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PARTITIONS OF SOME PLANAR GRAPHS INTO TWO LINEAR FORESTS

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Abstract

A linear forest is a forest in which every component is a path. It is known that the set of vertices V(G) of any outerplanar graph Gcan be partitioned into two disjoint subsets V_1, V_2 such that induced subgraphs $\langle V_1 \rangle$ and $\langle V_2 \rangle$ are linear forests (we say G has an $(\mathcal{LF}, \mathcal{LF})$ partition). In this paper, we present an extension of the above result to the class of planar graphs with a given number of *internal vertices* (i.e., vertices that do not belong to the external face at a certain fixed embedding of the graph G in the plane). We prove that there exists an $(\mathcal{LF}, \mathcal{LF})$ -partition for any plane graph G when certain conditions on the degree of the internal vertices and their neighbourhoods are satisfied.

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1. INTRODUCTION AND NOTATION

Let \mathcal{I} denote the set of all finite simple graphs. A graph property \mathcal{P} is a nonempty isomorphism-closed subclass of \mathcal{I} . We also say that a graph Ghas the property \mathcal{P} if $G \in \mathcal{P}$. A property \mathcal{P} of graphs is said to be (*induced*) hereditary if whenever $G \in \mathcal{P}$ and H is a (vertex induced) subgraph of G, then also $H \in \mathcal{P}$. A property \mathcal{P} is called *additive* if for each graph G all of whose components have the property \mathcal{P} it follows that G has the property \mathcal{P} , too. A hereditary property \mathcal{P} can be characterized in terms of forbidden subgraphs. The set of *minimal forbidden subgraphs* of \mathcal{P} is defined as follows:

$$\mathbf{F}(\mathcal{P}) = \{ G \in \mathcal{I} : G \notin \mathcal{P} \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P} \}.$$

In general, we use the notation and terminology of [1]. Let us mention selected hereditary properties of graphs:

 $\begin{aligned} \mathcal{O} &= \{ G \in \mathcal{I} : G \text{ is edgeless, i.e., } E(G) = \emptyset \}, \\ \mathcal{T}_k &= \{ G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2} \\ & \text{or } K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil} \}, \\ \mathcal{D}_k &= \{ G \in \mathcal{I} : G \text{ is } k\text{-degenerate} \}, \\ \mathcal{S}_k &= \{ G \in \mathcal{I} : \Delta(G) \leq k \}. \end{aligned}$

It is easy to see that $\mathcal{D}_1 = \mathcal{T}_1 = \{G : G \text{ is a forest}\}, \mathcal{LF} = \mathcal{D}_1 \cap \mathcal{S}_2$ is the linear forest, while \mathcal{T}_2 and \mathcal{T}_3 are the classes of all outerplanar and all planar graphs, respectively. For \mathcal{LF} the set of minimal forbidden subgraphs is given by

$$\boldsymbol{F}(\mathcal{LF}) = \{K_{1,3}, C_n \text{ with } n \ge 3\}.$$

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n, n > 1$ be any properties and let G belong to \mathcal{I} . A vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$ -partition of the graph G is a partition (V_1, V_2, \ldots, V_n) of V(G) such that each subgraph $\langle V_i \rangle$ of the graph G induced by V_i has the property $\mathcal{P}_i, i = 1, 2, \ldots, n$. A problem of partitioning planar graphs into linear forests has been extensively studied in many papers. Broere [3], Wang [8] and Mihók [6] proved that any outerplanar graph has an $(\mathcal{LF}, \mathcal{LF})$ -partition. Some extensions of the result given above and an algorithm can be found in [2]. The result of Poh [7] and Goddard [4] is that any planar graph has an $(\mathcal{LF}, \mathcal{LF}, \mathcal{LF})$ - partition (i.e., into three linear forests).

2. Results

Let W be a subset of the vertex set V(G) such that $\langle W \rangle$ is connected. By the operation of *contraction of the vertex set* W to the vertex u we will understand the removal of all the vertices belonging to W, addition of a new vertex u and all the edges required to satisfy the following condition $N(u) = \bigcup_{w \in W} N(w)$, where N(v) denotes the neighbourhood of the vertex v in G. Let us define a set Int(G) of all internal vertices of a planar graph G as a set of vertices not belonging to the external face at a certain fixed embedding of the graph G in the plane. Let int(G) = min |Int(G)| over all embeddings of the graph G in the plane. If int(G) = 0, then the graph G is outerplanar.

Theorem 1. Let G be a plane graph and $v \in V(G) \setminus Int(G)$ an arbitrarily chosen vertex. If the following conditions are satisfied:

- (i) for any $x, y \in \text{Int}(G)$, $(x, y) \notin E(G)$,
- (ii) for any vertex $x \in Int(G) \setminus N(v)$, d(x) > 4,
- (iii) for any vertex $x \in Int(G) \cap N(v), d(x) > 3$,

then there exists a (V_1, V_2) -partition of V(G) such that $\langle V_i \rangle \in \mathcal{LF}$ for i = 1, 2and $v \in V_1$, $N(v) \subseteq V_2$.

Proof. Without loss of generality, we assume that G is maximal in the sense that graph G + e does not satisfy one of the conditions (i)–(iii). The proof is by induction on the order of G. Let |V(G)| = 3. Then the Theorem is true. Assume that the Theorem holds for all graphs of order less than k. Let |V(G)| = k. Let the graph G^* be obtained from G by contraction of the set $\mathbb{N}[v] = \mathbb{N}(v) \cup \{v\}$ to the vertex w. We are going to prove that the graph G^* satisfies conditions (i)–(iii).

Claim 1. The graph G^* satisfies conditions (i)–(iii).

Proof. The proof falls into three cases.

Case 1. It is easy to see that for any $x, y \in \text{Int}(G^*)$ if $(x, y) \notin E(G)$, then $(x, y) \notin E(G^*)$, too. Thus the condition (i) is satisfied.

Case 2. From the definition of contraction of the set $\mathbb{N}[v]$ to the vertex w, it immediately follows that a degree of any vertex $x \in \mathrm{Int}(G)$ such that $\mathbb{N}(v) \cap \mathbb{N}(x) = \emptyset$ cannot be affected and $d_G(x) = d_{G^*}(x)$. Thus, for any $x \in \mathrm{Int}(G^*) \setminus \mathbb{N}(w)$, d(x) > 4 and the condition (ii) is satisfied.

Case 3. If for the vertex v there exists a vertex $x \in \text{Int}(G)$ such that $d_G(x) > 4$ and $N(v) \cap N(x) \neq \emptyset$, then $|N(v) \cap N(x)| \leq 2$. If $x \notin N(v)$, then an operation of contraction of the set N(v) may decrease the degree of the vertex x by at most 1. If $x \in N(v)$, then x will be contracted to the vertex $w \notin \text{Int}(G^*)$. Thus, for any $x \in \text{Int}(G^*) \cap N(w)$, d(x) > 3 and the condition (iii) is satisfied.

Hence, we get the graph G^* which satisfies conditions (i)–(iii).

Since $|V(G^*)| < k$ then G^* has a (V_1^*, V_2^*) -partition of $V(G^*)$ such that $\langle V_i^* \rangle \in \mathcal{LF}$ for i = 1, 2 and $w \in V_1^*$, $N(w) \subseteq V_2^*$. Let $V_1 = V_2^* \cup \{v\}$, $V_2 = (V_1^* \setminus \{w\}) \cup N(v)$. We are going to prove that V_1 and V_2 have the property \mathcal{LF} .

Claim 2. $\langle N(v) \rangle_G$ has the property \mathcal{LF} .

Proof. Assuming that $H = \langle N(v) \rangle_G$ has not the property \mathcal{LF} implies that H contains a cycle or a vertex of degree greater than 2. Thus, we have the following cases:

Case 1. Let us assume that H contains a cycle C_k of length $k \ge 3$. Since, $C_k + \langle \{v\} \rangle$, where + denotes the join, contains a subgraph homeomorphic to K_4 , then it is not outerplanar. Thus, there exists a vertex $x \in N(v)$ such that $x \in Int(G)$. From (iii) it follows that $d_G(x) > 3$, which implies an existence of the vertex y such that $(x, y) \in E(G)$ and $y \in Int(G)$, contrary to (i).

Case 2. Let us assume that there exists a vertex $u \in V(H)$ such that $d_H(u) > 2$ (i.e., H contains $K_{1,3}$ as a subgraph). Since $K_{1,3} + \langle \{v\} \rangle$ is not outerplanar, then there exists a vertex $x \in N(u)$ such that $x \in Int(G)$. From (iii) it follows that $d_G(x) > 3$, thus there exists a vertex y such that $(x, y) \in E(G)$ and $y \in Int(G)$, contrary to (i).

Thus, $\langle \mathbf{N}(v) \rangle_G$ has the property \mathcal{LF} .

Since $N(w) \subseteq V_2^*$, then no vertex from N(v) has the neighbour in the set $V_1^* \setminus \{w\}$. Hence, as V_1^* and $\langle N(v) \rangle_G$ both have the property \mathcal{LF} , it comes out that V_2 has the property \mathcal{LF} , too. Obviously, V_1 belongs to \mathcal{LF} and v has no adjacent vertex in V_1 . Thus, the partition (V_1, V_2) is the required $(\mathcal{LF}, \mathcal{LF})$ -partition of G.

Corollary 1. If a graph G is outerplanar, then for every vertex $v \in V(G)$ there exists an $(\mathcal{LF}, \mathcal{LF})$ -partition of G, say (V_1, V_2) , such that $v \in V_1$ and $N(v) \subseteq V_2$.

Theorem 2. Let G be a plane graph, $R \subseteq V(G)$ and Int(G) be a proper subset of R. If a subgraph of the graph G induced by R is a path, then the graph G has an $(\mathcal{LF}, \mathcal{LF})$ -partition.

Proof. Contracting the set R to a vertex w, we get an outerplanar graph G^* . Hence, G^* has an $(\mathcal{LF}, \mathcal{LF})$ -partition (V_1^*, V_2^*) of $V(G^*)$ such that $w \in V_1^*$ and $N(w) \subseteq V_2^*$. No vertex from R has a neighbour in the

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set $V_1^* \setminus \{w\}$. Let $V_1 = V_1^* \setminus \{w\} \cup R$, $V_2 = V_2^*$. Since $\langle V_1^* \rangle_G$ and $\langle R \rangle_G$ both have the \mathcal{LF} property, then V_1 belongs to \mathcal{LF} , too. Obviously, (V_1, V_2) is an $(\mathcal{LF}, \mathcal{LF})$ -partition of G.

Corollary 2. Let G be a maximal outerplanar graph with an outer-cycle C. Let $P \leq C$ be an induced path of C. Then G has an $(\mathcal{LF}, \mathcal{LF})$ -partition (V_1, V_2) of V(G) such that $V(P) \subseteq V_1$.

Theorem 3. Every planar graph G with $int(G) \leq 2$ has an $(\mathcal{LF}, \mathcal{LF})$ -partition.

Proof. The proof falls naturally into three cases.

Case 1. int(G) = 0.

If int(G) = 0, then the graph G is outerplanar and it has an $(\mathcal{LF}, \mathcal{LF})$ -partition.

Case 2. int(G) = 1.

Let $v \in \text{Int}(G)$ and $u \in N(v)$. It is easy to notice that $u \notin \text{Int}(G)$. According to Theorem 2, if $R = \{v, u\}$, then the graph G has an $(\mathcal{LF}, \mathcal{LF})$ -partition.

Case 3. int(G) = 2.

We can consider a maximal plane graph G with $Int(G) = \{r_1, r_2\}.$

Subcase 3.1. r_1 is adjacent to r_2 .

There exists a vertex u adjacent to r_1 such that u is not adjacent to r_2 . According to Theorem 2, if $R = \{u, r_1, r_2\}$, then the graph G has an $(\mathcal{LF}, \mathcal{LF})$ -partition.

Subcase 3.2. r_1 is not adjacent to r_2 .

Let R contain all the vertices belonging to the shortest path from r_1 to r_2 . Since $\langle R \rangle_G$ is a path and R contains at least one vertex not belonging to $\operatorname{Int}(G)$, i.e., $\operatorname{Int}(G)$ is a proper subgraph of R, then according to Theorem 2, the graph G has an $(\mathcal{LF}, \mathcal{LF})$ -partition.

Theorem 4. Let G be a planar graph of order $n \leq 9$ with int(G) = 3. Then G has an $(\mathcal{LF}, \mathcal{LF})$ -partition.

Proof. If $int(G) \leq 2$, then by Theorem 3, independently of the order n, the graph G has an $(\mathcal{LF}, \mathcal{LF})$ -partition.

Let us consider a planar graph with int(G) = 3, where $Int(G) = \{r_1, r_2, r_3\}$. Without loss of generality we assume that G is a near-triangulation, i.e., G is a plane graph which consists of an outer-cycle $C_k : v_1v_2...v_kv_1$ in

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clockwise order and vertices and edges inside C_k such that each bounded face is bounded by a triangle. Since the graph G is of order $n \leq 9$ and $\operatorname{int}(G) = 3$, then $3 \leq k \leq 6$. Let $S(r_i) = N(r_i) \cap V(C_k)$ be the set of the vertices of the cycle C_k that are adjacent to the vertex $r_i \in \operatorname{Int}(G)$ and $s_i = |S(r_i)|$. Then we have to consider four cases. Cases 1 and 2 are considered under the assumption that C_k is a chordless cycle. But, if there is a chord, then it divides the graph G into G_1 and G_2 , such that $\operatorname{Int}(G_1) = \{r_1, r_2, r_3\}$ and G_2 is outerplanar. Then an $(\mathcal{LF}, \mathcal{LF})$ -partition of G_1 can be easily extended to G.

Case 1. $\langle \text{Int}(G) \rangle = K_3$.

Let the vertices r_1, r_2, r_3 be in clockwise order and $s_1 \leq s_2$ and $s_1 \leq s_3$. If $S(r_1) = \{v_1, \ldots, v_{s_1}\}, S(r_2) = \{v_{s_1}, \ldots, v_{s_1+s_2-1}\}, S(r_3) = \{v_{s_1+s_2-1}, \ldots, v_1\}$, then the partition (V_1, V_2) can be obtained as follows:

$$V_1 = \{r_3, v_1, \dots, v_{s_1+s_2-2}\}, V_2 = \{r_1, r_2, v_{s_1+s_2-1}, \dots, v_k\}.$$

Case 2. $\langle \operatorname{Int}(G) \rangle = P_3$.

Let $N(r_2) \supseteq \{r_1, r_3\}$ and $s_1 \leq s_3$. Then we have two subcases.

Case 2.1. $s_3 = 2$.

If $s_3 = 2$, then there exists a vertex $v \in C_k$ such that $v \in N(r_1) \cap N(r_2) \setminus N(r_3)$. Then $V_1 = \{r_2, r_3, v\}$ and $V_2 = \{r_1\} \cup (C_k \setminus \{v\})$.

Case 2.2. $s_3 > 2$.

If $s_3 > 2$, then there exists a vertex $v \in C_k$ such that $v \in N(r_3) \setminus (N(r_1) \cup N(r_2))$. Then $V_1 = \{r_1, r_2, r_3, v\}$ and $V_2 = C_k \setminus \{v\}$.

Case 3. $(\operatorname{Int}(G)) = \overline{K_3}$.

It is easy to see that for any graph G considered in this case $|V(G)| \ge 8$ and the cycle C_k has at least two chords. If |V(G)| = 9, then we have eight graphs. Their $(\mathcal{LF}, \mathcal{LF})$ -partitions are shown in Figure 1.

There is only one graph H such that |V(H)| = 8. It is easy to see that H is a subgraph of two graphs presented at the bottom line of Figure 1.

Case 4. $\langle \operatorname{Int}(G) \rangle = \overline{P_3}$.

In this case C_k has at least one chord. Thus, G is divided by this chord into two graphs G_i with $int(G_i) = i$, i = 1, 2. Each of them has an $(\mathcal{LF}, \mathcal{LF})$ -partition which can be extended to the other one. The details are left to the reader.

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Figure 1

Theorem 5. For every integer $n \ge 10$ there exists a planar graph G of order n with int(G) = 3, which does not have an $(\mathcal{LF}, \mathcal{LF})$ -partition.

Proof.



Figure 2

Let us partition the set V(G) of the graph G in Figure 2 into two subsets V_1 and V_2 . Assuming that $a \in V_1$, at least one of the vertices from $\{b, c, d\}, \{e, f, g\}$ and $\{h, i, j\}$ must belong to V_1 . Otherwise, $\langle V_2 \rangle_G$ would have contained a cycle. But the set V_1 constructed in this way induces a subgraph containing a vertex of a degree greater than 2. Thus, any planar graph containing the graph G as its subgraph does not have an $(\mathcal{LF}, \mathcal{LF})$ -partition.

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