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POISSON CONVERGENCE OF NUMBERS OF VERTICES OF A GIVEN DEGREE IN RANDOM GRAPHS

Wojciech Kordecki

Institute of Mathematics, Technical University of Wrocław
Wybrzeże Wyspiańskiego 27, 50–370 Wrocław, Poland
e-mail: kordecki@im.pwr.wroc.pl

Abstract

The asymptotic distributions of the number of vertices of a given degree in random graphs, where the probabilities of edges may not be the same, are given. Using the method of Poisson convergence, distributions in a general and particular cases (complete, almost regular and bipartite graphs) are obtained.

Keywords: Random graphs, degrees of vertices, Poisson convergence.

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1. Introduction

Let $(\Omega, \mathcal{S}, \mathrm{Prob})$ be a fixed probability space such that all the considered random graphs and their integer-valued characteristics are random variables on this space. By a *random graph* one can mean a random matrix, measurable with respect to $(\Omega, \mathcal{S})$, being the matrix representation of the graph. In general, we shall assume that each edge arises with some prescribed probability and independently of all other edges. Sometimes in some auxiliary constructions the assumption of the independence may be omitted.

The main aim of our paper is to find distributions of numbers of vertices of given degrees in several particular cases deduced from a common general case. Properties of vertex degrees in random graphs have been extensively investigated in recent years. For a wide review we refer the reader to [8].

We accomplish our task using the Stein-Chen method (see Barbour, Holst and Janson [3]). In this paper we use the original version of the method, introduced in [1] and [2], which we summarize as follows.

Let $\varphi = \varphi_{\lambda,N} : \mathbb{Z}^+ \to \mathbb{R}$ be such that $\varphi(0) = 0$ and

$$\lambda\varphi(m+1) = \frac{\mathrm{Po}\,(\lambda, N \cap C_m) - \mathrm{Po}\,(\lambda, N)\,\mathrm{Po}\,(\lambda, C_m)}{\lambda^m} e^{\lambda} m!,$$

where $C_m = \{0, 1, \ldots, m\}$, N is any subset of the set $\mathbb{Z}^+$ of nonnegative integers and $\mathrm{Po}\,(\lambda, N)$ denotes the probability that the Poissonian random variable with expectation $\lambda > 0$, falls into N. Then for any nonnegative, integer-valued random variable X

$$\mathrm{Prob}(X \in N) - \mathrm{Po}\,(\lambda, N) = E\{\lambda\varphi(X+1) - X\varphi(X)\} \tag{1}$$

and

$$\Delta\varphi = \sup_{m \in \mathbb{Z}^+} |\varphi(m+1) - \varphi(m)| \leq \min\{1, \lambda^{-1}\}. \tag{2}$$

If

$$d_{TV}\,(X_n, \mathrm{Po}\,(\lambda_n)) = \sup_{N \subseteq \mathbb{Z}^+} |\mathrm{Prob}(X_n \in N) - \mathrm{Po}\,(\lambda_n, N)| \to 0$$

as $n \to \infty$, then we shall say that the sequence $\{X_n\}$ (or simply X_n) is Poisson convergent. If $\lambda_n \to \lambda < \infty$, then $X_n \to \mathrm{Po}\,(\lambda)$ and if $\lambda_n \to \infty$, then $(X_n - \lambda)\,/\sqrt{\lambda} \to \mathrm{N}(0,1)$.

In the next section several formal definitions of random graphs and their functions and related quantities will be given. Subsequently, in order to find conditions which guarantee the Poisson convergence, some estimations of those quantities will be considered. Finally, in the last two sections some special cases of random graphs such as regular, complete and bipartite ones will be considered.

This paper is an extension of [6]. Moreover, we give definitions of "random objects" rigorously, all as (multidimensional) random variables on a common probability space $(\Omega, \mathcal{S}, \mathrm{Prob})$.

2. Notation

Let A be a finite set of vertices and let each possible edge $\{a, b\}$ of the complete graph on A, be independently removed with the prescribed probability $q_{ab} = 1 - p_{ab}$. Then the resulting graph, denoted by $K(A, p)$, forms a random graph. Such a graph, may be considered as a random matrix on some probability space $(\Omega, \mathcal{S}, \mathrm{Prob})$. Assume that each probability p_{ab} depends not only on the edge $\{a, b\}$, but also on a set A, and assume that $p_{ab} = p_{ba}$, $p_{aa} = 0$.

Let $A = A_n$ depend on n in such a way that $A_n \subseteq A_{n+1}$ and $|A_n| \to \infty$ for $n \to \infty$.

Let $\deg^{(a)}(G)$ denote the degree of a in G. Then $\deg^{(a)}(K(A,p))$ is a random degree of a in $K(A,p)$.

Let

$$X^{(a)}(r, A) = \begin{cases} 1, & \text{if } \deg^{(a)}\left(K\left(A,p\right)\right) = r, \\ 0, & \text{otherwise.} \end{cases}$$

Put

$$X(r, A, B) = \sum_{a \in B} X^{(a)}(r, A),$$

where $B = B_n \subseteq A_n$, $B_n \subseteq B_{n+1}$ and $|B_n| \to \infty$ for $n \to \infty$. Denote $X(r, A, A)$ by simply $X(r, A)$.

Let

$$\alpha^{(a)}(r, A) = \mathrm{E}X^{(a)}(r, A) = \mathrm{Prob}\left(X^{(a)}(r, A) = 1\right),$$

$$\alpha(r, A, B) = \mathrm{E}X(r, A, B)$$

and

$$\alpha(r, A) = \mathrm{E}X(r, A).$$

If $a \neq b$, then put

$$Y^{(a)}(r, A, b) = \begin{cases} 1, & \text{if } \deg^{(a)}\left(K\left(A,p\right)\right) = r \text{ and } \{a, b\} \in K(A,p), \\ 0, & \text{otherwise,} \end{cases}$$

$$Y(r, A, B, b) = \sum_{a \in B \setminus b} Y^{(a)}(r, A, b),$$

$$\beta^{(a)}(r, A, b) = \mathrm{E}Y^{(a)}(r, A, b)$$

and

$$\beta(r, A, B, b) = \mathrm{E}Y(r, A, B, b).$$

To obtain the distribution of $X(r, A, B)$ we introduce a special type of a random graph. Let $U^{(a)}(A, p, s)$ be such a random graph that

$$\mathrm{Prob}\left(\deg^{(a)}\left(U^{(a)}(A, p, s)\right) = s\right) = 1$$

and

$$\mathrm{Prob}\left(\sum_{b\in A\setminus a}\deg^{(b)}\left(U^{(a)}(A,p,s)\right)=s\right)=1$$

i.e., the vertex a has degree s and any vertex $b\neq a$ in $U^{(a)}(K,p,s)$ has degree 1 and is joined with a or otherwise has degree 0.

The distribution of $U^{(a)}(A,p,s)$ is defined by the following condition:

$$\begin{aligned}&\mathrm{Prob}\left(\Gamma^{(a)}\left(U^{(a)}\left(A,p,s\right)\right)=B\right)\\&=\mathrm{Prob}\left(\Gamma^{(a)}\left(K\left(A,p\right)\right)=B|\deg^{(a)}\left(K\left(A,p\right)\right)=s\right),\end{aligned}$$

where $|A_n|=|A|$ and $\Gamma^{(a)}(G)$ denotes the set of all vertices joined to the vertex a in graph G.

Now we define an auxiliary random graph $K^{(a)}(A,p,s)$, closely related to $K(A,p)$ as follows

$$K^{(a)}(A,p,s)=K(A\setminus a,p)+U^{(a)}(A,p,s).$$

The more intuitive definition of $K^{(a)}(A,p,s)$ is as follows. $K^{(a)}(A,p,s)$ is a random graph such that on $A\setminus a$ is the same as $K(A\setminus a,p)$ and the vertex a is joined at random with exactly s vertices from the set $A\setminus a$, i.e.

$$\mathrm{Prob}\left(\{a,b\}\in K^{(a)}(A,p,s)\right)=\mathrm{Prob}\left(\{a,b\}\in K(A,p)|X^{(a)}(s,A)=1\right).$$

For $a\neq b$ define

$$Z^{(a)}(r,s,A,b)=\begin{cases}1, & \text{if } \deg^{(a)}\left(K^{(b)}\left(A,p,s\right)\right)=r \text{ and } \{a,b\}\in K^{(b)}(A,p,s),\\0, & \text{otherwise,}\end{cases}$$

$$Z(r,s,A,B,b)=\sum_{a\in B\setminus b}Z^{(a)}(r,s,A,b),$$

$$\gamma^{(a)}(r,s,A,b)=\ EZ^{(a)}(r,s,A,b),$$

and

$$\gamma(r,s,A,B,b)=\mathrm{E}Z(r,s,A,B,b).$$

Let $G=(A,E)$ be a graph on a set of vertices A and with a set of edges E. Random graphs are often defined by a formula

$$p_{ab}=\begin{cases}p, & \text{if } \{a,b\}\in E,\\0, & \text{otherwise,}\end{cases}$$

and $q = 1 - p$. Graph G is called the initial graph and will be denoted by ING .

In the last two sections the following special types of such random graphs will be discussed. The first type is a random regular graph. Its ING has, by definition, only vertices of degree d. If ING has vertices of degree d except some set D and $|D| = m = o(n)$ where $n = |A|$, we shall call such graphs almost regular. The special case of regular graphs where $m = 0$, and $d = n - 1$ are complete graphs, denoted by K_n.

If $K(A, p)$ is an almost regular graph, then we denote it by $R(n, d, m, p)$, where $ING = K(m, n)$, the complete $m \times n$ bipartite graphs.

The second type of random graphs are bipartite random graphs $K(m, n, p)$, where $ING = K(m, n)$, the complete $m \times n$ bipartite graphs.

3. Estimations

The aim of this section is to find an upper bound for

$$|\text{Prob}\,(X(r, A, B) \in N) - \text{Po}\,(\alpha(r, A, B), N)|\,.$$

First we note that the following equalities are obvious. If $a \in B \subseteq A$ then

$$X^{(a)}\,(r, A) = X\,(r, A, B) - X\,(r, A \setminus a, B) + Y\,(r + 1, A, B, a) - Y\,(r, A, B, a)$$

and

$$\begin{aligned}&\text{Prob}\left(X(r, A, B) = k|X^{(a)} = 1\right)\\ &= \text{Prob}\,(X(r, A \setminus a, B) + 1 + Z(r, r, A, B, a) - Z(r + 1, r, A, B, a) = k)\,.\end{aligned}$$

Let $B \subseteq A$. Hence we have the equality

$$\begin{aligned}&\text{E}\{X(r, A, B)\varphi\,(X(r, A, B))\}\\ &= \sum_{a\in B} \text{Prob}\left(X^{(a)}(r, A) = 1\right) \text{E}\{\varphi\,(X(r, A, B))\,|X^{(a)}(r, A) = 1\}\\ &= \sum_{a\in B} \alpha^{(a)}(r, A)\text{E}\{\varphi\,(X(r, A \setminus a, B) + 1 + Z(r, r, A, B, a)\\ &\quad -Z(r + 1, r, A, B, a))\}.\end{aligned}$$

Therefore, using relation (2) we obtain

$$\begin{aligned}&|\text{E}\{\alpha\,(r, A, B)\,\varphi\,(X(r, A, B) + 1) - X(r, A, B)\varphi\,(X(r, A, B))\}|\\ &= \ |\alpha(r, A, B)\text{E}\{\varphi(X(r, A, B) + 1) - \text{E}\sum_{a\in B} X^{(a)}(r, A)\varphi(X(r, A, B))\}|\end{aligned}$$

$$\begin{aligned}
&= |\alpha(r,A,B)\mathrm{E}\{\varphi(X(r,A,B)+1)\}\\
&\quad -\sum_{a\in B}\alpha^{(a)}(r,A)\mathrm{E}\{\varphi(X(r,A\setminus a,B)+1+Z(r,r,A,B,a)\\
&\quad - Z(r+1,r,A,B,a))\}|\\
&\le \max_{a\in B}\alpha(r,A,B)\mathrm{E}|\varphi(X(r,A,B)+1)\\
&\quad -\varphi(X(r,A\setminus a,B)+1-Z(r,r,A,B,a)+Z(r+1,r,A,B,a))|\\
&\le \max_{a\in B}\alpha(r,A,B)(\Delta\varphi)\mathrm{E}|X(r,A,B)-X(r,A\setminus a,B)\\
&\quad - Z(r,r,A,B,a)+Z(r+1,r,A,B,a)|\\
&\le \max_{a\in B}\alpha^{(a)}(r,A)+\max_{a\in B}\mathrm{E}\,|Z(r,r,A,B,a)-Y(r,A,B,a)|\\
&\quad + \max_{a\in B}E\,|Z(r+1,r,A,B,a)-Y(r+1,A,B,a)|\\
&\le \max_{a\in B}\alpha^{(a)}(r,A)+\max_{a\in B}\beta(r,A,B,a)+\max_{a\in B}\beta(r+1,A,B,a)\\
&\quad + \max_{a\in B}\gamma(r,r,A,B,a)+\max_{a\in B}\gamma(r+1,r,A,B,a).
\end{aligned}$$

Therefore, using equality (1) we get the following inequalities:

$$\begin{aligned}
&|\mathrm{Prob}\,(X(r,A,B)\in N)-\mathrm{Po}\,(\alpha(r,A,B),N)|\\
&\le \max_{a\in B}\alpha^{(a)}(r,A)+\max_{a\in B}\mathrm{E}|Z(r,r,A,B,a)-Y(r,A,B,a)|\\
&\quad +\max\nolimits_{a\in B}\mathrm{E}|Z(r+1,r,A,B,a)-Y(r+1,A,B,a)\\
&\le \max_{a\in B}\alpha^{(a)}(r,A)+\max_{a\in B}\beta(r,A,B,a)+\max_{a\in B}\beta(r+1,A,B,a)\\
&\quad +\max\nolimits_{a\in B}\gamma(r,r,A,B,a)+\max\nolimits_{a\in B}\gamma(r+1,r,A,B,a).
\end{aligned}\tag{3}$$

For further use we recall the following elementary inequalities. If $\{x_i\}$ is any real positive sequence, then

$$r!\sum_{\{i_1,\dots,i_r\}}\prod_{j=1}^{r}x_{i_j}\le\left(\sum_{i=1}^{n}x_i\right)^r,\tag{4}$$

$$\prod_{i=1}^{n}(1-x_i)\le\exp\left(-\sum_{i=1}^{n}x_i\right).\tag{5}$$

An easy computation shows that

$$\alpha^{(a)}(r,A)=\sum_{\substack{C\subseteq A\\ |C|=r}}\prod_{c\in C}p_{ac}\prod_{d\notin C}q_{ad},\tag{6}$$

$$\beta(k,A,B,b)=\sum_{a\in B}p_{ab}\alpha^{(a)}(k-1,A\setminus b),\tag{7}$$

$$\gamma(k,s,A,B,b)=\sum_{a\in B}p_{ab}\frac{\alpha^{(a)}(k-1,A\setminus b)\alpha^{(b)}(s-1,A\setminus a)}{\alpha^{(b)}(s,A)} \tag{8}$$

where $b\in B$.

Finally, by $\omega^{(a)}(A)$ we denote the expectation of the degree of a vertex a, i.e.,

$$\omega^{(a)}(A)=\sum_{b\in A}p_{ab}.$$

First we consider the case when all p_{ab}'s tend to zero.

Theorem 1. *Let, for $k=r$ and for $k=r+1$, the following asymptotic equalities be fulfilled:*

$$\min_{\substack{a\in B\\|C|=k}}\prod_{c\in C}q_{ac}=1+o(1), \tag{9}$$

$$\max_{a\in B}\frac{1}{k!}\left(\omega^{(a)}(A)\right)^k e^{-\omega^{(a)}(A)}=o(1), \tag{10}$$

$$\max_{b\in B}\frac{1}{(k-1)!}\sum_{a\in B}p_{ab}\left(\omega^{(a)}(A\setminus b)\right)^{k-1}e^{-\omega^{(a)}(A\setminus b)}=o(1). \tag{11}$$

If $r\geq 1$, $A=A_n$, $B=B_n$, $|B_n|\to\infty$ for $n\to\infty$, then $X(r,A,B)$ is Poisson convergent.

Proof. In order to prove this theorem we have to show that

$$\max_{a\in B}\alpha^{(a)}(k,A)=o(1), \tag{12}$$

$$\max_{a\in B}\beta(k,A,B,a)=o(1), \tag{13}$$

$$\max_{a\in B}\gamma(k,r,A,B,a)=o(1). \tag{14}$$

First we prove (12). From (6) and (9) we obtain that

$$\alpha^{(a)}(k,A)\sim\sum_{\substack{C\subseteq A\\|C|=k}}\prod_{c\in C}p_{ac}\prod_{d\in A}q_{ad}.$$

Next from (4) and (5) we have

$$\begin{aligned}\alpha^{(a)}(k,A) &\leq \frac{(1+o(1))}{k!}\left(\sum_{c\in A}p_{ac}\right)^r\prod_{d\in A}q_{ad}\\ &\sim \frac{1}{r}\left(\omega^{(a)}(A)\right)^r e^{-\omega^{(a)}(A)},\end{aligned}$$

and equality (12) follows from (10).

Now we prove (13). Formulae (7) and (9) give

$$\begin{aligned}\beta(k,A,B,b) &= \sum_{a\in B} p_{ab}\alpha^{(a)}(k-1,A\setminus b)\\ &\sim \sum_{a\in B} p_{ab} \sum_{\substack{C\subseteq A\setminus b\\ |C|=k-1}} \prod_{c\in C} p_{ac} \prod_{d\in A} q_{ad}.\end{aligned}$$

Applying (4), (5) and (10) we have

$$\beta(k,A,B,b) \le (1+o(1))\frac{1}{(k-1)!}\sum_{a\in B} p_{ab}(\omega^{(a)}(A\setminus b))^{k-1}e^{-\omega^{(a)}(A\setminus b)}$$

and (13) follows from (11).

Finally, note that

$$\begin{aligned}&p_{ab} \sum_{\substack{D\subseteq A\setminus a\\ |D|=r-1}} \prod_{c\in D} p_{bc} \prod_{\substack{d\notin D\\ d\neq a}} q_{bd}\\ &\le \sum_{\substack{D\subseteq A\\ |D|=r}} \prod_{c\in D} p_{bc} \prod_{d\notin D} q_{bd} - \sum_{\substack{D\subseteq A\setminus a\\ |D|=r}} \prod_{c\in D} p_{bc} \prod_{d\in A} q_{bd}.\end{aligned}$$

Hence from (4), (5) and (8) we have

$$\gamma(k,r,A,B,b) = \sum_{a\in B} p_{ab}\frac{\alpha^{(a)}(k-1,A\setminus b)\alpha^{(b)}(r-1,A\setminus a)}{\alpha^{(b)}(r,A)}$$

$$\le (1+o(1))\frac{1}{(k-1)!}(\omega^{(a)}(A\setminus b))^{k-1}e^{-\omega^{(a)}(A\setminus b)}\left(1-\frac{\sum_{\substack{D\subseteq A\setminus b\\ |D|=r}}\prod_{c\in D} p_{bc}}{\sum_{\substack{E\subseteq A\\ |E|=r}}\prod_{c\in E} p_{bc}}\right).$$

Now applying again (10) we obtain (14). ■

Remark. From the proof one can obtain not only equalities (12) – (14) giving Poisson convergence, but also a bound of the rate of such convergence. These bounds will be calculated effectively in Section 4 and Section 5 for two special classes of random graphs.

Theorem 2. *If, for $k=0$ and $k=1$, condition* (10) *is satisfied, then $X(0,A,B)$ is Poisson convergent.*

Proof. Note that $\gamma(k,0,A,B,a)\equiv 0$. Hence condition (11) is not needed and condition (9) is always satisfied. We only have to prove that $\alpha^{(a)}(k,A)=$

$o(1)$ and $\beta(k, A, B, a) = o(1)$ but the proof of these facts is the same as the corresponding part of the proof of Theorem 1. Observe only that $\beta(0, A, B, b) \equiv 0$. ■

At the end of this sections we consider some generalizations of the above results. If we omit the assumption that edges are independently removed, the random variables X, Y and Z remain well-defined. The estimations (3) remain true as well. Hence, from the proof of Theorem 1 one can deduce the following general (but fairly trivial) result.

Theorem 3. *If relations* (12) – (14) *hold, then* $X(r, A, B)$ *is Poisson convergent.* ■

Note that formulae (6) – (8) are not longer valid in this general case. As it was mentioned in the proof of Theorem 2, in the case $r = 0$ (i.e. for isolated vertices), we have $\gamma(0, A, B, b) \equiv 0$ and $\beta(0, A, B, b) \equiv 0$. Then we obtain the following result.

Theorem 4. *If*

$$\max_{a \in B} \alpha^{(a)}(0, A) = o(1)$$

and

$$\max_{a \in B} \beta(1, A, B, a) = o(1),$$

then $X(0, A, B)$ *is Poisson convergent.* ■

4. Regular Graphs

In this section we investigate a random graph $R(n, d, m, p)$ described in Section 2. Such a graph has been considered by Palka and Ruciński in [9], (see also [8]), where results concerning convergence of $X(r, n, d, m) = X(r, A, b)$ to the Poisson or normal distribution were obtained.

Let $0 < p < 1$ and $1 \leq r \leq d$. It is easy to check that

$$\begin{aligned}
\alpha^{(a)}(r, A) &= \binom{d}{r} p^r q^{d-r}, \\
\beta(k, A, B, b) &= d\binom{d-1}{k-1} p^k q^{d-k}, \\
\gamma(k, s, A, B, b) &= s\binom{d-1}{k-1} p^{k-1} q^{d-k},
\end{aligned}$$

for $|A| = n$, $B = A \setminus D$, $|B| = m$ and $a, b \in B$.

Note that from (3)

$$\begin{aligned} & d_{TV}\,(X(r,n,d,m), \mathrm{Po}\,(\alpha(r,n,d,m),A)) \\ \le\; & \binom{d}{r} p^r q^{d-r} + d\binom{d-1}{r-1} p^{r-1} q^{d-r} + \binom{d-1}{r-1} p^{r-1} q^{d-r} \\ & + d\binom{d-1}{r} p^{r+1} q^{d-r-1} + r\binom{d-1}{d-1} r p^r q^{d-r-1} \\ =\; & \binom{d-1}{r-1} p^{r-1} q^{d-r-1} \{\frac{d}{r} pq + dpq + rq + d\frac{d-r}{r} p^2 + (d-r)p\}. \end{aligned}$$

From a well-known inequality $\binom{n}{k} \le \frac{n^k}{k!}$, we obtain by substituting $\omega = dp$ the estimation:

$$\begin{aligned} & d_{TV}\,(X(r,n,d,m), \mathrm{Po}\,(\alpha(r,n,d,m),A)) \\ & \le \frac{\omega^{r-1}}{(r-1)!}(1-\frac{\omega}{d})^{d-r-1}\{3\omega + r + \frac{\omega^2}{r}\}. \end{aligned} \tag{15}$$

Corollary 1. *Assume $2 \le r \le d$. If $\omega = o(1)$ or $\omega \to \infty$ but $\omega/n^\varepsilon = o(1)$ for any $\varepsilon > 0$ and $r = o(d)$, then*

$$d_{TV}\,(X(r,n,d,m), \mathrm{Po}\,(\alpha(r,n,d,m),A)) = o(1). \tag{16}$$

Proof. It is easy to check that the left-hand-side of inequality (15) is smaller than $\frac{c_1}{(r-1)!}\omega^{r-1}$ if $\omega = o(1)$ and smaller than $\frac{c_2}{(r-1)!}\omega^{r+1}e^{-\omega}$ if $\omega \to \infty$, where c_1 and c_2 are some constants. ■

The cases $r = 1$ and $r = 0$ have to be considered separately. If $r = 1$, then the right-hand-side of estimation (15) can converge to zero only if $\omega \to \infty$. However, $\omega \to \infty$ implies $d \to \infty$. Hence we have

Corollary 2. *If $r = 1$, $\omega \to \infty$ but $\omega/n^\varepsilon = o(1)$ for any $\varepsilon > 0$, then formula* (16) *holds.* ■

If $r = 0$ and $0 < p < 1$ then we obtain the following result.

$$d_{TV}\,(X(0,n,d,m), \mathrm{Po}\,(\alpha(0,n,d,m),A)) \le \left(1 - \frac{\omega}{d}\right)^{d-1}(1+\omega)\,.$$

The proof of the above estimation is a simpler version of the proof of (15).

Corollary 3. *If $\omega \to \infty$ but $\omega/n^{\varepsilon} = o(1)$ for any $\varepsilon > 0$, then* (16) *holds with $r = 0$.* ■

The above theorems and corollaries are also valid if $d = n-1$ (a complete random graph $K(n,p)$). Asymptotic properties of vertex-degrees in such case are well-known and many papers are devoted to that problem. Particularly, paper [8] contains a wide survey on that. We only point out that the method of Poisson convergence applied in [4] and [6] also bonds the rate of convergence.

Note that if $\omega \to c = \text{const}$, then upper bounds in (12) and (15) do not tend to zero. Besides, note that if $\mathrm{E}X(r,n,d,m)$ is not asymptotically equal to $\operatorname{Var} X(r,n,d,m)$, then the estimations (15) and (16) cannot be improved.

The results given in above Corollaries 1 – 3 have been obtained by Palka and Ruciński [9], (see also [8]) using the so called method of moments. However, such a method does not bound the rate of convergence as in formulae (15) and (16).

5. Bichromatic Graphs

Let $K(m,n,p)$ denote a bichromatic random graph with m labelled vertices of one colour (say red) and n labelled vertices of another colour (say blue), where $m \le n$, and let each of the mn possible edges connecting only a red vertex with a blue one, occur with prescribed probability $p = 1-q$.

Let V denote the set of red vertices and W denote the set of blue ones, $|V| = m$, $|W| = n$, $m \le n$ and $m = m(n)$. Hence $A = V \cup W$.

Palka in [7] proved that under some assumptions a number of vertices of a given degree has asymptotically Poisson or normal distribution. In his paper, only the case $m = \theta n$, where $0 < \theta \le 1$ was considered. The aim of this section is to give a more complete description of the evolution of such a characteristic of bichromatic random graph.

From Theorem 2, we obtain the following results. Let $0 < p < 1$ and $r \ge 1$. Note that

$$\beta(k,A,B,b) = \begin{cases} n\binom{m}{k-1}p^k q^{m-k}, & \text{if } b \in V, \\ m\binom{n}{k-1}p^k q^{n-k}, & \text{if } b \in W, \end{cases}$$

$$\gamma(k,s,A,B,b) = \begin{cases} \frac{s}{m-1}\binom{m-1}{k-1}p^{k-1} q^{m-k+1}, & \text{if } b \in V, \\ \frac{s}{n-1}\binom{n-1}{k-1}p^{k-1} q^{n-k+1}, & \text{if } b \in W. \end{cases}$$

Hence, using (3), we have

$$d_{TV}(X(r,m,n),\mathrm{Po}\,(\alpha,n),A) \tag{17}$$
$$\le \begin{cases} \binom{n}{r}p^r q^{n-r}+\binom{m}{r}p^{r-1}q^{m-r-1}\Big\{\frac{r}{m-1}\big(\frac{r}{m}q^2+\frac{m-r}{m}pq\big)+n\Big(\frac{r}{m-r+1}pq+p^2\Big)\Big\}, \\ \binom{m}{r}p^r q^{m-r}+\binom{n}{r}p^{r-1}q^{n-r-1}\Big\{\frac{r}{n-1}\big(\frac{r}{n}q^2+\frac{n-r}{n}pq\big)+m\Big(\frac{r}{n-r+1}pq+p^2\Big)\Big\}. \end{cases}$$

Denote $\omega' = np$ and $\omega'' = mp$. Then

Corollary 4. *If $\omega' \to \infty$, $\omega'' \to \infty$ but $\omega''/n^\alpha = o(1)$ for every $\alpha > 0$ or if $\omega'' \to \infty$ but $\omega' = o(1)$ or if $\omega'' = o(1)$, then*

$$d_{TV}\,(X(r,m,n),\mathrm{Po}\,(\alpha,n),A) = o(1)$$

Proof. If $r \ge 2$, then we obtain from (17) that

$$d_{TV}\,(X(r,m,n),\mathrm{Po}\,(\alpha,n),A)$$
$$\le \begin{cases} a_1(\omega'')^r e^{-\omega''}/r! + a_2(\omega')^{r-1}e^{-\omega''}\{1/m+p+\omega'(1+\omega')\}/r!, \\ \text{or} \\ a_3(\omega'')^r e^{-\omega''}/r! + a_4(\omega')^{r-1}e^{-\omega''}\{1/n+p+\omega''(1+\omega'')\}/r! \end{cases}$$

for some constants a_1, a_2, a_3, a_4. In a similar way as in Section 4, we can modify the above consideration to obtain the result also for $r = 1$ and $r = 0$. ■

Denote
$$\alpha'(r,n) = \alpha(r,A,V), \quad \alpha''(r,n) = \alpha(r,A,W),$$
$$\alpha(r,n) = \alpha'(r,n) + \alpha''(r,n).$$

Corollary 4 gives the Poisson convergence of $X(r,m,n)$. Hence, under the assumptions of Corollary 4, we have $X(r,m,n) \to \mathrm{Po}\,(\alpha(r,n))$ if $\alpha(r,n) \to \alpha$ and $(X(r,m,n) - \alpha(r,n))/\sqrt{\alpha(r,n)} \to N(0,1)$ if $\alpha(r,n) \to \infty$.

Note that if $\omega'(n) \to c'$ or $\omega'' \to c''$, $c' = \mathrm{const}$, $c'' = \mathrm{const}$, then $\mathrm{E}X(r,m,n)$ is not asymptotically equal to $\mathrm{Var}X(r,m,n)$ and Poisson convergence of $X(r,m,n)$ is not possible.

Now we investigate the evolution of $K(n,m,p)$, where $m = m(n)$. At first we consider the case $m = \mathrm{const}$. If $0 < \varepsilon < p < 1-\varepsilon$, then vertices of degree r exist a.s. only in the set of one color. Then, let $p \to 0$ or $p \to 1$. Let $a_n = np^r = o(n)$ or $a_n = nq^{m-r} = o(n)$.

$$\alpha''(r,n) \sim \begin{cases} \binom{m}{r} a_n, & p = (a_n/n)^{1/r}, \quad r > 1, \\ \binom{m}{r} a_n, & q = (a_n/n)^{1/r}, \quad r \le 1. \end{cases}$$

Hence for $r > 1$ we have $\alpha'(r,n) = o(1)$. If $r = 1$, then

$$\alpha'(r,n) = mnpq^{n-1},$$

$$\alpha''(r,n) = mnpq^{m-1}$$

and

$$\alpha'(r,n) \sim ma_n(1 - a_n/n)^n \sim ma_n e^{-a_n}, \qquad \alpha''(r,n) = ma_n$$

Hence

$$\alpha(r,n) \sim ma_n(1 + e^{-a_n}).$$

Thus from (4) we have $X(r,m,n) \to \text{Po }(ma(1+e^{-a}))$.
If $m \to \infty$ and $r = \text{const}$, we consider the following three cases.

Case $\omega'' = o(1)$, $r \ge 2$.
Hence if $a_n = o(m^{1/r})$ and $p \sim a_n/nm^{1/r}$ then

$$\alpha'(r,n) \sim a_n^r/r!, \qquad \alpha''(r,n) \sim a_n^r \left(\frac{m}{n}\right)^{r-1} /r!$$

and

$$\alpha(r,n) \sim \frac{a_n^r \left((\frac{m}{n})^{r-1} + 1 \right)}{r!}.$$

For $m \sim \theta n$, $0 < \theta \le 1$, we obtain

$$\alpha(r,n) \sim a_n^r \left(\theta^{r-1} + 1\right) /r!$$

and for $m/n = o(1)$ we obtain

$$\alpha(r,n) \sim \alpha'(r,n) \sim a_n^r/r!.$$

Case $\omega'' \to \infty$, $p = o(1)$.
In this case let

$$\begin{aligned} np &= \log m + r\log\log m + a_n, \qquad a_n \to a \in [-\infty, \infty], \\ mp &= \log n + r\log\log n + b_n, \qquad b_n \to b \in [-\infty, \infty]. \end{aligned}$$

Hence

$$\alpha'(r,n) \sim e^{-a_n}/r!, \qquad \alpha''(r,n) \sim e^{-b_n}/r!$$

and

$$p = \frac{\log m + r\log\log m + a_n}{n} = \frac{\log n + r\log\log n + b_n}{m} \tag{18}$$

At first we calculate $b_n = b_n(a_n)$, next we calculate $a_n = a_n(b_n)$.
1°. From (18) we have

$$b_n = \frac{m}{n}\left(\log m + r\log\log m + a_n\right) - \log n - r\log\log n$$

and

$$e^{-b_n} = \left\{\frac{1}{m}\left(\frac{1}{\log m}\right)^r e^{-a_n}\right\}^{m/n} n(\log n)^r = \frac{n}{m^{m/n}}\left(\frac{\log n}{(\log m)^{m/n}}\right)^r e^{-ma_n/n}.$$

Under the assumption

$$(\log m + r\log\log m + a_n) \neq c + o(1)$$

we obtain

$$\alpha(r,n) \sim \frac{e^{-a_n} + \dfrac{n}{m^{m/n}}\left(\dfrac{\log n}{(\log m)^{m/n}}\right)^r e^{-ma_n/n}}{r!}.$$

For $m \sim \theta n$, $0 < \theta < 1$ we have

$$\alpha(r,n) \sim \frac{e^{-a_n} + \theta^{-\theta}\left(n\log n)^r\right)^{1-\theta} e^{-a_n}}{r!}$$

and if $a_n = x + o(1)$, $|x| < \infty$ then $\alpha(r,n) \sim \alpha''(r,n) \to \infty$ and

$$\alpha'(r,n) = e^{-x}/r! + o(1).$$

2°. From (18) we have

$$a_n = \frac{n}{m}(\log n + r\log\log n + b_n) - \log m - r\log\log m$$

and as previously

$$e^{-a_n} = \frac{m}{n^{n/m}} \left(\frac{\log m}{(\log n)^{n/m}} \right)^r e^{-nb_n/m}.$$

Under the assumption

$$(\log n + r \log\log n + a_n)^{n/m} \neq c + o(1)$$

we obtain

$$\alpha(r,n) \sim \frac{e^{-b_n} + \dfrac{m}{n^{n/m}} \left(\dfrac{\log m}{(\log n)^{n/m}} \right)^r e^{-nb_n/m}}{r!}.$$

For $m \sim \theta n$, $0 < \theta < 1$ or $m = o(n)$ we have

$$\alpha(r,n) \sim \frac{e^{-b_n} + \theta \left(n(\log n)^r\right)^{(\theta-1)/\theta} e^{-b_n}}{r!}$$

and if $b_n = x + o(1)$, $|x| < \infty$ then $\alpha(r,n) \sim \alpha'(r,n) \to o(1)$ and

$$\alpha(r,n) = \alpha''(r,n) = e^{-x}/r! + o(1).$$

Now let $m/n = 1 + o(1)$ and moreover let

$$\frac{m}{m^{n/m}} \left(\frac{\log m}{(\log n)^{n/m}} \right)^r = e^{-y} + o(1).$$

Then

$$n(\log n + r \log\log n)/m - \log m - r \log\log m \sim \frac{(n-m)\log m}{n} \sim y$$

and

$$\alpha(r,n) \sim e^{-x}(1 + e^{-y})/r!$$

where

$$\alpha'(r,n) \sim e^{-(x+y)}/r!, \qquad \alpha''(r,n) \sim e^{-x}/r!.$$

In the case $n/m = 1 + o(1)$ vertices of degree r are a.s. in both sets, blue and red. For $r = 0$ this case was investigated by Klee, Larman and Wright in [5].

Case $\omega' = o(1)$ and $\omega'' \to \infty$.
In this case the vertices of degree r occur only in the set of blue vertices. Hence, this case is similar to the case m=const.

In all the above mentioned cases we have from Corollary 4 that $X(r,m,n) \to \mathrm{Po}\,(\alpha(r,n))$ or $(X(r,m,n) - \alpha(r,n))\,/\sqrt{\alpha(r,n)} \to \mathrm{N}(0,1)$.

Finally, let us notice that if $m = o(n)$, then $K(m,n,p)$ is an almost regular random graph $R(m+n,m,m,p)$ which was investigated in Section 4.

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