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REMARKS ON 15-VERTEX (3,3)-RAMSEY GRAPHS NOT CONTAINING K_5

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Abstract

The paper gives an account of previous and recent attempts to determine the order of a smallest graph not containing K_5 and such that every 2-coloring of its edges results in a monochromatic triangle. A new 14-vertex K_4-free graph with the same Ramsey property in the vertex coloring case is found. This yields a new construction of one of the only two known 15-vertex (3,3)-Ramsey graphs not containing K_5.

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1. Introduction

Let G be a graph, and let k and l be positive integers. We write $G \to (k,l)^v$ ($G \to (k,l)^e$) if every red-blue coloring of the vertices (edges) of G forces a red complete subgraph K_k or a blue complete subgraph K_l in G. For $n > \max\{k,l\}$, let

$$\mathcal{G}^v(k,l;n) = \{G : G \to (k,l)^v \text{ and } K_n \not\subset G\}$$

and

$$\mathcal{G}^e(k,l;n) = \{G : G \to (k,l)^e \text{ and } K_n \not\subset G\}.$$

The graphs in $\mathcal{G}^v(k,l;n)$ are called *vertex-Folkman graphs* and the graphs in $\mathcal{G}^e(k,l;n)$ are called *edge-Folkman graphs*.

It is well known that $K_6 \to (3,3)^e$ and so $K_6 \in \mathcal{G}^e(3,3;n)$ for all $n > 6$. In 1967 Erdős and Hajnal [2] asked if $\mathcal{G}^e(3,3;6) \neq \emptyset$ and the following year Graham [6] answered this question showing that $K_8 - C_5 \in \mathcal{G}^e(3,3;6)$, where,

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for $q \leq p$, $K_p - C_q$ is the graph obtained by deleting the edges of a cycle C_q from K_p. In 1970 Folkman [4] showed that for all k, l and $n > \max(k, l)$ the families $\mathcal{G}^v(k, l; n)$ and $\mathcal{G}^e(k, l; n)$ are nonempty. One can ask what the minimum number of vertices of a vertex- or edge-Folkman graph is. This problem leads to the notion of Folkman numbers. Let us denote

$$F^v(k, l; n) = \min\{|V(G)| : G \in \mathcal{G}^v(k, l; n)\}$$

and

$$F^e(k, l; n) = \min\{|V(G)| : G \in \mathcal{G}^e(k, l; n)\},$$

where $V(G)$ is the vertex set of a graph G. These numbers are called *vertex-Folkman numbers* and *edge-Folkman numbers*, respectively. Observe that for $n > k + l - 1$ we have $F^v(k, l; n) = k + l - 1$ as a trivial consequence of the pigeon-hole principle. Since the clique on $R(k, l)$ vertices is the smallest graph G with the property $G \to (k, l)^e$ (here $R(k, l)$ is the Ramsey number), obviously we have $F^e(k, l; n) = R(k, l)$ for every $n > R(k, l)$. Very little is known about the edge-Folkman numbers in the case $n \leq R(k, l)$.

An edge-Folkman number that is still unknown but has been bounded reasonably is $F^e(3, 3; 5)$. The first proof of existence of this number is due to Pósa (unpublished). Schauble [15] in 1969 showed that $F^e(3, 3; 5) \leq 42$. The next upper bound was obtained in 1971 by Graham and Spencer [7]. They proved that $F^e(3, 3; 5) \leq 23$ and conjectured that $F^e(3, 3; 5) = 23$, but as they admitted, without much evidence. Their bound was pushed down to 18 by Irving [11] in 1973. In 1979 Hadziivanov and Nenov [8] showed a 16-vertex graph from $\mathcal{G}^e(3, 3; 5)$ and in 1981 Nenov [14] presented the first 15-vertex graph with that property proving that $F^e(3, 3; 5) \leq 15$. The second one was found in 1984 by Hadziivanov and Nenov [9]. The last three papers (written in Russian) were not generally noticed at that time. In 1993 Erickson [3] found a 17-vertex graph in $\mathcal{G}^e(3, 3; 5)$ and conjectured that $F^e(3, 3; 5) = 17$. This was recently disproved by Bukor [1], who came up with the same 16-vertex graph as in [8]. The author found independently the 15-vertex graph discovered in [9], but the construction is different. This will be shown below.

As far as the lower bound is concerned, in 1972 Lin [12] showed that $F^e(3, 3; 5) \geq 10$ and his result was later improved by Nenov [13] to $F^e(3, 3; 5) \geq 11$ and by Hadziivanov and Nenov [10] to $F^e(3, 3; 5) \geq 12$.

Much less is known about the number $F^e(3, 3; 4)$. Frankl and Rödl [5] proved that $F^e(3, 3; 4) \leq 10^{12}$ and later Spencer [16] squeezed out from their proof the inequality $F^e(3, 3; 4) \leq 10^{10}$. No reasonable lower bound for this Folkman number is known.

2. CONSTRUCTIONS

There were two general lines of the search for small $(3,3)$-Ramsey graphs not containing K_5. The first one, originated in the construction of Graham, was based on the following fact proved explicitly in [9]. The join $H + G$ of two vertex disjoint graphs H and G is the graph with the vertex set $V(H) \cup V(G)$ and the edge set $E(H) \cup E(G) \cup \{\{u, v\} : v \in V(H), u \in V(G)\}$.

Proposition 1 (Hadziivanov, Nenov, 1984). *Let P be a path of order 3. If $\chi(G) > 2$ and the edges of $P + G$ are 2-colored without monochromatic triangle, then P is monochromatic.* ■

This fact was used by Hadziivanov and Nenov [9] to build the following 15-vertex graph $G_1 \in \mathcal{G}^e(3, 3; 5)$. Let C be a 5-cycle contained in K_5. Let G_0 be the graph obtained by elementary subdividing each edge of C as shown in Figure 1.

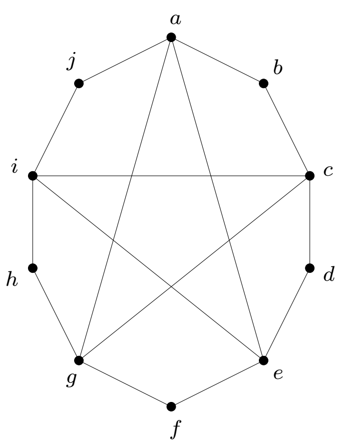

Figure 1

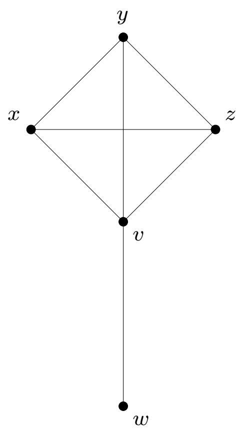

Figure 2

Observe that G_0 is a union of 3 edge-disjoint 5-cycles: $C_1 = \{a, b, c, d, e\}$, $C_2 = \{e, f, g, c, i\}$, $C_3 = \{g, h, i, j, a\}$. Consider the graph shown in Figure 2. It contains 3 paths of length 2: $P_1 = \{x, v, w\}$, $P_2 = \{y, v, w\}$, $P_3 = \{z, v, w\}$. Let G_1 be the union of the joins $C_1 + P_1$, $C_2 + P_2$ and $C_3 + P_3$. One can easily check that there is no K_5 in G_1. We shall now prove that $G_1 \rightarrow (3, 3)^e$. Suppose, on the contrary, that there exists a red-blue coloring of the edges of G_1 such that there is no monochromatic triangle. It follows from Proposition 1 that each path P_1, P_2 and P_3 is monochromatic. Thus, the edges $\{x,v\}, \{z,v\}, \{y,v\}$ have the same color, say red. Then the triangle $\{x,y,z\}$ cannot have a red edge, so it becomes blue. This contradiction proves that $G_1 \to (3,3)^e$ and, consequently, we have $G_1 \in \mathcal{G}^e(3,3;5)$.

The other method, going back to Pósa, constructs edge-Folkman graphs from vertex-Folkman graphs. Let $H + v$ denote the graph obtained from a graph H by adding a vertex v and all edges between v and H. The following result in case $k = l$ was proved in [11]. The idea of the proof below is basically taken from there.

Proposition 2. *Setting $m_1 = R(k-1,l)$ and $m_2 = R(k,l-1)$, if $H \in \mathcal{G}^v(m_1,m_2;n-1)$, then $H + K_1 \in \mathcal{G}^e(k,l;n)$.*
In particular,

$$F^e(k,l;n) \le F^v(m_1,m_2;n-1) + 1.$$

Proof. Let $H \in \mathcal{G}^v(m_1,m_2;n-1)$ and $G = H + v$. Of course, $K_n \not\subset G$. Let us consider any red-blue coloring of the edges of G. For every vertex $x \in V(H)$ we say that x is red if the edge $\{x,v\}$ is red, and it is blue if $\{x,v\}$ is blue. Since $H \in \mathcal{G}^v(m_1,m_2;n-1)$, there are two possibilities:

either there exists a K_{m_1} on red vertices of H
or there exists a K_{m_2} on blue vertices of H.

Assume that the first case is true. Then the red K_{m_1} contains a K_{k-1} with all edges red (so that this $K_{k-1} + v$ creates a K_k with all edges red), or it contains K_l with all edges blue. If there is a K_{m_2} on the blue vertices of H, then this K_{m_2} either contains a K_k with all edges red or it contains a K_{l-1} with all edges blue (so that this $K_{l-1} + v$ creates a K_l with all edges blue). Hence, one way or another, every red-blue coloring of the edges of G forces a red K_k or a blue K_l. ∎

We now present the other known 15-vertex graph G_2 belonging to $\mathcal{G}^e(3,3;5)$, constructed by this method. Figure 3 shows the graph F_1 from [14] which was the first 14-vertex graph discovered in the family $\mathcal{G}^v(3,3;4)$.

Claim 1. $F_1 \in \mathcal{G}^v(3,3;4)$.

Proof. One can very easily check that $K_4 \not\subset F_1$. Hence it is enough to prove that $F_1 \to (3,3)^v$. Suppose that there exists a red-blue coloring of the vertices of F_1 such that F_1 has no monochromatic triangle. Let F_0 denote

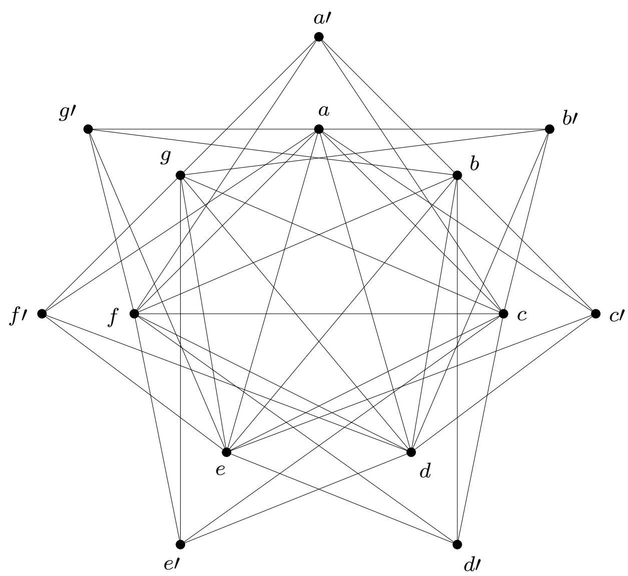

Figure 3. Graph F_1

the subgraph of F_1 induced by the vertices a, b, c, d, e, f, g. Since every 5 vertices of F_0 span a triangle, F_0 has at most 4 red vertices and at most 4 blue vertices. Without loss of generality, we may assume that it has precisely 3 red vertices and 4 blue vertices and that a and b are red. Now we consider four cases with respect to where the third red vertex might be.

(i) If c is the third red vertex, then a, c are red and d, g are blue, so we cannot color the vertex b'.

(ii) If d is red, then a, d are red and e, g are blue, and thus we cannot color the vertex f'.

(iii) If e is red, then a, e are red and d, g are blue so we have no color for the vertex f'.

(iv) Finally, if the vertex f (or g) is red, then we get the same situation as in case (ii) ((i) respectively) because of the symmetry of the graph F_1.

Thus no other vertex of F_0 can be red, a contradiction. Thus, such a coloring is impossible and $F_1 \in \mathcal{G}^v(3, 3; 4)$. ■

By Proposition 2, the join $G_2 = F_1 + K_1$ belongs to $\mathcal{G}^e(3, 3; 5)$, and this is the graph found by Nenov [14].

We shall now construct a 14-vertex graph $F_2 \in \mathcal{G}^v(3, 3; 4)$ different than Nenov's graph F_1 from Fig. 3. Let G_0 be the graph shown in Figure 1. We construct the required graph F_2 by adding four more vertices w, x, y, z and joining w to all vertices of G_0, x to all vertices of C_1, y to all vertices of C_2 and z to all vertices of C_3. Also we add the edges $\{x, y\}, \{y, z\}$ and $\{x, z\}$. Note that F_2 has 14 vertices.

Claim 2. $F_2 \in \mathcal{G}^v(3, 3; 4)$.

Proof. Let us first show that $K_4 \not\subset F_2$. Observe that $K_3 \not\subset G_0$ and hence $K_4 \not\subset G_0 + x$, $K_4 \not\subset G_0 + y$, $K_4 \not\subset G_0 + z$ and $K_4 \not\subset G_0 + w$. Moreover, $w \notin K_4$. Thus, if $K_4 \subset F_2$, then this K_4 must contain two or three vertices of the set $\{x, y, z\}$. The cycles C_1, C_2 and C_3 are edge-disjoint, so no two vertices of $\{x, y, z\}$ are in K_4. Thus, all x, y, z must be in K_4, but it is impossible because the cycles C_1, C_2, C_3 have no common vertex. Consequently, $K_4 \not\subset F_2$.

Assume that the vertices of F_2 are red-blue colored and there is no monochromatic triangle in F_2. Without loss of generality, we may assume that the vertex w is red. Each cycle C_1, C_2 and C_3 has at least two adjacent vertices of the same color. It must be blue since w is red. But then all x, y, z must be red and the triangle x, y, z becomes red. It is a contradiction proving that every red-blue coloring of vertices of F_2 forces a monochromatic triangle. Hence, the graph F_2 is the second known 14-vertex graph in $\mathcal{G}^v(3, 3; 4)$. ■

Note that the join $F_2 + K_1$ is isomorphic to graph G_1 described earlier. Thus, it turned out that both known 15-vertex $(3, 3)$-Ramsey graphs not containing K_5 can be viewed as a join of K_1 and a graph from $\mathcal{G}^v(3, 3; 4)$.

Open problem. Determine the precise value of the Folkman numbers $F^e(3, 3; 5)$ and $F^v(3, 3; 4)$, or tighten up the present estimates

$$11 \le F^v(3, 3; 4) \le 14,$$

$$12 \le F^e(3, 3; 5) \le 15.$$

(It follows from Proposition 2 that $F^e(3, 3; 5) \ge 12 \Rightarrow F^v(3, 3; 4) \ge 11$).

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