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PARTITIONS OF SOME PLANAR GRAPHS INTO TWO LINEAR FORESTS

Piotr Borowiecki

and

Mariusz Hałuszczak

Institute of Mathematics
Technical University of Zielona Góra
Podgórna 50, 65–246 Zielona Góra, Poland

e-mail: p.borowiecki@im.pz.zgora.pl
m.haluszczak@im.pz.zgora.pl

Abstract

A *linear forest* is a forest in which every component is a path. It is known that the set of vertices $V(G)$ of any outerplanar graph G can be partitioned into two disjoint subsets V_1, V_2 such that induced subgraphs $\langle V_1\rangle$ and $\langle V_2\rangle$ are linear forests (we say G has an $(\mathcal{LF}, \mathcal{LF})$-partition). In this paper, we present an extension of the above result to the class of planar graphs with a given number of *internal vertices* (i.e., vertices that do not belong to the external face at a certain fixed embedding of the graph G in the plane). We prove that there exists an $(\mathcal{LF}, \mathcal{LF})$-partition for any plane graph G when certain conditions on the degree of the internal vertices and their neighbourhoods are satisfied.

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1. Introduction and Notation

Let $\mathcal{I}$ denote the set of all finite simple graphs. A *graph property* $\mathcal{P}$ is a nonempty isomorphism-closed subclass of $\mathcal{I}$. We also say that a graph G has the property $\mathcal{P}$ if $G \in \mathcal{P}$. A property $\mathcal{P}$ of graphs is said to be (*induced*) *hereditary* if whenever $G \in \mathcal{P}$ and H is a (vertex induced) subgraph of G, then also $H \in \mathcal{P}$. A property $\mathcal{P}$ is called *additive* if for each graph G all of whose components have the property $\mathcal{P}$ it follows that G has the property $\mathcal{P}$, too. A hereditary property $\mathcal{P}$ can be characterized in terms of forbidden subgraphs. The set of *minimal forbidden subgraphs* of $\mathcal{P}$ is defined as follows:

$$\boldsymbol{F}(\mathcal{P}) = \{G \in \mathcal{I} : G \notin \mathcal{P} \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P}\}.$$

In general, we use the notation and terminology of [1]. Let us mention selected hereditary properties of graphs:

$$\begin{aligned}
\mathcal{O} &= \{G \in \mathcal{I} : G \text{ is edgeless, i.e., } E(G) = \emptyset\},\\
\mathcal{T}_k &= \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2}\\
&\qquad \text{or } K_{\lfloor\frac{k+3}{2}\rfloor,\lceil\frac{k+3}{2}\rceil}\},\\
\mathcal{D}_k &= \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate}\},\\
\mathcal{S}_k &= \{G \in \mathcal{I} : \Delta(G) \le k\}.
\end{aligned}$$

It is easy to see that $\mathcal{D}_1 = \mathcal{T}_1 = \{G : G \text{ is a forest}\}$, $\mathcal{LF} = \mathcal{D}_1 \cap \mathcal{S}_2$ is the linear forest, while $\mathcal{T}_2$ and $\mathcal{T}_3$ are the classes of all outerplanar and all planar graphs, respectively. For $\mathcal{LF}$ the set of minimal forbidden subgraphs is given by

$$\boldsymbol{F}(\mathcal{LF}) = \{K_{1,3}, C_n \text{ with } n \ge 3\}.$$

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, $n > 1$ be any properties and let G belong to $\mathcal{I}$. A *vertex* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partition* of the graph G is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that each subgraph $\langle V_i \rangle$ of the graph G induced by V_i has the property $\mathcal{P}_i, i = 1, 2, \ldots, n$. A problem of partitioning planar graphs into linear forests has been extensively studied in many papers. Broere [3], Wang [8] and Mihók [6] proved that any outerplanar graph has an $(\mathcal{LF}, \mathcal{LF})$-partition. Some extensions of the result given above and an algorithm can be found in [2]. The result of Poh [7] and Goddard [4] is that any planar graph has an $(\mathcal{LF}, \mathcal{LF}, \mathcal{LF})$- partition (i.e., into three linear forests).

2. Results

Let W be a subset of the vertex set $V(G)$ such that $\langle W \rangle$ is connected. By the operation of *contraction of the vertex set* W to the vertex u we will understand the removal of all the vertices belonging to W, addition of a new vertex u and all the edges required to satisfy the following condition $\mathrm{N}(u) = \bigcup_{w \in W} \mathrm{N}(w)$, where $\mathrm{N}(v)$ denotes the neighbourhood of the vertex v in G.

Let us define a set Int(G) of all internal vertices of a planar graph G as a set of vertices not belonging to the external face at a certain fixed embedding of the graph G in the plane. Let int(G) $=$ min $|\mathrm{Int}(G)|$ over all embeddings of the graph G in the plane. If int(G) $= 0$, then the graph G is outerplanar.

Theorem 1. *Let G be a plane graph and $v \in V(G)\backslash\mathrm{Int}(G)$ an arbitrarily chosen vertex. If the following conditions are satisfied:*

(i) *for any $x, y \in \mathrm{Int}(G)$, $(x, y) \notin E(G)$,*
(ii) *for any vertex $x \in \mathrm{Int}(G)\backslash\mathrm{N}(v)$, $d(x) > 4$,*
(iii) *for any vertex $x \in \mathrm{Int}(G) \cap \mathrm{N}(v)$, $d(x) > 3$,*

then there exists a (V_1, V_2)-partition of $V(G)$ such that $\langle V_i\rangle \in \mathcal{LF}$ for $i = 1, 2$ and $v \in V_1$, $\mathrm{N}(v) \subseteq V_2$.

Proof. Without loss of generality, we assume that G is maximal in the sense that graph $G + e$ does not satisfy one of the conditions (i)–(iii). The proof is by induction on the order of G. Let $|V(G)| = 3$. Then the Theorem is true. Assume that the Theorem holds for all graphs of order less than k. Let $|V(G)| = k$. Let the graph G^* be obtained from G by contraction of the set $\mathrm{N}[v]$ $=\mathrm{N}(v) \cup \{v\}$ to the vertex w. We are going to prove that the graph G^* satisfies conditions (i)–(iii).

Claim 1. The graph G^* satisfies conditions (i)–(iii).

Proof. The proof falls into three cases.

Case 1. It is easy to see that for any $x, y \in \mathrm{Int}(G^*)$ if $(x, y) \notin E(G)$, then $(x, y) \notin E(G^*)$, too. Thus the condition (i) is satisfied.

Case 2. From the definition of contraction of the set $\mathrm{N}[v]$ to the vertex w, it immediately follows that a degree of any vertex $x \in \mathrm{Int}(G)$ such that $\mathrm{N}(v)\cap \mathrm{N}(x) = \emptyset$ cannot be affected and $d_G(x) = d_{G^*}(x)$. Thus, for any $x \in \mathrm{Int}(G^*)\backslash\mathrm{N}(w)$, $d(x) > 4$ and the condition (ii) is satisfied.

Case 3. If for the vertex v there exists a vertex $x \in \mathrm{Int}(G)$ such that $d_G(x) > 4$ and $\mathrm{N}(v)\cap \mathrm{N}(x) \neq \emptyset$, then $|\mathrm{N}(v)\cap \mathrm{N}(x)| \leq 2$. If $x \notin \mathrm{N}(v)$, then an operation of contraction of the set $\mathrm{N}(v)$ may decrease the degree of the vertex x by at most 1. If $x \in \mathrm{N}(v)$, then x will be contracted to the vertex $w \notin \mathrm{Int}(G^*)$. Thus, for any $x \in \mathrm{Int}(G^*)\cap \mathrm{N}(w)$, $d(x) > 3$ and the condition (iii) is satisfied.

Hence, we get the graph G^* which satisfies conditions (i)–(iii).

Since $|V(G^*)| < k$ then G^* has a (V_1^*, V_2^*)-partition of $V(G^*)$ such that $\langle V_i^* \rangle \in \mathcal{LF}$ for $i = 1, 2$ and $w \in V_1^*$, $\mathrm{N}(w) \subseteq V_2^*$. Let $V_1 = V_2^* \cup \{v\}$, $V_2 = (V_1^* \backslash \{w\}) \cup \mathrm{N}(v)$. We are going to prove that V_1 and V_2 have the property $\mathcal{LF}$.

Claim 2. $\langle \mathrm{N}(v) \rangle_G$ has the property $\mathcal{LF}$.

Proof. Assuming that $H = \langle \mathrm{N}(v) \rangle_G$ has not the property $\mathcal{LF}$ implies that H contains a cycle or a vertex of degree greater than 2. Thus, we have the following cases:

Case 1. Let us assume that H contains a cycle C_k of length $k \geq 3$. Since, $C_k + \langle \{v\} \rangle$, where $+$ denotes the join, contains a subgraph homeomorphic to K_4, then it is not outerplanar. Thus, there exists a vertex $x \in \mathrm{N}(v)$ such that $x \in \mathrm{Int}(G)$. From (iii) it follows that $d_G(x) > 3$, which implies an existence of the vertex y such that $(x, y) \in E(G)$ and $y \in \mathrm{Int}(G)$, contrary to (i).

Case 2. Let us assume that there exists a vertex $u \in V(H)$ such that $d_H(u) > 2$ (i.e., H contains $K_{1,3}$ as a subgraph). Since $K_{1,3} + \langle \{v\} \rangle$ is not outerplanar, then there exists a vertex $x \in \mathrm{N}(u)$ such that $x \in \mathrm{Int}(G)$. From (iii) it follows that $d_G(x) > 3$, thus there exists a vertex y such that $(x, y) \in E(G)$ and $y \in \mathrm{Int}(G)$, contrary to (i).

Thus, $\langle \mathrm{N}(v) \rangle_G$ has the property $\mathcal{LF}$.

Since $\mathrm{N}(w) \subseteq V_2^*$, then no vertex from $\mathrm{N}(v)$ has the neighbour in the set $V_1^* \backslash \{w\}$. Hence, as V_1^* and $\langle \mathrm{N}(v) \rangle_G$ both have the property $\mathcal{LF}$, it comes out that V_2 has the property $\mathcal{LF}$, too. Obviously, V_1 belongs to $\mathcal{LF}$ and v has no adjacent vertex in V_1. Thus, the partition (V_1, V_2) is the required $(\mathcal{LF}, \mathcal{LF})$-partition of G. ■

Corollary 1. *If a graph G is outerplanar, then for every vertex $v \in V(G)$ there exists an $(\mathcal{LF}, \mathcal{LF})$-partition of G, say (V_1, V_2), such that $v \in V_1$ and* $\mathrm{N}(v) \subseteq V_2$.

Theorem 2. *Let G be a plane graph, $R \subseteq V(G)$ and* $\mathrm{Int}(G)$ *be a proper subset of R. If a subgraph of the graph G induced by R is a path, then the graph G has an $(\mathcal{LF}, \mathcal{LF})$-partition.*

Proof. Contracting the set R to a vertex w, we get an outerplanar graph G^*. Hence, G^* has an $(\mathcal{LF}, \mathcal{LF})$-partition (V_1^*, V_2^*) of $V(G^*)$ such that $w \in V_1^*$ and $\mathrm{N}(w) \subseteq V_2^*$. No vertex from R has a neighbour in the set $V_1^*\backslash\{w\}$. Let $V_1 = V_1^*\backslash\{w\} \cup R$, $V_2 = V_2^*$. Since $\langle V_1^*\rangle_G$ and $\langle R\rangle_G$ both have the $\mathcal{LF}$ property, then V_1 belongs to $\mathcal{LF}$, too. Obviously, (V_1, V_2) is an $(\mathcal{LF}, \mathcal{LF})$-partition of G. ■

Corollary 2. *Let G be a maximal outerplanar graph with an outer-cycle C. Let $P \leq C$ be an induced path of C. Then G has an $(\mathcal{LF}, \mathcal{LF})$-partition (V_1, V_2) of $V(G)$ such that $V(P) \subseteq V_1$.*

Theorem 3. *Every planar graph G with* $\mathrm{int}(G) \leq 2$ *has an $(\mathcal{LF}, \mathcal{LF})$-partition.*

Proof. The proof falls naturally into three cases.

Case 1. $\mathrm{int}(G) = 0$.
If $\mathrm{int}(G) = 0$, then the graph G is outerplanar and it has an $(\mathcal{LF}, \mathcal{LF})$-partition.

Case 2. $\mathrm{int}(G) = 1$.
Let $v \in \mathrm{Int}(G)$ and $u \in \mathrm{N}(v)$. It is easy to notice that $u \notin \mathrm{Int}(G)$. According to Theorem 2, if $R = \{v, u\}$, then the graph G has an $(\mathcal{LF}, \mathcal{LF})$-partition.

Case 3. $\mathrm{int}(G) = 2$.
We can consider a maximal plane graph G with $\mathrm{Int}(G) = \{r_1, r_2\}$.

Subcase 3.1. r_1 is adjacent to r_2.
There exists a vertex u adjacent to r_1 such that u is not adjacent to r_2. According to Theorem 2, if $R = \{u, r_1, r_2\}$, then the graph G has an $(\mathcal{LF}, \mathcal{LF})$-partition.

Subcase 3.2. r_1 is not adjacent to r_2.
Let R contain all the vertices belonging to the shortest path from r_1 to r_2. Since $\langle R\rangle_G$ is a path and R contains at least one vertex not belonging to $\mathrm{Int}(G)$, i.e., $\mathrm{Int}(G)$ is a proper subgraph of R, then according to Theorem 2, the graph G has an $(\mathcal{LF}, \mathcal{LF})$-partition. ■

Theorem 4. *Let G be a planar graph of order $n \leq 9$ with* $\mathrm{int}(G) = 3$. *Then G has an $(\mathcal{LF}, \mathcal{LF})$-partition.*

Proof. If $\mathrm{int}(G) \leq 2$, then by Theorem 3, independently of the order n, the graph G has an $(\mathcal{LF}, \mathcal{LF})$-partition.
Let us consider a planar graph with $\mathrm{int}(G) = 3$, where $\mathrm{Int}(G) = \{r_1, r_2, r_3\}$. Without loss of generality we assume that G is a near-triangulation, i.e., G is a plane graph which consists of an outer-cycle $C_k : v_1 v_2 \ldots v_k v_1$ in

clockwise order and vertices and edges inside C_k such that each bounded face is bounded by a triangle. Since the graph G is of order $n \leq 9$ and $\mathrm{int}(G) = 3$, then $3 \leq k \leq 6$. Let $S(r_i) = \mathrm{N}(r_i) \cap V(C_k)$ be the set of the vertices of the cycle C_k that are adjacent to the vertex $r_i \in \mathrm{Int}(G)$ and $s_i = |S(r_i)|$. Then we have to consider four cases. Cases 1 and 2 are considered under the assumption that C_k is a chordless cycle. But, if there is a chord, then it divides the graph G into G_1 and G_2, such that $\mathrm{Int}(G_1) = \{r_1, r_2, r_3\}$ and G_2 is outerplanar. Then an $(\mathcal{LF}, \mathcal{LF})$-partition of G_1 can be easily extended to G.

Case 1. $\langle \mathrm{Int}(G) \rangle = K_3$.
Let the vertices r_1, r_2, r_3 be in clockwise order and $s_1 \leq s_2$ and $s_1 \leq s_3$. If $S(r_1) = \{v_1, \ldots, v_{s_1}\}$, $S(r_2) = \{v_{s_1}, \ldots, v_{s_1+s_2-1}\}$, $S(r_3) = \{v_{s_1+s_2-1}, \ldots, v_1\}$, then the partition (V_1, V_2) can be obtained as follows:

$$V_1 = \{r_3, v_1, \ldots, v_{s_1+s_2-2}\}\ ,\ V_2 = \{r_1, r_2, v_{s_1+s_2-1}, \ldots, v_k\}.$$

Case 2. $\langle \mathrm{Int}(G) \rangle = P_3$.
Let $\mathrm{N}(r_2) \supseteq \{r_1, r_3\}$ and $s_1 \leq s_3$. Then we have two subcases.

Case 2.1. $s_3 = 2$.
If $s_3 = 2$, then there exists a vertex $v \in C_k$ such that $v \in \mathrm{N}(r_1) \cap \mathrm{N}(r_2) \backslash \mathrm{N}(r_3)$. Then $V_1 = \{r_2, r_3, v\}$ and $V_2 = \{r_1\} \cup (C_k \backslash \{v\})$.

Case 2.2. $s_3 > 2$.
If $s_3 > 2$, then there exists a vertex $v \in C_k$ such that $v \in \mathrm{N}(r_3) \backslash (\mathrm{N}(r_1) \cup \mathrm{N}(r_2))$. Then $V_1 = \{r_1, r_2, r_3, v\}$ and $V_2 = C_k \backslash \{v\}$.

Case 3. $\langle \mathrm{Int}(G) \rangle = \overline{K_3}$.
It is easy to see that for any graph G considered in this case $|V(G)| \geq 8$ and the cycle C_k has at least two chords. If $|V(G)| = 9$, then we have eight graphs. Their $(\mathcal{LF}, \mathcal{LF})$-partitions are shown in Figure 1.

There is only one graph H such that $|V(H)| = 8$. It is easy to see that H is a subgraph of two graphs presented at the bottom line of Figure 1.

Case 4. $\langle \mathrm{Int}(G) \rangle = \overline{P_3}$.
In this case C_k has at least one chord. Thus, G is divided by this chord into two graphs G_i with $\mathrm{int}(G_i) = i$, $i = 1, 2$. Each of them has an $(\mathcal{LF}, \mathcal{LF})$-partition which can be extended to the other one. The details are left to the reader. ■

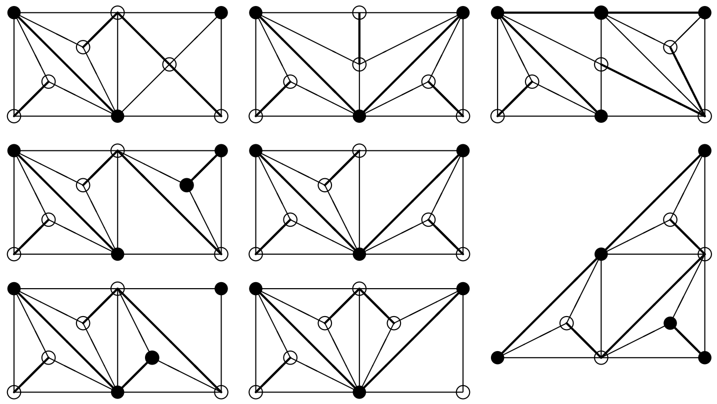

Figure 1

Theorem 5. *For every integer $n \geq 10$ there exists a planar graph G of order n with* $\operatorname{int}(G) = 3$*, which does not have an* $(\mathcal{LF}, \mathcal{LF})$*-partition.*

Proof.

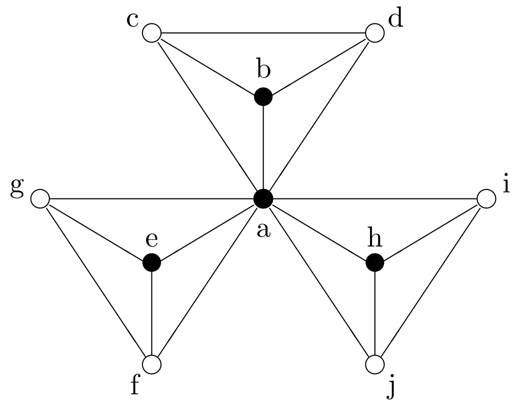

Figure 2

Let us partition the set $V(G)$ of the graph G in Figure 2 into two subsets V_1 and V_2. Assuming that $a \in V_1$, at least one of the vertices from $\{b, c, d\}, \{e, f, g\}$ and $\{h, i, j\}$ must belong to V_1. Otherwise, $\langle V_2 \rangle_G$ would have contained a cycle. But the set V_1 constructed in this way induces a subgraph containing a vertex of a degree greater than 2. Thus, any planar graph containing the graph G as its subgraph does not have an $(\mathcal{LF}, \mathcal{LF})$-partition. ∎

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