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A NOTE ON STRONG AND CO-STRONG PERFECTNESS OF THE X-JOIN OF GRAPHS

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Abstract

Strongly perfect graphs were introduced by C. Berge and P. Duchet in [1]. In [4], [3] the following was studied: the problem of strong perfectness for the Cartesian product, the tensor product, the symmetrical difference of n, $n \geq 2$, graphs and for the generalized Cartesian product of graphs. Co-strong perfectness was first studied by G. Ravindra and D. Basavayya [5]. In this paper we discuss strong perfectness and co-strong perfectness for the generalized composition (the lexicographic product) of graphs named as the X-join of graphs.

Keywords: strongly perfect graphs, co-strongly perfect graphs, the X-join of graphs.

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1. Introduction

Let G be a finite undirected connected simple graph. By $V(G)$ and $E(G)$ we denote its vertex set and edge set, respectively. The notation $H = < V_0 >_G$, $V_0 \subseteq V(G)$ means that H is the subgraph of G induced by V_0. A subset $S \subset V(G)$ is said to be *stable* in G if no two distinct vertices of S are adjacent in G. A subset $Q \subseteq V(G)$ is a *clique* of G if $< Q >_G$ is a complete subgraph of G. If the stable set S meets every maximal (with respect to the set inclusion) clique Q, then we will call it a *stable* *transversal* of G. A graph G is called *strongly perfect* ([1]) if its every induced subgraph (including G itself) has a stable transversal. We call G *co-strongly perfect* ([5]) if G and the complementary graph $\overline{G}$ to G are strongly perfect. Let $G_1, \ldots, G_n$, $n \geq 2$, be graphs of the same order $m \geq 2$ with the vertex sets $V(G_i) = V = \{y_1, \ldots, y_m\}$ for $i = 1, \ldots, n$ and X be a graph such that $V(X) = \{x_1, \ldots, x_n\}$. The *X-join* ([2]) of the sequence of graphs $G_1, \ldots, G_n$ and the graph X is the graph $X[G_1, \ldots, G_n]$ with the vertex set $V(X) \times V$ and the edge set $\{[(x_j, y_p), (x_k, y_q)] : j = k$ and $[y_p, y_q] \in E(G_i)$ or $[x_j, x_k] \in E(X)\}$.

Observe that if $G_1 = G_2 = \ldots = G_n = Y$, then we obtain the *composition* (the *lexicographic product*) of graphs Y and X denoted by $X[Y]$.

Let $V_0 \subseteq V(X) \times V$. By the *projection* $Pr_X V_0$ of the subset V_0 on the graph X we mean the set $Pr_X V_0 = \{x \in V(X)$: there exists $y \in V(G_i)$, $1 \leq i \leq n$, that $(x, y) \in V_0\}$.

2. Results

Put $G = X[G_1, \ldots, G_n]$, for convenience. Let H be a connected induced subgraph of G such that it is not isomorphic to any induced subgraph H' of the graph X or G_i, for $i = 1, \ldots n$.

Let $Pr_X V(H) = \{x_{i1}, \ldots, x_{ip}\}$, $2 \leq p \leq n$.

We partition the set $V(H)$ on p-disjoint sets $V_{ij}(H)$ such that $Pr_X V_{ij}(H) = \{x_{ij}\}$ for $j = 1, \ldots, p$. For an arbitrary subset $R \subseteq V(H)$, in a natural way we can write $R = \bigcup_{j=1}^{t} R \cap V_{ij}(H)$, where $1 \leq t \leq p$.

For G and H given above it follows immediately.

Lemma 1. *If Q is a maximal clique of H, then $Pr_X Q$ is a maximal clique of $< Pr_X V(H) >$.*

Lemma 2. *A subset $Q \subseteq V(H)$ is a maximal clique of H if and only if*

(1) *$Q \cap V_{ij}(H)$ is a maximal clique of $< V_{ij}(H) >$ for $j = 1, \ldots, p$ or $Q \cap V_{ij}(H) = \emptyset$ and*

(2) *$Pr_X Q$ is a maximal clique of $< Pr_X V(H) >$.*

Proof. I. Assume that Q is a maximal clique of H. We can write $Q = \bigcup_{j=1}^{t} Q \cap V_{ij}(H)$ where $1 \leq t \leq p$ with $Q \cap V_{ij}(H) \neq \emptyset$ for each $j = 1, \ldots, t$. Moreover, each of the sets $Q \cap V_{ij}(H)$ must be a clique of $< V_{ij}(H) >$. Suppose there exists j, $1 \leq j \leq t$ such that $Q \cap V_{ij}(H)$, is not maximal. In consequence, there exists a vertex $(x_{ij}, y_r) \in V_{ij}(H) \backslash Q \cap V_{ij}(H), 1 \leq j \leq t$

(of course $(x_{ij}, y_r) \notin Q$) which is adjacent to each vertex from $Q \cap V_{ij}(H)$. Moreover, by the definition of G and from the fact that $Q \cap V_{ij}(H) \subset Q$ it follows that (x_{ij}, y_r) must be adjacent to each vertex from $Q \backslash Q \cap V_{ij}(H)$. In consequence, (x_{ij}, y_r) is adjacent to all vertices from Q and $(x_{ij}, y_r) \notin Q$, a contradiction with the assumption that Q is a maximal clique of H. This shows that the condition in (1) holds.

Condition (2) follows from Lemma 1.

II. Suppose that conditions (1) and (2) hold. We can write $Q = \bigcup_{j=1}^{t} Q \cap V_{ij}(H)$, $1 \leq t \leq p$. Note that $\mid Q \mid > 1$, by the asumption about H. Firstly, we shall show that Q is a clique of H. Let $(x_{ij}, y_r), (x_{ik}, y_s)$ be two distinct vertices from Q. If $j = k$, then they belong to $Q \cap V_{ij}(H)$ and are adjacent by (1). If $j \neq k$, then $x_{ij}, x_{ik} \in Pr_X Q$ and by (2) they are adjacent in X. Thus, by the definition of G the vertices $(x_{ij}, y_r), (x_{ik}, y_s)$ are adjacent in G. This proves that Q is a clique of H.

Assume that Q is not maximal. This means that there exists $(x_{il}, y_r) \notin Q$ but it is adjacent to each vertex from Q. Moreover, by the definition of G, the vertex x_{il} is adjacent to all vertices from $Pr_X Q$. This implies that $x_{il} \in Pr_X Q$ by (2). In consequence, it must be that $(x_{il}, y_r) \in V_{il}(H) \backslash Q \cap V_{il}(H)$ (evidently $(x_{il}, y_r) \notin Q \cap V_{il}(H)$). Since $Q \cap V_{il}(H) \subset Q$ and (x_{il}, y_r) is adjacent to each vertex from Q, then it is adjacent to each vertex from $Q \cap V_{il}(H)$. Hence by (1) it must be that $(x_{il}, y_r) \in Q \cap V_{il}(H)$, a contradiction. Hence, Q is a maximal clique of H and this completes the proof of the lemma. ■

Using the same method as in the proof of Lemma 2 we prove.

Lemma 3. *A subset $S \subset V(H)$ is a maximal stable set of H if and only if*
- (1) *$S \cap V_{ij}(H)$ is a maximal stable set of $< V_{ij}(H) >$ for $j = 1, \ldots, s$ or $S \cap V_{ij}(H) = \emptyset$ and*
- (2) *$Pr_X S$ is a maximal stable set of $< Pr_X V(H) >$.*

Lemma 4 follows directly from the definition of the graph $X[G_1, \ldots, G_n]$.

Lemma 4. $\overline{X[G_1, \ldots, G_n]} = \overline{X}[\overline{G_1}, \ldots, \overline{G_n}]$.

Theorem 1. *$X[G_1, \ldots, G_n]$ is strongly perfect if and only if the graphs $X, G_1, \ldots, G_n$ are strongly perfect.*

Proof. I. Let $X[G_1, \ldots, G_n]$ be strongly perfect. Then $X, G_1, \ldots, G_n$ are strongly perfect since they are isomorphic to some induced subgraphs of G.

II. Suppose that the graphs $X, G_1, \ldots, G_n$ are strongly perfect. We shall show that G is strongly perfect. Let H be a connected induced subgraph of G. We shall prove that H has a stable transversal.

If H is an induced subgraph of X or $G_i, 1 \leq i \leq n$, then H has a stable transversal, by the asumption that $X, G_1, \ldots, G_n$ are strongly perfect.

Let H be not induced subgraph of $X, G_i, i = 1, \ldots, n$. Assume that H does not have a stable transversal, i.e., for every maximal stable set $S \subseteq V(H)$ there exists a maximal clique $Q \subseteq V(H)$ such that $S \cap Q = \emptyset$. Moreover, by the definition of G and Lemmas 2, 3 we have that for every maximal stable set $Pr_X S$ of $< Pr_X V(H) >$ there exists a maximal clique $Pr_X Q$ of $< Pr_X V(H) >$ such that $Pr_X S \cap Pr_X Q = \emptyset$. This is a contradiction, since $< Pr_X V(H) >$ has a stable transversal.

This proves that $X[G_1, \ldots, G_n]$ is strongly perfect and the proof is complete. ■

For $G_1 = G_2 = \ldots = G_n = Y$ we obtain

Corollary 1. *The composition $X[Y]$ of graphs X and Y is strongly perfect if and only if both X and Y are strongly perfect.*

Using Lemma 4 and Theorem 1 we obtain

Corollary 2. $\overline{X[G_1, \ldots, G_n]}$ *is strongly perfect if and only if the graphs* $\overline{X}, \overline{G_1}, \ldots, \overline{G_n}$ *are strongly perfect.*

In consequence, it follows immediately

Theorem 2. $X[G_1, \ldots, G_n]$ *is co-strongly perfect if and only if the graphs* $X, G_1, \ldots, G_n$ *are co-strongly perfect.*

References

[1] C. Berge and P. Duchet, *Strongly perfect graphs*, Ann. Disc. Math. **21** (1984) 57–61.

[2] M. Borowiecki and A. Szelecka, *One factorizations of the generalized Cartesian product and of the X-join of regular graphs*, Discussiones Mathematicae **13** (1993) 15–19.

[3] M. Kwaśnik and A. Szelecka, *Strong perfectness of the generalized Cartesian product of graphs*, accepted for publication in the special issue of Discrete Math., devoted to the Second Kraków Conference on Graph Theory, Zakopane 1994.

[4] E. Mandrescu, *Strongly perfect product of graphs*, Czech. Math. Journal, **41** (116) (1991) 368–372.

[5] G. Ravindra and D. Basavayya, *Co-strongly perfect bipartite graphs*, Jour. Math. Phy. Sci. **26** (1992) 321–327.

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