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# ON SOME VARIATIONS OF EXTREMAL GRAPH PROBLEMS

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**Abstract**

A set $\mathcal{P}$ of graphs is termed *hereditary property* if and only if it contains all subgraphs of any graph $G$ belonging to $\mathcal{P}$. A graph is said to be *maximal* with respect to a hereditary property $\mathcal{P}$ (shortly $\mathcal{P}$-*maximal*) whenever it belongs to $\mathcal{P}$ and none of its proper supergraphs of the same order has the property $\mathcal{P}$. A graph is $\mathcal{P}$-*extremal* if it has a the maximum number of edges among all $\mathcal{P}$-maximal graphs of given order. The number of its edges is denoted by $\mathrm{ex}(n,\mathcal{P})$. If the number of edges of a $\mathcal{P}$-maximal graph is minimum, then the graph is called $\mathcal{P}$-*saturated* and its number of edges is denoted by $\mathrm{sat}(n,\mathcal{P})$.

In this paper, we consider two famous problems of extremal graph theory. We shall translate them into the language of $\mathcal{P}$-maximal graphs and utilize the properties of the lattice of all hereditary properties in order to establish some general bounds and particular results. Particularly, we shall investigate the behaviour of $\mathrm{sat}(n,\mathcal{P})$ and $\mathrm{ex}(n,\mathcal{P})$ in some interesting intervals of the mentioned lattice.

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## 1. Hereditary Properties of Graphs

All graphs considered in this paper are ordinary and finite. The nature of our considerations allows us to restrict our attention to the set $\mathcal{I}$ of all mutually nonisomorphic graphs. For the sake of brevity, we shall say "a graph $G$ contains a subgraph $H$" instead of "a graph $G$ contains a subgraph isomorphic to a graph $H$".

A nonempty subset $\mathcal{P}$ of $\mathcal{I}$ is called *hereditary property*, whenever it is closed under subgraphs. In other words, if $G$ is any graph from $\mathcal{P}$ and $H$ is its subgraph, then $H$ also belongs to $\mathcal{P}$. A hereditary property is named *additive*, whenever it is closed under disjoint union of graphs.

In what follows we shall deal with the following examples of hereditary properties:

$$\begin{aligned}
\mathcal{O} &= \{G \in \mathcal{I} : G \text{ is totally disconnected}\},\\
\mathcal{O}_k &= \{G \in \mathcal{I} : \text{ each component of } G \text{ has at most } k+1 \text{ vertices}\},\\
\mathcal{D}_k &= \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate}\},\\
\mathcal{T}_k &= \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2} \text{ or}\\
&\qquad K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\},\\
\mathcal{I}_k &= \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\}.
\end{aligned}$$

Any hereditary property $\mathcal{P}$ can be uniquely determined either by the set of graphs not appearing in $\mathcal{P}$ (even as a subgraphs) or by the set of maximal admissible graphs (for details see e.g. [1]). More precisely, let us define the sets $\mathbf{F}(\mathcal{P})$ of *minimal forbidden subgraphs* and $\mathbf{M}(\mathcal{P})$ of $\mathcal{P}$-maximal graphs in the following manner:

$$\begin{aligned}
\mathbf{F}(\mathcal{P}) &= \{F \in \mathcal{I} \setminus \mathcal{P} : \text{any proper subgraph } F^* \text{ of } F \text{ belongs to } \mathcal{P}\},\\
\mathbf{M}(\mathcal{P}) &= \bigcup_{n=1}^{\infty} \mathbf{M}(n, \mathcal{P}),\\
\mathbf{M}(n, \mathcal{P}) &= \{G \in \mathcal{P} : |V(G)| = n \text{ and } G + e \notin \mathcal{P} \text{ for any edge } e \in E(\overline{G})\}.
\end{aligned}$$

In the next sections, we shall often need the following useful lemmas.

**Lemma 1.** *Let $\mathcal{P}_1, \mathcal{P}_2$ be any hereditary properties. Then the following statements are mutually equivalent:*

1. $\mathcal{P}_1 \subseteq \mathcal{P}_2$;
2. *for each $H \in \mathbf{F}(\mathcal{P}_2)$ there exists $H' \in \mathbf{F}(\mathcal{P}_1)$ such that $H' \subseteq H$;*
3. *for any positive integer $n$ and an arbitrary $G \in \mathbf{M}(n, \mathcal{P}_1)$ there is $G' \in \mathbf{M}(n, \mathcal{P}_2)$ such that $G \subseteq G'$.*

***Proof.*** (1) $\Rightarrow$ (2). Let $H \in \mathbf{F}(\mathcal{P}_2)$. Then $H \notin \mathcal{P}_1$, and clearly $H$ is not a subgraph of any $G \in \mathcal{P}_1$. Hence, there exists $H' \in \mathbf{F}(\mathcal{P}_1)$ such that $H' \subseteq H$.

(2) $\Rightarrow$ (3). If $G \in \mathbf{M}(n, \mathcal{P}_1)$, then $G$ does not possess any $H' \in \mathbf{F}(\mathcal{P}_1)$. Thus $G$ does not contain any $H \in \mathbf{F}(\mathcal{P}_2)$ and therefore either $G \in \mathbf{M}(n, \mathcal{P}_2)$ or there exists $G' \in \mathbf{M}(n, \mathcal{P}_2)$ such that $G \subseteq G'$.

(3) $\Rightarrow$ (1). This implication follows immediately from the definitions. ■

**Lemma 2.** *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any hereditary properties of graphs. If $\mathcal{P}_1 \subseteq \mathcal{P}_2$, $G \in \mathbf{M}(n, \mathcal{P}_2)$ and $G \in \mathcal{P}_1$, then $G$ belongs to $\mathbf{M}(n, \mathcal{P}_1)$.*

***Proof.*** If $G \in \mathbf{M}(n, \mathcal{P}_2)$, then for each edge $e$ of the complement of $G$ we have $G + e \notin \mathcal{P}_2$. Hence, $G + e \notin \mathcal{P}_1$ for any edge $e \in E(\overline{G})$. Then since $G \in \mathcal{P}_1$, we get $G \in \mathbf{M}(n, \mathcal{P}_1)$. ■

It is not so difficult to see that for any hereditary property $\mathcal{P}$, which is distinct from $\mathcal{I}$, there exists the number $c(\mathcal{P})$ (called the *completeness* of $\mathcal{P}$) defined as follows: $c(\mathcal{P}) = \max\{k : K_{k+1} \in \mathcal{P}\}$.

Given an arbitrary property $\mathcal{P}$, we define the *chromatic number of* $\mathcal{P}$ as the minimum of the chromatic numbers of forbidden subgraphs of $\mathcal{P}$ and we denote it by $\chi(\mathcal{P})$. It is clear, that for each additive hereditary property $\mathcal{P}$ the value $\chi(\mathcal{P})$ is at least two.

The following results describe the structure of additive hereditary properties of graphs.

**Theorem 1** [1]**.** *Let $\mathbb{L}$ be the set of all hereditary properties. Then $(\mathbb{L}, \subseteq)$ is a complete and distributive lattice in which the join and the meet correspond to the set-union and the set-intersection, respectively.*

**Theorem 2** [1]**.** *For every nonnegative $k$, $\mathbb{L}_k = \{\mathcal{P} \in \mathbb{L} | c(\mathcal{P}) = k\}$ is a complete distributive sublattice of $(\mathbb{L}, \subseteq)$ with the least element $\mathcal{O}_k$ and the greatest element $\mathcal{I}_k$.*

## 2. Two Extremal Graph Problems

Many problems in graph theory involve optimization. One of them could be formulated in the following way: for a graph of given order a certain type of subgraphs is prohibited, and one is to determine the maximum possible number of edges in such a graph. A problem of this type was first formulated by Turán and his original problem asked for the maximum number of edges in any graph of order $n$ which does not contain the complete graph $K_p$ (i.e., he was interested in the number $\mathrm{ex}(n, \mathcal{I}_{p-2})$, see [2], [3], [9], [10], [12], [13]).

A general extremal problem, in our terminology, can be formulated as follows. Given a family $\mathbf{F}(\mathcal{P})$ of forbidden subgraphs, find the number

$$\mathrm{ex}(n, \mathcal{P}) = \max\{|E(G)| : G \in \mathbf{M}(n, \mathcal{P})\}.$$

The set of $\mathcal{P}$-maximal graphs of order $n$ with exactly $\mathrm{ex}(n, \mathcal{P})$ edges is denoted by $\mathrm{Ex}(n, \mathcal{P})$. The members of $\mathrm{Ex}(n, \mathcal{P})$ are called $\mathcal{P}$-*extremal* graphs.

It is natural to investigate also the "opposite side", and therefore we define the number

$$\mathrm{sat}(n, \mathcal{P}) = \min\{|E(G)| : G \in \mathbf{M}(n, \mathcal{P})\}.$$

By the symbol $\mathrm{Sat}(n, \mathcal{P})$ we shall denote the set of all $\mathcal{P}$-maximal graphs on $n$ vertices with $\mathrm{sat}(n, \mathcal{P})$ edges. These graphs are called $\mathcal{P}$-*saturated.*

From the definitions immediately follows

**Proposition 1.** *Let* $\mathcal{P}, \mathcal{P}_1$ *and* $\mathcal{P}_2$ *be arbitrary hereditary properties and* $G \in \mathbf{M}(n, \mathcal{P})$. *Then*

1. $\mathrm{sat}(n, \mathcal{P}) \leq |E(G)| \leq \mathrm{ex}(n, \mathcal{P})$;
2. *if* $1 \leq n \leq c(\mathcal{P}) + 1$, *then* $\mathrm{sat}(n, \mathcal{P}) = \mathrm{ex}(n, \mathcal{P}) = \binom{n}{2}$;
3. $\mathrm{ex}(n, \mathcal{P}) \leq ex(n+1, \mathcal{P})$ *for every* $n$;
4. *if* $\mathcal{P}_1 \subseteq \mathcal{P}_2$, *then* $\mathrm{ex}(n, \mathcal{P}_1) \leq \mathrm{ex}(n, \mathcal{P}_2)$ *for every* $n$;
5. $\mathrm{ex}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = \max\{\mathrm{ex}(n, \mathcal{P}_1), \mathrm{ex}(n, \mathcal{P}_2)\}$ *for* $n \geq 1$;
6. $\mathrm{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2) \leq \min\{\mathrm{ex}(n, \mathcal{P}_1), \mathrm{ex}(n, \mathcal{P}_2)\}$ *for* $n \geq 1$.

In [14] examples are presented, which demonstrate that unlike the number $\mathrm{ex}(n, \mathcal{P})$, the behaviour of $\mathrm{sat}(n, \mathcal{P})$ is not monotone in general.

The following theorems present some fundamental results of extremal graph theory. The symbol $\alpha(G)$ denotes the number of vertices in a maximum independent set of $G$.

**Theorem 3** [11]. *If* $\mathcal{P}$ *is a hereditary property with chromatic number* $\chi(\mathcal{P})$, *then*

$$\mathrm{ex}(n, \mathcal{P}) = \left(1 - \frac{1}{\chi(\mathcal{P}) - 1}\right)\binom{n}{2} + o(n^2).$$

**Theorem 4** [14]**.** *If $\mathcal{P}$ is a given hereditary property and*

$$\begin{aligned} u = u(\mathcal{P}) &= \min\Big\{|V(F)| - \alpha(F) - 1 : F \in \mathcal{P}\Big\} \\ d = d(\mathcal{P}) &= \min\Big\{|E(F')| : F' \subseteq F \in \mathbf{F}(\mathcal{P}) \text{ is induced by a set } S \cup \{x\}, \\ &\quad S \subseteq V(F) \text{ is independent and } |S| = |V(F)| - u - 1, \\ &\quad x \in V(F) \setminus S\Big\}, \end{aligned}$$

*then*

$$\operatorname{sat}(n, \mathcal{P}) \le un + \frac{1}{2}(d-1)(n-u) - \binom{u+1}{2},$$

*if $n$ is large enough.*

One can observe that in the case when the structure of $\mathbf{F}(\mathcal{P})$ is not known, the evaluation of the bound for $\operatorname{sat}(n, \mathcal{P})$ is much more complicated as the evaluation of the bound for $\operatorname{ex}(n, \mathcal{P})$. As a matter of fact, we can present hom-properties of graphs which were studied from this point of view in [4]. For that reason, in Section 3 we shall try to obtain another type of bounds for $\operatorname{sat}(n, \mathcal{P})$.

However, as a consequence of the previous two theorems, we immediately have

**Corollary 1.** *If $\mathcal{P}$ is a hereditary property of graphs and* $\operatorname{sat}(n, \mathcal{P}) = \operatorname{ex}(n, \mathcal{P})$ *for every positive $n$, then $\chi(\mathcal{P}) = 2$.*

## 3. Intervals of Monotonicity

In spite of the fact that $\operatorname{sat}(n, \mathcal{P})$ is not monotone, we can prove some inequalities and estimations using the properties of the lattice of all hereditary properties. It will be shown that the class of $k$-degenerate graphs plays a very important role since, $\operatorname{sat}(n, \mathcal{D}_k) = \operatorname{ex}(n, \mathcal{D}_k) = kn - \binom{k+1}{2}$ (see e.g. [5]).

**Lemma 3.** *Let $\mathcal{P}_1, \mathcal{P}_2$ be any hereditary properties and let $\mathcal{P}_1 \subseteq \mathcal{P}_2$. If* $\operatorname{sat}(n, \mathcal{P}_2) = \operatorname{ex}(n, \mathcal{P}_2)$*, then* $\operatorname{sat}(n, \mathcal{P}_1) \le \operatorname{sat}(n, \mathcal{P}_2)$*.*

***Proof.*** If $G \in \mathbf{M}(n, \mathcal{P}_1)$ then, by Lemma 1, there exists a graph $H \in \mathbf{M}(n, \mathcal{P}_2)$ such that $G \subseteq H$. Hence, $|E(G)| \le |E(H)|$. Since $\operatorname{ex}(n, \mathcal{P}_2) = |E(H)| = \operatorname{sat}(n, \mathcal{P}_2)$, we obtain $|E(G)| \le \operatorname{sat}(n, \mathcal{P}_2)$. Therefore, $\operatorname{sat}(n, \mathcal{P}_1) \le |E(G)| \le \operatorname{sat}(n, \mathcal{P}_2)$. ■

**Theorem 5.** *If* $\mathcal{O}_k \subseteq \mathcal{P} \subseteq \mathcal{D}_k$, $n \geq k+1$, *then* $\mathrm{sat}(n, \mathcal{P}) \leq kn - \binom{k+1}{2}$.

***Proof.*** As already pointed out, $\mathrm{sat}(n, \mathcal{D}_k) = \mathrm{ex}(n, \mathcal{D}_k) = kn - \binom{k+1}{2}$. Hence, by Lemma 3, we have $\mathrm{sat}(n, \mathcal{P}) \leq kn - \binom{k+1}{2}$. ■

The following lemmas describe two other criteria of monotonicity in $\mathbb{L}$.

**Lemma 4.** *Let* $\mathcal{P}_1, \mathcal{P}_2$ *be any hereditary properties. Then*

$$\mathrm{sat}(n, \mathcal{P}_1 \cup \mathcal{P}_2) \geq \min\{\mathrm{sat}(n, \mathcal{P}_1), \mathrm{sat}(n, \mathcal{P}_2)\}.$$

***Proof.*** It is not difficult to see that $\mathbf{M}(n, \mathcal{P}_1 \cup \mathcal{P}_2)$ is a subset of $\mathbf{M}(n, \mathcal{P}_1) \cup \mathbf{M}(n, \mathcal{P}_2)$. Thus, $\mathrm{sat}(n, \mathcal{P}_1 \cup \mathcal{P}_2)$ cannot be less than the minimum of $\mathrm{sat}(n, \mathcal{P}_2)$ and $\mathrm{sat}(n, \mathcal{P}_1)$. ■

**Lemma 5.** *Let* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *be any hereditary properties of graphs,* $\mathcal{P}_1 \subseteq \mathcal{P}_2$, *and let* $G$ *be a graph of order* $n$. *If* $G \in \mathcal{P}_1$ *and* $G$ *is* $\mathcal{P}_2$*-saturated, then* $\mathrm{sat}(n, \mathcal{P}_1) \leq \mathrm{sat}(n, \mathcal{P}_2)$.

***Proof.*** Lemma 2 yields $G \in \mathbf{M}(n, \mathcal{P}_1)$. Hence, by an application of Statement (1) of Proposition 1, we get $\mathrm{sat}(n, \mathcal{P}_1) \leq |E(G)| = \mathrm{sat}(n, \mathcal{P}_2)$. ■

Theorem 5 provides an upper bound for $\mathrm{sat}(n, \mathcal{P})$ for the first part of interval $(\mathcal{O}_k, \mathcal{I}_k)$ in $\mathbb{L}_k$. The next theorem covers the rest of this interval. In order to prove it, we have to recall that in [6] it was proved that for any $F \in \mathbf{F}(\mathcal{D}_k)$ holds $\delta(F) \geq k+1$.

**Theorem 6.** *If* $\mathcal{D}_k \subseteq \mathcal{P} \subseteq \mathcal{I}_k$, $n \geq k+1$, *then* $\mathrm{sat}(n, \mathcal{P}) \leq kn - \binom{k+1}{2}$.

***Proof.*** Since $c(\mathcal{P}) = k$, we observe that $K_{k+2} \notin \mathcal{P}$. Hence, by Lemma 1, there exist graphs $F \in \mathbf{F}(\mathcal{D}_k)$ and $H \in \mathbf{F}(\mathcal{P})$ such that $F \subseteq H \subseteq K_{k+2}$. But, as it was mentioned above, $\delta(F) \geq k+1$ and therefore $F = H = K_{k+2}$. In addition, according to the definition of $\mathbf{F}(\mathcal{P})$, no graph of $\mathbf{F}(\mathcal{P})$ is properly contained in $K_{k+2}$, which implies $|V(F)| \geq k+2$ for any $F \in \mathbf{F}(\mathcal{P})$.

Now, let us define the graph $G_n^k$ with the vertex set $V(G_n^k) = \{v_1, v_2, \ldots v_n\}$ in the following way (the symbol $N(u)$ stands for the neighbourhood of the vertex $u$):

$$\begin{aligned} N(v_i) &= \{v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots v_n\}, \quad & i &= 1, 2, \ldots, k, \\ N(v_i) &= \{v_1, v_2, \ldots, v_k\}, & i &= k+1, k+2, \ldots, n. \end{aligned}$$

The graph $G_n^k$ does not contain a subgraph isomorphic to $K_{k+2}$, but it is easy to see that after adding any edge $e \in E(\overline{G_n^k})$ a copy of $K_{k+2}$ must appear in $G_n^k + e$. Hence, $G_n^k \in \mathbf{M}(n, \mathcal{I}_k)$. Furthermore, $G_n^k \in \mathcal{D}_k$ and then, applying Lemma 2, $G_n^k \in \mathbf{M}(n, \mathcal{P})$. This implies, using Lemma 5, Proposition 1, that $\mathrm{sat}(n, \mathcal{P}) \leq |E(G_n^k)| = kn - \binom{k+1}{2}$. ■

## 4. Some Estimations of $\mathrm{sat}(n, \mathcal{P})$ and $\mathrm{ex}(n, \mathcal{P})$

In the previous section we have established a bound for $\mathrm{sat}(n, \mathcal{P})$ in the part of the interval $(\mathcal{O}_k, \mathcal{I}_k)$ of the sublattice $\mathbb{L}_k$. The following theorem presents the exact value of $\mathrm{sat}(n, \mathcal{P})$ in one specific case. By the invariant $\kappa(\mathcal{P})$ we understand the minimum of the numbers $\kappa(F)$, the vertex-connectivity number of $F$, running over all graphs $F$ from $\mathbf{F}(\mathcal{P})$. We shall use the fact, proved in [7], that for any $\mathcal{P}$-maximal graph $G$ the value $\kappa(G)$ is at least $\kappa(\mathcal{P}) - 1$.

**Theorem 7.** *Let $\mathcal{P}$ be a hereditary property and let $\mathcal{D}_1 \subseteq \mathcal{P} \subseteq \mathcal{I}_1$. If $\kappa(\mathcal{P}) \geq 1$, then* $\mathrm{sat}(n, \mathcal{P}) = n - 1$.

***Proof.*** By Theorem 6, we have $\mathrm{sat}(n, \mathcal{P}) \leq n - 1$. An application of the fact, that the minimum degree of a graph from $\mathbf{F}(\mathcal{D}_1)$ is 2, and Lemma 1 yields that any $F \in \mathbf{F}(\mathcal{P})$ has a subgraph isomorphic to $C_n$ for some $n \geq 3$ (the symbol $C_n$ stands for the cycle on $n$ vertices). We distinguish two cases.

*Case* 1. Let $\kappa(\mathcal{P}) = 1$. Suppose indirectly that $\mathrm{sat}(n, \mathcal{P}) \leq n - 2$ for some $n$. Then there exists a graph $G \in \mathbf{M}(n, \mathcal{P})$ with at most $n - 2$ edges. It is easy to see that $G$ is disconnected. Let us denote by $G_1, G_2, \ldots, G_s$, $s \geq 2$, the components of $G$ and let $r_i = |V(G_i)|$ for $i = 1, 2, \ldots, s$. Since each $G_i$ has at least $r_i - 1$ edges and $\sum_{i=1}^{s} r_i = n$, it follows that at least two components of $G$, say $G_1, G_2$, are trees. Then after adding any edge $e = \{u, v\}$, $u \in V(G_1)$, $v \in V(G_2)$, some $F \in \mathbf{F}(\mathcal{P})$ must appear in $G + e$. Since $\kappa(G) = 1$, we obtain $F \subseteq (G_1 \cup G_2) + e$. But $(G_1 \cup G_2) + e$ is a tree which contradicts the fact that $F$ contains a cycle.

*Case* 2. Let $\kappa(\mathcal{P}) \geq 2$. If $G \in \mathbf{M}(n, \mathcal{P})$ then G is connected. Hence $G$ has at least $n - 1$ edges. Therefore $\mathrm{sat}(n, \mathcal{P}) = n - 1$. ■

The set of $k$-degenerate graphs is one with $\mathrm{sat}(n, \mathcal{P}) = \mathrm{ex}(n, \mathcal{P})$. It is widely known that the properties $\mathcal{T}_2$ (to be an outerplanar graph) and $\mathcal{T}_3$ (to be a planar graph) are other examples of such properties. We show that such properties have an exceptional position in the lattice $\mathbb{L}$ of all hereditary properties.

**Lemma 6.** *Let* $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$ *be any hereditary properties of graphs and let* $f : \{1, 2, \ldots\} \to \{0, 1, \ldots\}$ *be a mapping. If* $\mathrm{ex}(n, \mathcal{P}_1) = \mathrm{ex}(n, \mathcal{P}_3) = f(n)$*, then* $\mathrm{sat}(n, \mathcal{P}_2) \leq f(n)$ *and* $\mathrm{ex}(n, \mathcal{P}_2) = f(n)$*.*

***Proof.*** By Statement (4) of Proposition 1, we have $f(n) = \mathrm{ex}(n, \mathcal{P}_1) \leq \mathrm{ex}(n, \mathcal{P}_2) \leq \mathrm{ex}(n, \mathcal{P}_3) = f(n)$, which implies that $\mathrm{ex}(n, \mathcal{P}_2) = f(n)$.
Since $\mathrm{sat}(n, \mathcal{P}_2) \leq \mathrm{ex}(n, \mathcal{P}_2)$ the assertion $\mathrm{sat}(n, \mathcal{P}_2) \leq f(n)$ is also valid. ■

**Theorem 8.** *If* $\mathcal{P}$ *is a hereditary property,* $\mathcal{T}_2 \subseteq \mathcal{P} \subseteq \mathcal{D}_2$*, then* $\mathrm{sat}(n, \mathcal{P}) \leq 2n - 3$ *and* $\mathrm{ex}(n, \mathcal{P}) = 2n - 3$ *for* $n \geq 3$*.*

***Proof.*** The proof follows from the fact that $\mathcal{T}_2 \subseteq \mathcal{D}_2$ and the number of edges of all $\mathcal{T}_2$-maximal and $\mathcal{D}_2$-maximal graphs of order $n \geq 3$ is exactly $2n - 3$. ■

**Lemma 7.** *Let* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *be any hereditary properties of graphs and let* $f : \{1, 2, \ldots\} \to \{0, 1, \ldots\}$ *be a mapping. If* $\mathrm{sat}(n, \mathcal{P}_1) = \mathrm{sat}(n, \mathcal{P}_2) = \mathrm{ex}(n, \mathcal{P}_1) = \mathrm{ex}(n, \mathcal{P}_2) = f(n)$*, then*

1. $\mathrm{sat}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = \mathrm{ex}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = f(n)$*;*
2. $\mathrm{sat}(n, \mathcal{P}_1 \cap \mathcal{P}_2) \leq f(n)$ *and* $\mathrm{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2) \leq f(n)$*.*

*Furthermore, if there exists a graph* $G \in \mathbf{M}(n, \mathcal{P}_1) \cap \mathbf{M}(n, \mathcal{P}_2)$*, then* $\mathrm{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2) = f(n)$*.*

***Proof.*** (1) From the fact $\mathbf{M}(n, \mathcal{P}_1 \cup \mathcal{P}_2) \subseteq \mathbf{M}(n, \mathcal{P}_1) \cup \mathbf{M}(n, \mathcal{P}_2)$ it follows that $\mathrm{sat}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = \mathrm{ex}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = f(n)$.
(2) By Proposition 1, we have $\mathrm{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2) \leq \min\{\mathrm{ex}(n, \mathcal{P}_1), \mathrm{ex}(n, \mathcal{P}_2)\} = f(n)$. Since $\mathrm{sat}(n, \mathcal{P}_1 \cap \mathcal{P}_2) \leq \mathrm{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2)$, we obtain the desired inequality.

Moreover, if there exists a graph $G \in \mathbf{M}(n, \mathcal{P}_1) \cap \mathbf{M}(n, \mathcal{P}_2)$, then $G \in \mathbf{M}(n, \mathcal{P}_1 \cap \mathcal{P}_2)$. Clearly, $|E(G)| = f(n)$. It immediately follows that $\mathrm{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2) = f(n)$. ■

It is easy to see that $\mathcal{T}_3$ and $\mathcal{D}_3$ are incomparable in the lattice $\mathbb{L}$. So we can examine the lattice interval between $\mathcal{T}_3 \cap \mathcal{D}_3$ and $\mathcal{T}_3 \cup \mathcal{D}_3$.

**Lemma 8.** *If* $n$ *is a positive integer,* $n \geq 4$*, then*

1. $\mathrm{sat}(n, \mathcal{T}_3 \cup \mathcal{D}_3) = \mathrm{ex}(n, \mathcal{T}_3 \cup \mathcal{D}_3) = 3n - 6$*;*
2. $\mathrm{sat}(n, \mathcal{T}_3 \cap \mathcal{D}_3) \leq 3n - 6$ *and* $\mathrm{ex}(n, \mathcal{T}_3 \cap \mathcal{D}_3) = 3n - 6$*.*

***Proof.*** As $\mathrm{ex}(n, \mathcal{T}_3) = \mathrm{ex}(n, \mathcal{D}_3) = 3n - 6$ for $n \geq 4$, we have, by Lemma 7, that $\mathrm{sat}(n, \mathcal{T}_3 \cup \mathcal{D}_3) = \mathrm{ex}(n, \mathcal{T}_3 \cup \mathcal{D}_3) = 3n - 6$ and $\mathrm{sat}(n, \mathcal{T}_3 \cap \mathcal{D}_3) \leq 3n - 6$.

It is easy to see that there exists a graph $G$ with $3n - 6$ edges which is planar and 3-degenerate. It means $G \in \mathbf{M}(n, \mathcal{D}_3)$ and simultaneously

$G \in \mathbf{M}(n, \mathcal{T}_3)$. Hence, $G \in \mathbf{M}(n, \mathcal{D}_3 \cup \mathcal{T}_3)$ and $G \in \mathbf{M}(n, \mathcal{D}_3 \cap \mathcal{T}_3)$. Therefore, by Lemma 7, $\mathrm{ex}(n, \mathcal{T}_3 \cup \mathcal{D}_3) = 3n - 6$. ■

The next theorem is an immediate consequence of the previous two lemmas.

**Theorem 9.** *Let $\mathcal{P}$ be a hereditary property such that $\mathcal{T}_3 \cap \mathcal{D}_3 \subseteq \mathcal{P} \subseteq \mathcal{T}_3 \cup \mathcal{D}_3$. Then $\mathrm{ex}(n, \mathcal{P}) = 3n - 6$ and $\mathrm{sat}(n, \mathcal{P}) \leq 3n - 6$ for $n \geq 4$.*

## 5. Reducible Hereditary Properties

A generalization of a colouring of graphs leads us to the concept of reducible hereditary properties.

Given hereditary properties $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$, a *vertex* $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$-*partition* of a graph $G \in \mathcal{I}$ is a partition $(V_1, V_2, \dots, V_n)$ of $V(G)$ such that for each $i = 1, 2, \dots, n$ the induced subgraph $G[V_i]$ has the property $\mathcal{P}_i$. A property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ is defined as the set of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$-partition (for more details see [1], [8]).

The structure of extremal graphs with respect to reducible hereditary property is described by the following lemma.

**Lemma 9.** *If a graph $G$ belongs to $\mathrm{Ex}(n, \mathcal{P}_1 \circ \mathcal{P}_2)$, then for each $(\mathcal{P}_1, \mathcal{P}_2)$-partition of $V(G)$ into two disjoint sets $V_1, V_2$ the following holds: the induced subgraph $G[V_1]$ is $\mathcal{P}_1$-extremal, $G[V_2]$ is $\mathcal{P}_2$-extremal and $G = G[V_1] + G[V_2]$.*

***Proof.*** If $G$ is $\mathcal{P}_1 \circ \mathcal{P}_2$-extremal, then obviously for any $(\mathcal{P}_1, \mathcal{P}_2)$-partition of $V(G)$ into $V_1$ and $V_2$ holds $G = G[V_1] + G[V_2]$ (otherwise we can add at least one edge, which is a contradiction to the extremality of $G$). Furthermore, if the graph $G[V_1]$ is not $\mathcal{P}_1$-extremal, then then there exists a graph $G^* \in \mathcal{P}_1$ of the same order with greater number of edges as $G[V_1]$. Clearly, $G^* + G[V_2] \in \mathcal{P}_1 \circ \mathcal{P}_2$ and moreover, $|E(G^* + G[V_2])| > |E(G[V_1] + G[V_2])|$, which is again a contradiction. Thereby $G[V_1]$ is $\mathcal{P}_1$-extremal. Analogous arguments work for $G[V_2]$ and that is why $G[V_2]$ is a $\mathcal{P}_2$-extremal graph. ■

As in [7] it was shown that $\chi(\mathcal{P}_1 \circ \mathcal{P}_2) = \chi(\mathcal{P}_1) + \chi(\mathcal{P}_2) - 1$, we immediately have

**Theorem 10.** *If $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ is a reducible hereditary property, then*

$$\mathrm{ex}(n, \mathcal{R}) = \left(1 - \frac{1}{\chi(\mathcal{P}_1) + \chi(\mathcal{P}_2) - 2}\right) \binom{n}{2} + o(n^2).$$

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