Discussiones Mathematicae
Graph Theory 17 (1997) 67–76

ON SOME VARIATIONS OF EXTREMAL GRAPH PROBLEMS

Gabriel Semanišin[1]

Department of Geometry and Algebra
Faculty of Science, P.J. Šafárik University
Jesenná 5, 041 54 Košice, Slovak Republic
e-mail: semanisi@turing.upjs.sk

Abstract

A set $\mathcal{P}$ of graphs is termed *hereditary property* if and only if it contains all subgraphs of any graph G belonging to $\mathcal{P}$. A graph is said to be *maximal* with respect to a hereditary property $\mathcal{P}$ (shortly $\mathcal{P}$-*maximal*) whenever it belongs to $\mathcal{P}$ and none of its proper supergraphs of the same order has the property $\mathcal{P}$. A graph is $\mathcal{P}$-*extremal* if it has a the maximum number of edges among all $\mathcal{P}$-maximal graphs of given order. The number of its edges is denoted by $\mathrm{ex}(n, \mathcal{P})$. If the number of edges of a $\mathcal{P}$-maximal graph is minimum, then the graph is called $\mathcal{P}$-*saturated* and its number of edges is denoted by $\mathrm{sat}(n, \mathcal{P})$.

In this paper, we consider two famous problems of extremal graph theory. We shall translate them into the language of $\mathcal{P}$-maximal graphs and utilize the properties of the lattice of all hereditary properties in order to establish some general bounds and particular results. Particularly, we shall investigate the behaviour of $\mathrm{sat}(n, \mathcal{P})$ and $\mathrm{ex}(n, \mathcal{P})$ in some interesting intervals of the mentioned lattice.

Keywords: hereditary properties of graphs, maximal graphs, extremal graphs, saturated graphs.

1991 Mathematics Subject Classification: 05C15, 05C35.

---

[1]Research supported in part by the Slovak VEGA Grant.

1. Hereditary Properties of Graphs

All graphs considered in this paper are ordinary and finite. The nature of our considerations allows us to restrict our attention to the set $\mathcal{I}$ of all mutually nonisomorphic graphs. For the sake of brevity, we shall say "a graph G contains a subgraph H" instead of "a graph G contains a subgraph isomorphic to a graph H".

A nonempty subset $\mathcal{P}$ of $\mathcal{I}$ is called *hereditary property*, whenever it is closed under subgraphs. In other words, if G is any graph from $\mathcal{P}$ and H is its subgraph, then H also belongs to $\mathcal{P}$. A hereditary property is named *additive*, whenever it is closed under disjoint union of graphs.

In what follows we shall deal with the following examples of hereditary properties:

$$\begin{aligned}
\mathcal{O} &= \{G \in \mathcal{I} : G \text{ is totally disconnected}\},\\
\mathcal{O}_k &= \{G \in \mathcal{I} : \text{ each component of } G \text{ has at most } k+1 \text{ vertices}\},\\
\mathcal{D}_k &= \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate}\},\\
\mathcal{T}_k &= \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2} \text{ or}\\
&\quad\; K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\},\\
\mathcal{I}_k &= \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\}.
\end{aligned}$$

Any hereditary property $\mathcal{P}$ can be uniquely determined either by the set of graphs not appearing in $\mathcal{P}$ (even as a subgraphs) or by the set of maximal admissible graphs (for details see e.g. [1]). More precisely, let us define the sets $\mathbf{F}(\mathcal{P})$ of *minimal forbidden subgraphs* and $\mathbf{M}(\mathcal{P})$ of $\mathcal{P}$-maximal graphs in the following manner:

$$\begin{aligned}
\mathbf{F}(\mathcal{P}) &= \{F \in \mathcal{I} \setminus \mathcal{P} : \text{any proper subgraph } F^* \text{ of } F \text{ belongs to } \mathcal{P}\},\\
\mathbf{M}(\mathcal{P}) &= \bigcup_{n=1}^{\infty} \mathbf{M}(n, \mathcal{P}),\\
\mathbf{M}(n, \mathcal{P}) &= \{G \in \mathcal{P} : |V(G)| = n \text{ and } G + e \notin \mathcal{P} \text{ for any edge } e \in E(\overline{G})\}.
\end{aligned}$$

In the next sections, we shall often need the following useful lemmas.

Lemma 1. *Let $\mathcal{P}_1, \mathcal{P}_2$ be any hereditary properties. Then the following statements are mutually equivalent:*

1. $\mathcal{P}_1 \subseteq \mathcal{P}_2$;
2. *for each $H \in \mathbf{F}(\mathcal{P}_2)$ there exists $H' \in \mathbf{F}(\mathcal{P}_1)$ such that $H' \subseteq H$;*
3. *for any positive integer n and an arbitrary $G \in \mathbf{M}(n, \mathcal{P}_1)$ there is $G' \in \mathbf{M}(n, \mathcal{P}_2)$ such that $G \subseteq G'$.*

Proof. (1) $\Rightarrow$ (2). Let $H \in \mathbf{F}(\mathcal{P}_2)$. Then $H \notin \mathcal{P}_1$, and clearly H is not a subgraph of any $G \in \mathcal{P}_1$. Hence, there exists $H' \in \mathbf{F}(\mathcal{P}_1)$ such that $H' \subseteq H$.

(2) $\Rightarrow$ (3). If $G \in \mathbf{M}(n, \mathcal{P}_1)$, then G does not possess any $H' \in \mathbf{F}(\mathcal{P}_1)$. Thus G does not contain any $H \in \mathbf{F}(\mathcal{P}_2)$ and therefore either $G \in \mathbf{M}(n, \mathcal{P}_2)$ or there exists $G' \in \mathbf{M}(n, \mathcal{P}_2)$ such that $G \subseteq G'$.

(3) $\Rightarrow$ (1). This implication follows immediately from the definitions. ■

Lemma 2. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any hereditary properties of graphs. If $\mathcal{P}_1 \subseteq \mathcal{P}_2$, $G \in \mathbf{M}(n, \mathcal{P}_2)$ and $G \in \mathcal{P}_1$, then G belongs to $\mathbf{M}(n, \mathcal{P}_1)$.*

Proof. If $G \in \mathbf{M}(n, \mathcal{P}_2)$, then for each edge e of the complement of G we have $G + e \notin \mathcal{P}_2$. Hence, $G + e \notin \mathcal{P}_1$ for any edge $e \in E(\overline{G})$. Then since $G \in \mathcal{P}_1$, we get $G \in \mathbf{M}(n, \mathcal{P}_1)$. ■

It is not so difficult to see that for any hereditary property $\mathcal{P}$, which is distinct from $\mathcal{I}$, there exists the number $c(\mathcal{P})$ (called the *completeness* of $\mathcal{P}$) defined as follows: $c(\mathcal{P}) = \max\{k : K_{k+1} \in \mathcal{P}\}$.

Given an arbitrary property $\mathcal{P}$, we define the *chromatic number of* $\mathcal{P}$ as the minimum of the chromatic numbers of forbidden subgraphs of $\mathcal{P}$ and we denote it by $\chi(\mathcal{P})$. It is clear, that for each additive hereditary property $\mathcal{P}$ the value $\chi(\mathcal{P})$ is at least two.

The following results describe the structure of additive hereditary properties of graphs.

Theorem 1 [1]**.** *Let $\mathbb{L}$ be the set of all hereditary properties. Then $(\mathbb{L}, \subseteq)$ is a complete and distributive lattice in which the join and the meet correspond to the set-union and the set-intersection, respectively.*

Theorem 2 [1]**.** *For every nonnegative k, $\mathbb{L}_k = \{\mathcal{P} \in \mathbb{L} | c(\mathcal{P}) = k\}$ is a complete distributive sublattice of $(\mathbb{L}, \subseteq)$ with the least element $\mathcal{O}_k$ and the greatest element $\mathcal{I}_k$.*

2. Two Extremal Graph Problems

Many problems in graph theory involve optimization. One of them could be formulated in the following way: for a graph of given order a certain type of subgraphs is prohibited, and one is to determine the maximum possible number of edges in such a graph. A problem of this type was first formulated by Turán and his original problem asked for the maximum number of edges in any graph of order n which does not contain the complete graph K_p (i.e., he was interested in the number $\mathrm{ex}(n, \mathcal{I}_{p-2})$, see [2], [3], [9], [10], [12], [13]).

A general extremal problem, in our terminology, can be formulated as follows. Given a family $\mathbf{F}(\mathcal{P})$ of forbidden subgraphs, find the number

$$\mathrm{ex}(n, \mathcal{P}) = \max\{|E(G)| : G \in \mathbf{M}(n, \mathcal{P})\}.$$

The set of $\mathcal{P}$-maximal graphs of order n with exactly $\mathrm{ex}(n, \mathcal{P})$ edges is denoted by $\mathrm{Ex}(n, \mathcal{P})$. The members of $\mathrm{Ex}(n, \mathcal{P})$ are called $\mathcal{P}$-*extremal* graphs.

It is natural to investigate also the "opposite side", and therefore we define the number

$$\mathrm{sat}(n, \mathcal{P}) = \min\{|E(G)| : G \in \mathbf{M}(n, \mathcal{P})\}.$$

By the symbol $\mathrm{Sat}(n, \mathcal{P})$ we shall denote the set of all $\mathcal{P}$-maximal graphs on n vertices with $\mathrm{sat}(n, \mathcal{P})$ edges. These graphs are called $\mathcal{P}$-*saturated.*

From the definitions immediately follows

Proposition 1. *Let $\mathcal{P}, \mathcal{P}_1$ and $\mathcal{P}_2$ be arbitrary hereditary properties and $G \in \mathbf{M}(n, \mathcal{P})$. Then*

1. $\mathrm{sat}(n, \mathcal{P}) \leq |E(G)| \leq \mathrm{ex}(n, \mathcal{P})$;
2. *if* $1 \leq n \leq c(\mathcal{P}) + 1$, *then* $\mathrm{sat}(n, \mathcal{P}) = \mathrm{ex}(n, \mathcal{P}) = \binom{n}{2}$;
3. $\mathrm{ex}(n, \mathcal{P}) \leq ex(n + 1, \mathcal{P})$ *for every* n;
4. *if* $\mathcal{P}_1 \subseteq \mathcal{P}_2$, *then* $\mathrm{ex}(n, \mathcal{P}_1) \leq \mathrm{ex}(n, \mathcal{P}_2)$ *for every* n;
5. $\mathrm{ex}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = \max\{\mathrm{ex}(n, \mathcal{P}_1), \mathrm{ex}(n, \mathcal{P}_2)\}$ *for* $n \geq 1$;
6. $\mathrm{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2) \leq \min\{\mathrm{ex}(n, \mathcal{P}_1), \mathrm{ex}(n, \mathcal{P}_2)\}$ *for* $n \geq 1$.

In [14] examples are presented, which demonstrate that unlike the number $\mathrm{ex}(n, \mathcal{P})$, the behaviour of $\mathrm{sat}(n, \mathcal{P})$ is not monotone in general.

The following theorems present some fundamental results of extremal graph theory. The symbol $\alpha(G)$ denotes the number of vertices in a maximum independent set of G.

Theorem 3 [11]**.** *If $\mathcal{P}$ is a hereditary property with chromatic number $\chi(\mathcal{P})$, then*

$$\mathrm{ex}(n, \mathcal{P}) = \left(1 - \frac{1}{\chi(\mathcal{P}) - 1}\right)\binom{n}{2} + o(n^2).$$

Theorem 4 [14]**.** *If $\mathcal{P}$ is a given hereditary property and*

$$\begin{aligned} u = u(\mathcal{P}) &= \min\Big\{|V(F)| - \alpha(F) - 1 : F \in \mathcal{P}\Big\} \\ d = d(\mathcal{P}) &= \min\Big\{|E(F')| : F' \subseteq F \in \mathbf{F}(\mathcal{P}) \textit{ is induced by a set } S \cup \{x\}, \\ &\quad S \subseteq V(F) \textit{ is independent and } |S| = |V(F)| - u - 1, \\ &\quad x \in V(F) \setminus S\Big\}, \end{aligned}$$

then

$$\operatorname{sat}(n, \mathcal{P}) \leq un + \frac{1}{2}(d-1)(n-u) - \binom{u+1}{2},$$

if n is large enough.

One can observe that in the case when the structure of $\mathbf{F}(\mathcal{P})$ is not known, the evaluation of the bound for $\operatorname{sat}(n, \mathcal{P})$ is much more complicated as the evaluation of the bound for $\operatorname{ex}(n, \mathcal{P})$. As a matter of fact, we can present hom-properties of graphs which were studied from this point of view in [4]. For that reason, in Section 3 we shall try to obtain another type of bounds for $\operatorname{sat}(n, \mathcal{P})$.

However, as a consequence of the previous two theorems, we immediately have

Corollary 1. *If $\mathcal{P}$ is a hereditary property of graphs and* $\operatorname{sat}(n, \mathcal{P}) = \operatorname{ex}(n, \mathcal{P})$ *for every positive n, then $\chi(\mathcal{P}) = 2$.*

3. Intervals of Monotonicity

In spite of the fact that $\operatorname{sat}(n, \mathcal{P})$ is not monotone, we can prove some inequalities and estimations using the properties of the lattice of all hereditary properties. It will be shown that the class of k-degenerate graphs plays a very important role since, $\operatorname{sat}(n, \mathcal{D}_k) = \operatorname{ex}(n, \mathcal{D}_k) = kn - \binom{k+1}{2}$ (see e.g. [5]).

Lemma 3. *Let $\mathcal{P}_1, \mathcal{P}_2$ be any hereditary properties and let $\mathcal{P}_1 \subseteq \mathcal{P}_2$. If* $\operatorname{sat}(n, \mathcal{P}_2) = \operatorname{ex}(n, \mathcal{P}_2)$*, then* $\operatorname{sat}(n, \mathcal{P}_1) \leq \operatorname{sat}(n, \mathcal{P}_2)$*.*

Proof. If $G \in \mathbf{M}(n, \mathcal{P}_1)$ then, by Lemma 1, there exists a graph $H \in \mathbf{M}(n, \mathcal{P}_2)$ such that $G \subseteq H$. Hence, $|E(G)| \leq |E(H)|$. Since $\operatorname{ex}(n, \mathcal{P}_2) = |E(H)| = \operatorname{sat}(n, \mathcal{P}_2)$, we obtain $|E(G)| \leq \operatorname{sat}(n, \mathcal{P}_2)$. Therefore, $\operatorname{sat}(n, \mathcal{P}_1) \leq |E(G)| \leq \operatorname{sat}(n, \mathcal{P}_2)$. ■

Theorem 5. *If* $\mathcal{O}_k \subseteq \mathcal{P} \subseteq \mathcal{D}_k$, $n \geq k+1$, *then* $\mathrm{sat}(n,\mathcal{P}) \leq kn - \binom{k+1}{2}$.

Proof. As already pointed out, $\mathrm{sat}(n,\mathcal{D}_k) = \mathrm{ex}(n,\mathcal{D}_k) = kn - \binom{k+1}{2}$. Hence, by Lemma 3, we have $\mathrm{sat}(n,\mathcal{P}) \leq kn - \binom{k+1}{2}$. ■

The following lemmas describe two other criteria of monotonicity in $\mathbb{L}$.

Lemma 4. *Let* $\mathcal{P}_1, \mathcal{P}_2$ *be any hereditary properties. Then*

$$\mathrm{sat}(n,\mathcal{P}_1 \cup \mathcal{P}_2) \geq \min\{\mathrm{sat}(n,\mathcal{P}_1), \mathrm{sat}(n,\mathcal{P}_2)\}.$$

Proof. It is not difficult to see that $\mathbf{M}(n,\mathcal{P}_1 \cup \mathcal{P}_2)$ is a subset of $\mathbf{M}(n,\mathcal{P}_1) \cup \mathbf{M}(n,\mathcal{P}_2)$. Thus, $\mathrm{sat}(n,\mathcal{P}_1 \cup \mathcal{P}_2)$ cannot be less than the minimum of $\mathrm{sat}(n,\mathcal{P}_2)$ and $\mathrm{sat}(n,\mathcal{P}_1)$. ■

Lemma 5. *Let* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *be any hereditary properties of graphs,* $\mathcal{P}_1 \subseteq \mathcal{P}_2$, *and let* G *be a graph of order* n. *If* $G \in \mathcal{P}_1$ *and* G *is* $\mathcal{P}_2$*-saturated, then* $\mathrm{sat}(n,\mathcal{P}_1) \leq \mathrm{sat}(n,\mathcal{P}_2)$.

Proof. Lemma 2 yields $G \in \mathbf{M}(n,\mathcal{P}_1)$. Hence, by an application of Statement (1) of Proposition 1, we get $\mathrm{sat}(n,\mathcal{P}_1) \leq |E(G)| = \mathrm{sat}(n,\mathcal{P}_2)$. ■

Theorem 5 provides an upper bound for $\mathrm{sat}(n,\mathcal{P})$ for the first part of interval $(\mathcal{O}_k, \mathcal{I}_k)$ in $\mathbb{L}_k$. The next theorem covers the rest of this interval. In order to prove it, we have to recall that in [6] it was proved that for any $F \in \mathbf{F}(\mathcal{D}_k)$ holds $\delta(F) \geq k+1$.

Theorem 6. *If* $\mathcal{D}_k \subseteq \mathcal{P} \subseteq \mathcal{I}_k$, $n \geq k+1$, *then* $\mathrm{sat}(n,\mathcal{P}) \leq kn - \binom{k+1}{2}$.

Proof. Since $c(\mathcal{P}) = k$, we observe that $K_{k+2} \notin \mathcal{P}$. Hence, by Lemma 1, there exist graphs $F \in \mathbf{F}(\mathcal{D}_k)$ and $H \in \mathbf{F}(\mathcal{P})$ such that $F \subseteq H \subseteq K_{k+2}$. But, as it was mentioned above, $\delta(F) \geq k+1$ and therefore $F = H = K_{k+2}$. In addition, according to the definition of $\mathbf{F}(\mathcal{P})$, no graph of $\mathbf{F}(\mathcal{P})$ is properly contained in K_{k+2}, which implies $|V(F)| \geq k+2$ for any $F \in \mathbf{F}(\mathcal{P})$.

Now, let us define the graph G_n^k with the vertex set $V(G_n^k) = \{v_1, v_2, \ldots v_n\}$ in the following way (the symbol $N(u)$ stands for the neighbourhood of the vertex u):

$$\begin{aligned} N(v_i) &= \{v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots v_n\}, & i &= 1, 2, \ldots, k, \\ N(v_i) &= \{v_1, v_2, \ldots, v_k\}, & i &= k+1, k+2, \ldots, n. \end{aligned}$$

The graph G_n^k does not contain a subgraph isomorphic to K_{k+2}, but it is easy to see that after adding any edge $e \in E(\overline{G_n^k})$ a copy of K_{k+2} must appear in $G_n^k + e$. Hence, $G_n^k \in \mathbf{M}(n, \mathcal{I}_k)$. Furthermore, $G_n^k \in \mathcal{D}_k$ and then, applying Lemma 2, $G_n^k \in \mathbf{M}(n, \mathcal{P})$. This implies, using Lemma 5, Proposition 1, that $\mathrm{sat}(n, \mathcal{P}) \leq |E(G_n^k)| = kn - \binom{k+1}{2}$. ■

4. Some Estimations of $\mathrm{sat}(n, \mathcal{P})$ and $\mathrm{ex}(n, \mathcal{P})$

In the previous section we have established a bound for $\mathrm{sat}(n, \mathcal{P})$ in the part of the interval $(\mathcal{O}_k, \mathcal{I}_k)$ of the sublattice $\mathbb{L}_k$. The following theorem presents the exact value of $\mathrm{sat}(n, \mathcal{P})$ in one specific case. By the invariant $\kappa(\mathcal{P})$ we understand the minimum of the numbers $\kappa(F)$, the vertex-connectivity number of F, running over all graphs F from $\mathbf{F}(\mathcal{P})$. We shall use the fact, proved in [7], that for any $\mathcal{P}$-maximal graph G the value $\kappa(G)$ is at least $\kappa(\mathcal{P}) - 1$.

Theorem 7. *Let $\mathcal{P}$ be a hereditary property and let $\mathcal{D}_1 \subseteq \mathcal{P} \subseteq \mathcal{I}_1$. If $\kappa(\mathcal{P}) \geq 1$, then* $\mathrm{sat}(n, \mathcal{P}) = n - 1$.

Proof. By Theorem 6, we have $\mathrm{sat}(n, \mathcal{P}) \leq n - 1$. An application of the fact, that the minimum degree of a graph from $\mathbf{F}(\mathcal{D}_1)$ is 2, and Lemma 1 yields that any $F \in \mathbf{F}(\mathcal{P})$ has a subgraph isomorphic to C_n for some $n \geq 3$ (the symbol C_n stands for the cycle on n vertices). We distinguish two cases.

Case 1. Let $\kappa(\mathcal{P}) = 1$. Suppose indirectly that $\mathrm{sat}(n, \mathcal{P}) \leq n - 2$ for some n. Then there exists a graph $G \in \mathbf{M}(n, \mathcal{P})$ with at most $n - 2$ edges. It is easy to see that G is disconnected. Let us denote by $G_1, G_2, \ldots, G_s$, $s \geq 2$, the components of G and let $r_i = |V(G_i)|$ for $i = 1, 2, \ldots, s$. Since each G_i has at least $r_i - 1$ edges and $\sum_{i=1}^{s} r_i = n$, it follows that at least two components of G, say G_1, G_2, are trees. Then after adding any edge $e = \{u, v\}$, $u \in V(G_1)$, $v \in V(G_2)$, some $F \in \mathbf{F}(\mathcal{P})$ must appear in $G + e$. Since $\kappa(G) = 1$, we obtain $F \subseteq (G_1 \cup G_2) + e$. But $(G_1 \cup G_2) + e$ is a tree which contradicts the fact that F contains a cycle.

Case 2. Let $\kappa(\mathcal{P}) \geq 2$. If $G \in \mathbf{M}(n, \mathcal{P})$ then G is connected. Hence G has at least $n - 1$ edges. Therefore $\mathrm{sat}(n, \mathcal{P}) = n - 1$. ■

The set of k-degenerate graphs is one with $\mathrm{sat}(n, \mathcal{P}) = \mathrm{ex}(n, \mathcal{P})$. It is widely known that the properties $\mathcal{T}_2$ (to be an outerplanar graph) and $\mathcal{T}_3$ (to be a planar graph) are other examples of such properties. We show that such properties have an exceptional position in the lattice $\mathbb{L}$ of all hereditary properties.

Lemma 6. *Let $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$ be any hereditary properties of graphs and let $f : \{1, 2, \ldots\} \to \{0, 1, \ldots\}$ be a mapping. If* $\mathrm{ex}(n, \mathcal{P}_1) = \mathrm{ex}(n, \mathcal{P}_3) = f(n)$, *then* $\mathrm{sat}(n, \mathcal{P}_2) \leq f(n)$ *and* $\mathrm{ex}(n, \mathcal{P}_2) = f(n)$.

Proof. By Statement (4) of Proposition 1, we have $f(n) = \mathrm{ex}(n, \mathcal{P}_1) \leq \mathrm{ex}(n, \mathcal{P}_2) \leq \mathrm{ex}(n, \mathcal{P}_3) = f(n)$, which implies that $\mathrm{ex}(n, \mathcal{P}_2) = f(n)$.
Since $\mathrm{sat}(n, \mathcal{P}_2) \leq \mathrm{ex}(n, \mathcal{P}_2)$ the assertion $\mathrm{sat}(n, \mathcal{P}_2) \leq f(n)$ is also valid. ■

Theorem 8. *If $\mathcal{P}$ is a hereditary property, $\mathcal{T}_2 \subseteq \mathcal{P} \subseteq \mathcal{D}_2$, then* $\mathrm{sat}(n, \mathcal{P}) \leq 2n - 3$ *and* $\mathrm{ex}(n, \mathcal{P}) = 2n - 3$ *for $n \geq 3$.*

Proof. The proof follows from the fact that $\mathcal{T}_2 \subseteq \mathcal{D}_2$ and the number of edges of all $\mathcal{T}_2$-maximal and $\mathcal{D}_2$-maximal graphs of order $n \geq 3$ is exactly $2n - 3$. ■

Lemma 7. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any hereditary properties of graphs and let $f : \{1, 2, \ldots\} \to \{0, 1, \ldots\}$ be a mapping. If* $\mathrm{sat}(n, \mathcal{P}_1) = \mathrm{sat}(n, \mathcal{P}_2) = \mathrm{ex}(n, \mathcal{P}_1) = \mathrm{ex}(n, \mathcal{P}_2) = f(n)$, *then*

1. $\mathrm{sat}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = \mathrm{ex}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = f(n)$*;*
2. $\mathrm{sat}(n, \mathcal{P}_1 \cap \mathcal{P}_2) \leq f(n)$ *and* $\mathrm{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2) \leq f(n)$.

Furthermore, if there exists a graph $G \in \mathbf{M}(n, \mathcal{P}_1) \cap \mathbf{M}(n, \mathcal{P}_2)$, then $\mathrm{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2) = f(n)$.

Proof. (1) From the fact $\mathbf{M}(n, \mathcal{P}_1 \cup \mathcal{P}_2) \subseteq \mathbf{M}(n, \mathcal{P}_1) \cup \mathbf{M}(n, \mathcal{P}_2)$ it follows that $\mathrm{sat}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = \mathrm{ex}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = f(n)$.
(2) By Proposition 1, we have $\mathrm{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2) \leq \min\{\mathrm{ex}(n, \mathcal{P}_1), \mathrm{ex}(n, \mathcal{P}_2)\} = f(n)$. Since $\mathrm{sat}(n, \mathcal{P}_1 \cap \mathcal{P}_2) \leq \mathrm{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2)$, we obtain the desired inequality.

Moreover, if there exists a graph $G \in \mathbf{M}(n, \mathcal{P}_1) \cap \mathbf{M}(n, \mathcal{P}_2)$, then $G \in \mathbf{M}(n, \mathcal{P}_1 \cap \mathcal{P}_2)$. Clearly, $|E(G)| = f(n)$. It immediately follows that $\mathrm{ex}(n, \mathcal{P}_1 \cap \mathcal{P}_2) = f(n)$. ■

It is easy to see that $\mathcal{T}_3$ and $\mathcal{D}_3$ are incomparable in the lattice $\mathbb{L}$. So we can examine the lattice interval between $\mathcal{T}_3 \cap \mathcal{D}_3$ and $\mathcal{T}_3 \cup \mathcal{D}_3$.

Lemma 8. *If n is a positive integer, $n \geq 4$, then*

1. $\mathrm{sat}(n, \mathcal{T}_3 \cup \mathcal{D}_3) = \mathrm{ex}(n, \mathcal{T}_3 \cup \mathcal{D}_3) = 3n - 6$*;*
2. $\mathrm{sat}(n, \mathcal{T}_3 \cap \mathcal{D}_3) \leq 3n - 6$ *and* $\mathrm{ex}(n, \mathcal{T}_3 \cap \mathcal{D}_3) = 3n - 6$.

Proof. As $\mathrm{ex}(n, \mathcal{T}_3) = \mathrm{ex}(n, \mathcal{D}_3) = 3n - 6$ for $n \geq 4$, we have, by Lemma 7, that $\mathrm{sat}(n, \mathcal{T}_3 \cup \mathcal{D}_3) = \mathrm{ex}(n, \mathcal{T}_3 \cup \mathcal{D}_3) = 3n - 6$ and $\mathrm{sat}(n, \mathcal{T}_3 \cap \mathcal{D}_3) \leq 3n - 6$.

It is easy to see that there exists a graph G with $3n - 6$ edges which is planar and 3-degenerate. It means $G \in \mathbf{M}(n, \mathcal{D}_3)$ and simultaneously

$G \in \mathbf{M}(n, \mathcal{T}_3)$. Hence, $G \in \mathbf{M}(n, \mathcal{D}_3 \cup \mathcal{T}_3)$ and $G \in \mathbf{M}(n, \mathcal{D}_3 \cap \mathcal{T}_3)$. Therefore, by Lemma 7, $\mathrm{ex}(n, \mathcal{T}_3 \cup \mathcal{D}_3) = 3n - 6$. ■

The next theorem is an immediate consequence of the previous two lemmas.

Theorem 9. *Let $\mathcal{P}$ be a hereditary property such that $\mathcal{T}_3 \cap \mathcal{D}_3 \subseteq \mathcal{P} \subseteq \mathcal{T}_3 \cup \mathcal{D}_3$. Then $\mathrm{ex}(n, \mathcal{P}) = 3n - 6$ and $\mathrm{sat}(n, \mathcal{P}) \leq 3n - 6$ for $n \geq 4$.*

5. Reducible Hereditary Properties

A generalization of a colouring of graphs leads us to the concept of reducible hereditary properties.

Given hereditary properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, a *vertex* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*partition* of a graph $G \in \mathcal{I}$ is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that for each $i = 1, 2, \ldots, n$ the induced subgraph $G[V_i]$ has the property $\mathcal{P}_i$. A property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ is defined as the set of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition (for more details see [1], [8]).

The structure of extremal graphs with respect to reducible hereditary property is described by the following lemma.

Lemma 9. *If a graph G belongs to $\mathrm{Ex}(n, \mathcal{P}_1 \circ \mathcal{P}_2)$, then for each $(\mathcal{P}_1, \mathcal{P}_2)$-partition of $V(G)$ into two disjoint sets V_1, V_2 the following holds:
the induced subgraph $G[V_1]$ is $\mathcal{P}_1$-extremal, $G[V_2]$ is $\mathcal{P}_2$-extremal and $G = G[V_1] + G[V_2]$.*

Proof. If G is $\mathcal{P}_1 \circ \mathcal{P}_2$-extremal, then obviously for any $(\mathcal{P}_1, \mathcal{P}_2)$-partition of $V(G)$ into V_1 and V_2 holds $G = G[V_1] + G[V_2]$ (otherwise we can add at least one edge, which is a contradiction to the extremality of G). Furthermore, if the graph $G[V_1]$ is not $\mathcal{P}_1$-extremal, then then there exists a graph $G^* \in \mathcal{P}_1$ of the same order with greater number of edges as $G[V_1]$. Clearly, $G^* + G[V_2] \in \mathcal{P}_1 \circ \mathcal{P}_2$ and moreover, $|E(G^* + G[V_2])| > |E(G[V_1] + G[V_2])|$, which is again a contradiction. Thereby $G[V_1]$ is $\mathcal{P}_1$-extremal. Analogous arguments work for $G[V_2]$ and that is why $G[V_2]$ is a $\mathcal{P}_2$-extremal graph. ■

As in [7] it was shown that $\chi(\mathcal{P}_1 \circ \mathcal{P}_2) = \chi(\mathcal{P}_1) + \chi(\mathcal{P}_2) - 1$, we immediately have

Theorem 10. *If $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ is a reducible hereditary property, then*

$$\mathrm{ex}(n, \mathcal{R}) = \left(1 - \frac{1}{\chi(\mathcal{P}_1) + \chi(\mathcal{P}_2) - 2}\right) \binom{n}{2} + o(n^2).$$

References

[1] M. Borowiecki and P. Mihók, *Hereditary properties of graphs*, in: V.R. Kulli, ed., Advances in Graph Theory (Vishwa Intern. Publication, Gulbarga, 1991) 41–68.

[2] P. Erdös, *Some recent results on extremal problems in graph theory*, Results in: P. Rosentstiehl, ed., Theory of Graphs (Gordon and Breach New York; Dunod Paris, 1967) 117–123; MR**37**#2634.

[3] P. Erdös, *On some new inequalities concerning extremal properties of graphs*, in: P. Erdös and G. Katona, eds., Theory of Graphs (Academic Press, New York, 1968) 77–81; MR**38**#1026.

[4] J. Kratochvíl, P. Mihók and G. Semanišin, *Graphs maximal with respect to hom-properties*, Discussiones Mathematicae Graph Theory **17** (1997) 77–88.

[5] R. Lick and A. T. White, *k-degenerate graphs*, Canadian J. Math. **22** (1970) 1082–1096; MR**42**#1715.

[6] P. Mihók, *On graphs critical with respect to vertex partition numbers*, Discrete Math. **37** (1981) 123–126.

[7] P. Mihók and G. Semanišin, *On the chromatic number of reducible hereditary properties* (submitted).

[8] P. Mihók and G. Semanišin, *Reducible properties of graphs*, Discussiones Math. Graph Theory **15** (1995) 11–18; MR96c:05149.

[9] M. Simonovits, *A method for solving extremal problems in graph theory, stability problems*, in: P. Erdös and G. Katona, eds., Theory of Graphs (Academic Press, New York, 1968) 279–319; MR**38**#2056.

[10] M. Simonovits, *Extremal graph problems with symmetrical extremal graphs. Additional chromatical conditions*, Discrete Math. **7** (1974) 349–376; MR**49**#2459.

[11] M. Simonovits, *Extremal graph theory*, in: L.W. Beineke and R.J. Wilson, eds., Selected Topics in Graph Theory vol. 2 (Academic Press, London, 1983) 161–200.

[12] P. Turán, *On an extremal problem in graph theory*, Mat. Fiz. Lapok **48** (1941) 436–452 (Hungarian); MR**8**,284j.

[13] P. Turán, *On the theory of graph*, Colloquium Math. **3** (1954) 19–30; MR**15**,476b.

[14] L. Kászonyi and Z. Tuza, *Saturated graphs with minimal number of edges*, J. Graph Theory **10** (1986) 203–210.

Received 3 January 1997