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OBSERVATIONS ON MAPS AND Δ-MATROIDS

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Abstract

Using a Δ-matroid associated with a map, Anderson et al (*J. Combin. Theory* (B) **66** (1996) 232–246) showed that one can decide in polynomial time if a medial graph (a 4-regular, 2-face colourable embedded graph) in the sphere, projective plane or torus has two Euler tours that each never cross themselves and never use the same transition at any vertex. With some simple observations, we extend this to the Klein bottle and the sphere with 3 crosscaps and show that the argument does not work in any other surface. We also show there are other Δ-matroids that one can associate with an embedded graph.

Keywords: Δ-matroids, graph embeddings, A-trails.

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1. Introduction

In [2], Bouchet introduced a Δ-matroid associated with a graph G embedded in a surface Σ. (Precise definitions will be given in Section 2.) This Δ-matroid captures precisely the noncrossing Euler tours in the medial graph of G. In [1], the authors observed that this Δ-matroid could be used to determine in polynomial time whether the medial graph of G has two noncrossing Euler tours which never use the same two consecutive edges, assuming Σ is either the sphere, the real projective plane or the torus. This is because, in these cases, each of the layers of the Δ-matroid is a matroid.

With a few simple observations, we show that if Σ is the sphere with two or three crosscaps, then the layers of the Δ-matroid are again matroids and so in these cases there is again a polynomial time algorithm for determining whether there are two noncrossing Euler tours which never use the same two consecutive edges. For no other surface is it true that all the layers of the Δ-matroid are necessarily matroids.

Part of the interest in these noncrossing Euler tours arises from the fact that such tours correspond to a partition of the ground set of the Δ-matroid into disjoint feasible sets of the Δ-matroid. This is an instance of the 2-*covering problem:* Given two Δ-matroids Δ_1 and Δ_2 on the same ground set V, do there exist complementary subsets D_1 and D_2 of V such that, for $i = 1, 2$, D_i is a feasible set in Δ_i?

As observed in [3] the 2-covering problem in general cannot be solved in polynomial time, as the parity problem for matroids is a special case.

In Section 3, we exhibit many other Δ-matroids that can be associated with an embedded graph. In these cases, the Δ-matroid has the very special structure of a g-matroid and so every layer is necessarily a matroid. Finally, in Section 4, we discuss whether a partition into noncrossing Euler tours of certain types implies the existence of a second partition into noncrossing Euler tours of other types.

2. A-Trails

Every graph G embedded in a surface Σ has an associated *medial graph*. This is a 4-regular 2-face colourable embedded graph $m(G)$ in Σ obtained by placing a vertex in the middle of each edge of G and joining two such vertices e and e' by an edge whenever the edges e and e' make a corner in the embedding (cf [1,5]). The faces of $m(G)$ correspond naturally to the vertices and faces of G and every edge of $m(G)$ separates a "vertex-face" from a "face-face," so the dual of $m(G)$ is bipartite. We colour the faces of $m(G)$ black and white.

It is classical that the converse holds: every 4-regular 2-face colourable embedded graph is the medial of some embedded graph and, moreover, if G^* is the dual of G, then $m(G^*)$ is isomorphic to $m(G)$.

An *A-trail* in the medial graph is an Euler tour in which consecutive edges always are a corner of the embedding. Kotzig [4] has proved that if, for each vertex v of a 4-regular graph we select a set B_v of two edges incident with v, then there is an Euler tour of G that never has consecutive edges from any B_v. In particular, every medial graph has an A-trail.

The main question answered in [1] is: which planar graphs have orthogonal A-trails? Orthogonal here means that no two consecutive edges of one Euler tour are consecutive in the other.

In [1] the authors ask: is there a polynomial time algorithm for determining if a medial graph has orthogonal A-trails? An affirmative answer was given when Σ is the sphere, real projective plane and torus. This is extended here to include the sphere with 2 or 3 crosscaps.

The main result of [2] is the following. If G is a graph embedded in a surface Σ and H is a subgraph of G, then $\bar{H}$ denotes the subgraph of G induced by the edges of $E(G) \setminus E(H)$ and H^* denotes the subgraph of the dual G^* of G induced by the edges dual to the edges of H.

Theorem 1 [2]. *Let G be a graph embedded in a surface Σ. Then the set $\Delta(G) = \{E(H) \mid H$ is a subgraph of G and $\Sigma \setminus (H \cup \bar{H}^*)$ is connected $\}$ is a Δ-matroid.* ∎

A Δ-*matroid* is a pair $(V, \mathcal{F})$ consisting of a finite set V and a collection $\mathcal{F}$ of subsets of V satisfying the symmetric exchange property: if $F, F' \in \mathcal{F}$ and $e \in F \triangle F'$, then there is an $e' \in F \triangle F'$ such that $F \triangle \{e, e'\} \in \mathcal{F}$.

In this definition, possibly $e' = e$, in which case $\{e, e'\} = \{e\}$. It is an easy exercise to show that the minimal elements of $\mathcal{F}$ are the bases of a matroid. A duality theory exists (cf. [3]), so that the maximal elements are also bases of a matroid. The feasible sets, therefore, partition nicely by cardinality into layers; the layer of the smallest cardinality sets is a matroid, as is the layer of the largest cardinality sets.

The relevance of Theorem 1 to the current discussion is that by picking transitions of $m(G)$ so as not to cross the edges of H and not to cross the edges of $\bar{H}^*$, an A-trail of $m(G)$ is created if and only if $H \in \Delta(G)$ [2]. Thus the A-trails of $m(G)$ are in 1-1 correspondence with certain spanning subgraphs of G. Another description of $\Delta(G)$ was given in [5].

Theorem 2 [5]. *Let G be a graph embedded in a surface Σ. Then $H \in \Delta(G)$ if and only if H is a spanning connected subgraph of G having only one face which has only one boundary component.* ∎

The smallest feasible sets of $\Delta(G)$ are the spanning trees of G and the largest feasible sets are the complements of spanning trees of the dual G^*. If Σ is the sphere, then these are the same and there is only one layer. If Σ is the real projective plane or the torus, these are the only two non-empty layers.

Let $\varepsilon(\Sigma)$ denote either $2g$ if Σ is the sphere with g handles or k if Σ is the sphere with k crosscaps. The sets in $\Delta(G)$ are each of the form "spanning tree with t edges" and $0 \leq t \leq \varepsilon(\Sigma)$. For orientable surfaces, as discussed in [2], t must be even. This explains the preceding observations in general terms.

If Σ is the Klein bottle, however, there can be three non-empty layers, corresponding to $t = 0, 1, 2$. A subgraph H in the layer $t = 1$ consists of a tree plus one edge and so has a unique cycle C. If C is orientation-preserving (i.e., 2-sided), then each side of C yields a boundary walk of H. Therefore, either H has 2 faces or the one face of H has two boundary components. By Theorem 2, such an H is not in $\Delta(G)$. On the other hand, if C is orientation-reversing (i.e., one-sided), then H has only one face and this face has only one boundary component. Thus, H is in $\Delta(G)$. Summarizing, we have the following.

Proposition 3. *The level $t = 1$ of $\Delta(G)$ is precisely the set of spanning connected subgraphs of G with a unique cycle and that cycle is orientation-reversing.* ■

In [7] Zaslavsky has shown that the set of spanning connected subgraphs of G with a unique cycle which is orientation-reversing is the set of bases of a matroid. Therefore, the $t = 1$ layer of $\Delta(G)$ is always a matroid (in the case of orientable surfaces, it is empty). By duality, the layer corresponding to $t = \varepsilon(\Sigma) - 1$ is also a matroid. In the case of the Klein bottle, these two layers, $t = 1$ and $t = \varepsilon(\Sigma) - 1$, are the same. In the sphere with 3 crosscaps, the four layers are $t = 0, 1, 2$ and 3. Thus, these are all matroids.

However, for no other surface is it true that all the layers are necessarily matroids. To show this, we present examples in each such surface. If Σ is the orientable surface with $g > 1$ handles, then we consider the standard embedding of the graph with one vertex and $2g$ loops $e_1, e_2, \ldots, e_{2g}$ given by the rotation scheme

$$e_1, e_2, e_1, e_2, e_3, e_4, e_3, e_4, \ldots, e_{2g-1}, e_{2g}, e_{2g-1}, e_{2g}.$$

For each $i = 1, 2, \ldots, g$, any member $E(H)$ of $\Delta(G)$ contains either both or neither of e_{2i-1} and e_{2i}, for if it contained only e_{2i}, say, then e_{2i} by itself is one boundary walk of H and the other side of e_{2i} is in another boundary walk. Therefore, for $g > 1$ and fixed layer $t \neq 0,\ \varepsilon(\Sigma)$, t even, these are not the bases of a matroid.

On the other hand, the same graph embeds in the same surface with rotation $e_1, e_2, \ldots, e_{2g}, e_1, e_2, \ldots, e_{2g}$. In this case, every even set of edges is a member of $\Delta(G)$ and, therefore, every layer is a matroid.

For the nonorientable surface with k crosscaps, we have the graph with one vertex and k loops $e_1, e_2, \ldots, e_k$, with rotation

$$e_1, e_2, e_1, e_2, e_3, e_3, e_4, e_4, \ldots, e_k, e_k,$$

with e_1 and e_2 being orientation preserving (signed with $+$) and $e_3, \ldots, e_k$ being orientation reversing (signed with $-$). Again, any feasible set contains either both or neither of e_1 and e_2 so that, for $k > 3$ and any fixed layer $t \neq 0,\ 1,\ \varepsilon(\Sigma) - 1,\ \varepsilon(\Sigma)$, these are not the bases of a matroid.

The same graph embeds in the same surface in such a way that each loop is orientation-reversing and the rotation is $e_1, e_1, e_2, e_2, \ldots, e_k, e_k$. In this case, every even set of edges is a member of $\Delta(G)$ and so again every layer is a matroid.

As mentioned in [1], for the cases where $\Delta(G)$ has matroids in each layer, Edmonds' Matroid Intersection Theorem (and associated polynomial time algorithm) can be used to determine if there are disjoint feasible sets that partition V. Thus, for any surface Σ with $\varepsilon(\Sigma) \leq 3$, and any medial graph $m(G)$ in Σ, there is a polynomial time algorithm to determine if $m(G)$ has orthogonal A-trails. For other surfaces, the existence of such an algorithm is still open.

We conclude this section by remarking that not every matroid with three consecutive layers has the middle layer a matroid. We can take the simple example, with ground set $\{a, b, c, d\}$ and feasible sets all singletons and 3-tuples as well as the two sets $\{a, b\}$ and $\{c, d\}$.

3. MORE Δ-MATROIDS FROM MAPS

In this section, we describe another family of Δ-matroids obtained from maps. In fact, these are g-matroids, which were introduced by Tardos [6]. If V is a finite set and $\mathcal{F}$ is a family of subsets of V, then $\mathcal{F}$ is a *g-matroid* if:

(1) there are two matroids M_1 and M_2 on V such that $\mathcal{F} = \{\, F \mid \exists B_1 \in M_1,\ B_2 \in M_2$ such that $B_1 \subseteq F \subseteq B_2\}$ and

(2) if r_i is the rank function of M_i, $i = 1, 2$, then $r_2 - r_1$ is a nondecreasing set function, i.e., if $F \subseteq F'$, then $(r_2 - r_1)(F) \leq (r_2 - r_1)(F')$.

It can be shown that every g-matroid is a Δ-matroid and that every level of a g-matroid is a matroid. Let G be an embedded graph and let p and q be positive integers. Define the set

$$\begin{aligned}\mathcal{H}(G, p, q) = \{H \subset G \mid\ & H \text{ has at most } p \text{ components} \\ & \text{and } \bar{H}^* \text{ has at most } q \text{ components}\}.\end{aligned}$$

We assume that $V(H) = V(G)$ and that $V(\bar{H}^*) = V(G^*)$.

Proposition. *For any embedded graph G and any positive integers p and q, $\mathcal{H}(G, p, q)$ is a g-matroid.*

Proof. Set $p^* = \min\{p, |V(G)|\}$ and $q^* = \min\{q, |V(G^*)|\}$. Clearly we have $\mathcal{H}(G, p^*, q^*) = \mathcal{H}(G, p, q)$. The minimal elements of $\mathcal{H}(G, p, q)$ are those subgraphs H which are spanning forests having exactly p^* components. The maximal elements are those subgraphs H for which $\bar{H}^*$ is a spanning forest of G^* with exactly q^* components. Thus, the minimal and maximal sets in $\mathcal{H}(G,p,q)$ are the bases of matroids on $E(G)$ and anything in between is in $\mathcal{H}(G,p,q)$.

Let M_1 be the matroid on $E(G)$ whose bases are the spanning forests of G having p^* components and let M_2 be the matroid dual to the matroid M_2^* whose bases are the spanning forests of the dual G^* having q^* components. For $i=1,2$, let r_i denote the rank function of M_i and let r_2^* denote the rank function of M_2^*. For a subset X of $E(G)$, let $\omega(X)$ denote the number of components of the spanning subgraph $(V(G),X)$ of G. Similarly, $\omega^*(X)$ is the number of components of $(V(G^*),X)$. (We identify the labels of edges of G and G^* that are dual to each other. Thus, $E(G)=E(G^*)$.)

It is easy to check that

$$\begin{aligned} r_1(X) &= |V(G)| - \max\{\omega(X),p^*\} \\ r_2^*(X) &= |V(G^*)| - \max\{\omega^*(X),q^*\} \quad \text{and} \\ r_2(X) &= |X| + q^* - \max\{\omega^*(\overline{X}),q^*)\}, \end{aligned}$$

where $\overline{X}=E(G)\setminus X$.

In order to show r_2-r_1 is nondecreasing, it suffices to show for any $X\subset E(G)$ and any $e\in E(G)\setminus X$, $r_2(X+e)-r_1(X+e)-r_2(X)+r_1(X)\geq 0$. Using the formulae of the previous paragraph, this is equivalent to

$$\begin{aligned} &1-\max\{\omega^*(\overline{X+e}),q^*\}+\max\{\omega^*(\overline{X}),q^*\} \\ &+\max\{\omega(X+e),p^*\}-\max\{\omega(X),p^*\}\geq 0. \end{aligned}$$

Thus, it is enough to show that if $\omega(X+e)<\omega(X)$, then $\omega^*(\overline{X+e})=\omega^*(\overline{X})$. This is actually very straightforward.

Suppose $\omega(X+e)<\omega(X)$. Then the ends of e lie in different components H_1 and H_2 of the subgraph $(V(G),X)$ of G. In particular, e is an edge of the edge-cut $\delta(V(H_1))$, consisting of the edges with exactly one end in $V(H_1)$. The set of edges of G^* dual to the edges of $\delta(V(H_1))$ is an element of the cycle space of G^* and so is the edge-disjoint union of circuits of G^*. These edges are all in $\overline{X}$, so e is in a circuit of the subgraph $(V(G^*),\overline{X})$ of G^*. Thus, deleting e does not increase the number of components, so $\omega^*(\overline{X+e})=\omega^*(\overline{X})$, as required. ■

We note that $\mathcal{H}(G,1,1)$ contains $\Delta(G)$. My original misunderstanding of $\Delta(G)$ led me to consider $\mathcal{H}(G,1,1)$ and ask whether it is a Δ-matroid. The answer is a strong yes.

4. Another Example

One might ask if there are any restrictions as to how $\Delta(G)$ might partition into two feasible sets. Given $t, t' \in \{0,1,\ldots,\varepsilon(\Sigma)\}$, there is the obvious restriction that if Σ is orientable, then t and t' must both be even. Other than this, there are no restrictions, as can be seen in the examples shown in Figure 1 (the orientable case) and Figure 2 (the nonorientable case). We may suppose that $0 \le t \le t' \le \varepsilon(\Sigma)$ and, by duality, that $t + t' \le \varepsilon(\Sigma)$. (In Figure 1, each of the handles is represented by two circles with the same label whose interiors are deleted and whose boundaries are indentified. In Figure 2, each of the crosscaps is represented by a shaded circle whose interior is deleted and whose boundary points are indentified antipodally.)

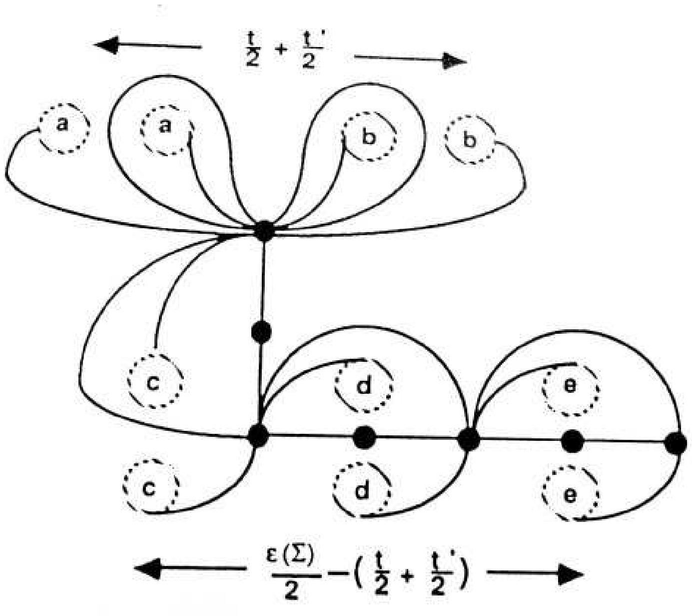

Figure 1

There do seem to be some possibilities for inferring the existence of one kind of partition from the existence of another kind of partition. For example, the proof of Theorem 7 in [1] seems to imply that if G is a graph embedded in the Klein bottle and there is a partition of the edges of G as $T_1 + e_1$ and $T_2 + e_2$, where each T_i is a spanning tree and the unique cycle in $T_i + e_i$

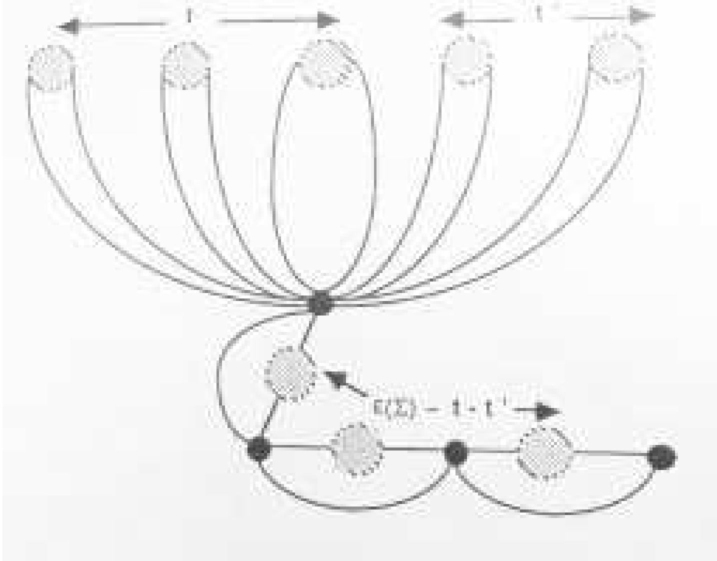

Figure 2

is orientation-reversing, then there is also a partition of the edges of G as T_1' and $T_2' + \{e_1', e_2'\}$, where each T_i' is a spanning tree and the subgraph $T_2' + \{e_1', e_2'\}$ has just one face. That is, if there is a partition of the edges of G into two bases at level $t = 1$, then there is another partition into bases, one at level $t = 0$ and the other at level $t = 2$.

One can ask, if there is a partition into levels t_1 and t_2, when is there necessarily a partition into levels t_1' and t_2'? One obvious necessary condition is that $t_1' + t_2' = t_1 + t_2$. However, in the nonorientable surface Σ with $\varepsilon(\Sigma)$ even, there is an example in which every partition into levels t and t' has t and t' both even. This is shown in Figure 3. In the example, the loops e_{2k-1} are orientation-preserving and so any feasible set that contains e_{2k-1} must also contain e_{2k}.

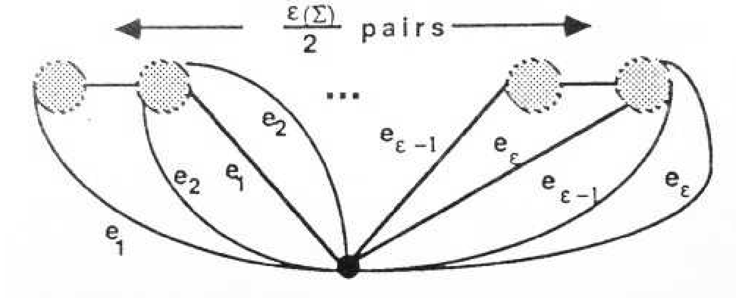

Figure 3

For example, in the Klein bottle the example has a partition into a feasible set at level $t = 0$ and a feasible set at level $t = 2$, but no partition into two feasible sets at level $t = 1$.

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