs of graphs in $\mathcal{P}'$. Then $\mathcal{P}^*$ is a hereditary property. We shall prove that $\mathcal{P} = \mathcal{Q} \circ \mathcal{P}^*$.

Suppose $G \in \boldsymbol{M}(\mathcal{P})$. Then, by our assumption, G has a $(\mathcal{Q}, \mathcal{P})$-extendible set. Let S be such a set. Then $G - S \in \mathcal{P}^*$, by the definition of $\mathcal{P}^*$. Thus $\{S, G - S\}$ is a $(\mathcal{Q}, \mathcal{P}^*)$-partition of G, so that $G \in \mathcal{Q}\circ\mathcal{P}^*$. This proves that $\boldsymbol{M}(\mathcal{P}) \subseteq \mathcal{Q}\circ\mathcal{P}^*$. But $\mathcal{Q}\circ\mathcal{P}^*$ is a hereditary property by Lemma 2, and hence $\mathcal{P} \subseteq \mathcal{Q}\circ\mathcal{P}^*$.

Now suppose $G \in \boldsymbol{M}(\mathcal{Q}\circ\mathcal{P}^*)$. Let $\{V_1, V_2\}$ be a $(\mathcal{Q}, \mathcal{P}^*)$-partition of G. Then it folllows from Lemma 2 that $G[V_1] \in \boldsymbol{M}(\mathcal{Q})$, $G[V_2] \in \boldsymbol{M}(\mathcal{P}^*)$ and $G = G[V_1] + G[V_2]$. By the definition of $\mathcal{P}^*$ there exists a $\mathcal{P}$-maximal graph F and a $(\mathcal{Q}, \mathcal{P})$-extendible set S of F such that $G[V_2] \subseteq F - S$. But then, since $G[V_1] \in \mathcal{Q}$, we have $G[V_1] + F - S \in \mathcal{P}$. But $G \subseteq G[V_1] + F - S$, and hence $G \in \mathcal{P}$. This proves that $\mathcal{Q}\circ\mathcal{P}^* \subseteq \mathcal{P}$. ■

Theorem 4. *Let $\mathcal{P}, \mathcal{Q} \in \mathbb{L}^a$ and let $\mathcal{Q}$ be a degenerate property. Then $\boldsymbol{U}(\mathcal{Q}\circ\mathcal{P}) \neq \emptyset$ if and only if $\mathcal{Q}$ does not divide $\mathcal{P}$.*

Proof. If $\boldsymbol{U}(\mathcal{Q}\circ\mathcal{P}) \neq \emptyset$ then, by Lemma 3, $\mathcal{Q}$ does not divide $\mathcal{P}$.
To prove the converse, suppose $\mathcal{Q}$ does not divide $\mathcal{P}$. Then it follows from Theorem 3 and Lemma 4 that there exists a $\mathcal{P}$-strict graph H that contains no $(\mathcal{Q}, \mathcal{P})$-extendible set. Let

$$Z = \{S | S \subseteq V(H) \text{ and } H[S] \in \mathcal{Q}\}.$$

Then, for every $S \in Z$, there exists a $\mathcal{Q}$-strict graph $T(S)$ such that

$$T(S) + (H - S) \notin \mathcal{P}.$$

Now let

$$T = \cup_{S \in Z} T(S).$$

Since $\mathcal{Q}$ is a degenerate property, there is an integer q such that $K_{q,q} \notin \mathcal{Q}$. Let

$$G_1 = qT, \ \ G_2 = qH, \text{ and } G = G_1 + G_2.$$

Since $\mathcal{P}$ and $\mathcal{Q}$ are additive properties, $G_1 \in \mathcal{P}$ and $G_2 \in \mathcal{Q}$, and thus $G \in \mathcal{P}\circ\mathcal{Q}$.

Now let $\{W_1, W_2\}$ be any $(\mathcal{P}, \mathcal{Q})$-partition of G. Suppose each of the q copies of H in G_2 has at least one vertex in W_1. Then

$$|V(G_2) \cap W_1| \geq q.$$

Now let H_0 be a specific copy of H in G_2, and let $S_0 = V(H_0) \cap W_1$. Then $H_0[S_0] \in \mathcal{Q}$ and hence, by the definition of T, we have

$$T + H_0 - S_0 \notin \mathcal{P}.$$

Since $V(H_0) - S_0 \in W_2$, it follows that none of the q copies of T in G_1 has all its vertices in W_2. Thus

$$|V(G_1) \cap W_1)| \geq q.$$

But then $K_{q,q} \subseteq G[W_1]$. This contradiction proves that at least one of the q copies of H in G_2 has all its vertices in W_2. Since H is $\mathcal{P}$-strict, it follows that $W_2 \cap V(G_1) = \emptyset$. But G_1 is $\mathcal{Q}$-strict, and hence $W_1 = G(V_1)$, which implies that $\{V(G_1), V(G_2)\}$ is the only $(\mathcal{Q}, \mathcal{P})$-partition of G. Thus $G \in \boldsymbol{U}(\mathcal{Q}\circ\mathcal{P})$. ■

4. Construction of Uniquely $(\mathcal{P}_1, \ldots \mathcal{P}_n)$-Partitionable Graphs for Degenerate Properties

Uniquely $(\mathcal{P}^n)$-partitionable graphs have been proved to exist for several specific degenerate properties $\mathcal{P}$ (see [5], [23], [28]). The following theorem generalizes those results.

Theorem 5. *Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$, be degenerate, additive, hereditary properties of graphs. Then there exists a uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graph.*

Proof. We may assume, without loss of generality, that the properties $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are ordered in such a way that $\mathcal{P}_i \not\subset \mathcal{P}_j$ if $i < j$ and, if $\mathcal{P}_i = \mathcal{P}_j$ and $i < k < j$, then $\mathcal{P}_i = \mathcal{P}_k$. Then there exist graphs $H_1, \ldots, H_n$ such that H_i is $\mathcal{P}_i$-strict for $i = 1, \ldots, n$ and, if $i < j$, then $H_i \notin \mathcal{P}_j$ unless $\mathcal{P}_i = \mathcal{P}_j$. Since $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are degenerate properties, there exists an integer q such that $K_{q,q} \notin \mathcal{P}_i$ for $i = 1, \ldots, n$. Now let

$$G_i = (n(q-1)+1)H_i \text{ for } i = 1, \ldots, n.$$

and

$$G = G_1 + \cdots + G_n.$$

We shall prove, by induction on n, that the graph G thus constructed is uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable.

The result is true for $n = 1$.

Now let $n \geq 2$. Put

$$V_i = V(G_i), \quad i = 1, \ldots, n.$$

Let $\{W_1, \ldots, W_n\}$ be any $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of G. Since $|V(G_i)| \geq n(q-1)+1$ for each $i = 1, \ldots, n$, we have that, for each $i \in \{1, \ldots, n\}$

$$|V_i \cap W_j| \geq q \text{ for at least one } j \in \{1, \ldots, n\}.$$

Now suppose two different members of $\{W_1, \ldots, W_n\}$ each contain at least q vertices of V_1. Then there are at least $n+1$ sets of the form $V_i \cap W_j$ whose cardinality is at least q. Then, by Dirichlet's principle, there exist integers $i, r, s \in \{1, \ldots, n\}$ with $r \neq s$ such that

$$|W_i \cap V_r| \geq q \text{ and } |W_i \cap V_s| \geq q,$$

and thus

$$K_{q,q} \subseteq G[W_i].$$

This contradiction proves that only one of the W_i, say W_t, contains at least q vertices of V_1. Since $G[V_1]$ contains $n(q-1)+1$ copies of H_1, at least one of these copies has all its vertices in W_t. Our assumption on the ordering of the properties $\mathcal{P}_1, \ldots, \mathcal{P}_n$, implies that $H_1 \notin \mathcal{P}_i$ for $i = 2, \ldots, n$ unless $\mathcal{P}_i = \mathcal{P}_1$. We may therefore assume, without loss of generality, that $t = 1$. Since H_1 is $\mathcal{P}_1$-strict, it then follows that

$$W_1 \cap V_i = \emptyset \text{ for } i = 2, \ldots, n.$$

By our induction hypothesis, the graph $G_2 + \cdots + G_n$ is uniquely $(\mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable, so that $\{V_2, \ldots, V_n\}$ is the only $(\mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of $G_2 + \cdots + G_n$. Thus, for each $i \in \{2, \ldots, n\}$, we have that $W_i \supseteq V_i$ and hence, since G_i is $\mathcal{P}_i$-strict, $W_i = V_i$. This implies that $\{W_1, \ldots, W_n\}$ $=\{V_1, \ldots, V_n\}$, and hence G is uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable. ∎

References

[1] J.A. Andrews and M.S. Jacobson, *On a generalization of chromatic number*, Congressus Numerantium **47** (1985) 33–48.

[2] G. Benadé, I. Broere and J.I. Brown, *A construction of uniquely C_4-free colourable graphs*, Questiones Mathematicae **13** (1990) 259–264.

[3] G. Benadé, I. Broere, B. Jonck and M. Frick, *Uniquely $(m,k)^\tau$-colourable graphs and $k-\tau$-saturated graphs*, Discrete Math. **162** (1996) 13–22.

[4] B. Bollobás and N. Sauer, *Uniquely colourable graphs with large girth*, Canad. J. Math. **28** (1976) 1340–1344.

[5] B. Bollobás and A.G. Thomason, *Uniquely partitionable graphs*, J. London Math. Soc. **16** (1977) 403–410.

[6] M. Borowiecki and P. Mihók, *Hereditary properties of graphs*, in: V.R. Kulli, eds., Advances in Graph Theory, (Vishwa International Publication, Gulbarga 1991) 42–69.

[7] I. Broere and M. Frick, *On the order of uniquely* (k,m)*-colourable graphs*, Discrete Math. **82** (1990) 225–232.

[8] J.I. Brown and D.G. Corneil, *On generalized graph colourings*, J. Graph Theory **11** (1987) 86–99.

[9] J.I. Brown, D. Kelly, J. Schoenheim and R.E. Woodrow, *Graph coloring satisfying restraints*, Discrete Math. (1990) 123–143.

[10] S.A. Burr and M.S. Jacobson, *On inequalities involving vertex-partition parameters of graphs*, Congressus Numerantium **70** (1990) 159–170.

[11] L.J. Cowen, R.H. Cowen and D.R. Woodall, *Defective colorings of graphs in surfaces; partitions into subgraphs of bounded valency*, J. Graph Theory **10** (1986) 187–195.

[12] M. Frick, *A survey of* (m,k)*-colourings*, in: J. Gimbel c.a, eds., Quo Vadis, Graph Theory? Annals of Discrete Mathematics **55** (North-Holland, Amsterdam, 1993) 45–58.

[13] M. Frick and M.A. Henning, *Extremal results on defective colorings of graph*, Discrete Math. **126** (1994) 151–158.

[14] D. Gernet, *Forbidden and unavoidable subgraphs*, Ars Combinatoria **27** (1989) 165–176.

[15] R.L. Graham, M. Grötschel and L. Lovász, Handbook of Combinatorics (Elsevier Science B.V., Amsterdam 1995).

[16] D.L. Greenwell, R.L. Hemminger and J. Klerlein, *Forbidden subgraphs*, in: Proc. 4th S-E Conf. Combinatorics, Graph Theory and Computing, (Utilitas Math., Winnipeg, Man., 1973) 389–394.

[17] F. Harary, S.T. Hedetniemi and R.W. Robinson, *Uniquely colourable graphs*, J. Combin. Theory **6** (1969) 264–270.

[18] G. Chartrand, D. Geller and S. Hedetniemi, *Graphs with forbidden subgraphs*, J. Combin. Theory (B) **10** (1971) 12–41.

[19] G. Chartrand and J.P. Geller, *Uniquely colourable planar graphs*, J. Combin. Theory **6** (1969) 271–278.

[20] T.R. Jensen and B. Toft, Graph Colouring Problems (Wiley-Interscience Publications, New York 1995).

[21] R.P. Jones, *Hereditary properties and* P*-chromatic number*, in: T.P. McDonough and V.C. Marron, eds., Combinatorics, Proc. Brit. Comb. Conf. (London Math. Soc. Lecture Note Ser., No.13, Cambridge Univ. Press, London 1974) 83–88.

[22] L. Lovász, *On decomposition of graphs*, Studia Sci. Math. Hungar **1** (1966) 237–238.

[23] P. Mihók, *Additive hereditary properties and uniquely partitionable graphs*, in: M. Borowiecki and Z. Skupien, eds., Graphs, Hypergraphs and Matroids (Zielona Góra, 1985) 49–58.

[24] J. Mitchem, *Maximal k-degenerate graphs*, Utilitas Math. **11** (1977) 101–106.

[25] F.S. Roberts, *From garbage to rainbows: generalizations of graph colouring and their applications*, in: Y. Alavi, G. Chartrand, O.R. Oellermann, and A.J. Schwenk, eds., Graph Theory, Combinatorics and Applications, (Wiley, New York, 1991) 1031–1052.

[26] F.S. Roberts, *New directions in graph theory* (*with an emphasis on the role of applications*), in: J. Gimbel, J.W. Kennedy, and L.V. Quintas, eds., Quo Vadis Graph Theory, (North-Holland, Amsterdam, 1993) 13–44.

[27] J.M.S. Simões-Pereira, *Joins of n-degenerate graphs and uniquely (m,n)-partitionable graphs*, J. Combin. Theory (B) **21** (1976) 21–29.

[28] J.M.S. Simões-Pereira, *On graphs uniquely partitionable into n-degenerate subgraphs*, in: Infinite and Finite Sets Colloquia Math. Soc. J. Bólyai **10** (North-Holland, Amsterdam, 1975) 1351–1364.

[29] M. Simonovits, *Extremal graph theory*, in: L.W. Beineke and R.J. Wilson, eds., Selected Topics in Graph Theory, **2** (Academic Press, London, 1983) 161–200.

[30] M. Simonovits, *Extremal graph problems and graph products*, in: Studies in Pure Math. (dedicated to the memory of P. Turán) (1983) 669–680.

[31] M. Weaver and D.B. West, *Relaxed chromatic numbers of graphs*, Graphs and Combinatorics **10** (1994) 75–93.

[32] D. Woodall, *Improper colorings of graphs*, in: R. Nelson and R.J. Wilson, eds., Graph Colorings (Longman, New York, 1990) 45–64.

[33] X. Zhu, *Uniquely H-colorable graphs with large girth*, J. Graph Theory **23** (1996) 33–41.

Received 3 January 1997
Revised 25 April 1997