Discussiones Mathematicae
Graph Theory 16 (1996) 119–122

A NOTE ON (k, l)-KERNELS IN B-PRODUCTS OF GRAPHS

Iwona Włoch

Department of Mathematics, Technical University of Rzeszów
W. Pola 2, 35–959 Rzeszów, Poland
e-mail: iwloch@ewa.prz.rzeszow.pl

Abstract

B-products of graphs and their generalizations were introduced in [4]. We determined the parameters k, l of (k, l)-kernels in generalized B-products of graphs. These results are generalizations of theorems from [2].

Keywords: kernels, distance in graphs, products of graphs.

1991 Mathematics Subject Classification: 05C12, 05C75.

1. Definitions and Notation

By G we mean a finite connected graph without loops and multiple edges with the vertex set $V(G)$ and the edge set $E(G)$. The number $d_G(x, y)$ denotes the length of the shortest path connecting x and y in G. Note that $d_G(x, y)$ is finite and $d_G(x, y) \geq 1$ if $x \neq y$.

Let k, l be integers, $k \geq 2$ and $l \geq 1$. $J \subset V(G)$ is called a (k, l)-kernel of G if and only if

(1) for distinct $x, y \in J, d_G(x, y) \geq k$ and

(2) for each $x \notin J$ there exists $y \in J$ such that $d_G(x, y) \leq l$.

For $k = 2,\ l = 1$ we obtain a kernel in Berge's sense.

The Cartesian product of two graphs G_1, G_2 is the graph $G_1 \times G_2$ with the vertex set $V(G_1) \times V(G_2)$ and the edge set $E(G_1 \times G_2)$, such that $[(x', y'), (x, y)] \in E(G_1 \times G_2)$ if and only if $[x', x] \in E(G_1)$ and $y = y'$ or $[y, y'] \in E(G_2)$ and $x = x'$.

The normal product of two graphs G_1, G_2 is the graph $G_1 \cdot G_2$, such that $V(G_1 \cdot G_2) = V(G_1) \times V(G_2)$ and $[(x', y'), (x, y)] \in E(G_1 \cdot G_2)$ if and only if $[x', x] \in E(G_1)$ and $y = y'$ or $[y', y] \in E(G_2)$ and $x = x'$ or $[x', x] \in E(G_1)$ and $[y', y] \in E(G_2)$.

So-called B-products of graphs were defined in [4] as follows.

Let $B \subset N \times N - \{(0,0)\}$, where N is the set of non-negative integers. Then the *B-product* of the graphs G_1, G_2 is the graph $B(G_1, G_2)$ with $V(B(G_1,G_2)) = V(G_1) \times V(G_2)$ and $E(B(G_1,G_2)) = \{[(i,j),(i',j')] : (d_{G_1}(i,i'), d_{G_2}(j,j')) \in B\}$. The set B is called the *basic set* of the B-product.

The *generalized Cartesian product* $B_m^n(G_1,G_2)$ and the *generalized normal product* $B_{mn}(G_1,G_2)$ are defined by the basic sets $B_m^n = \{(i,0) : 1 \leq i \leq m\} \cup \{(0,j) : 1 \leq j \leq n\}$, $B_{mn} = \{(i,j) : 0 \leq i \leq m$ and $1 \leq j \leq n$ or $1 \leq i \leq m$ and $0 \leq j \leq m\}$, respectively.

If $m = 1$ and $n = 1$, then $B_1^1(G_1,G_2) = G_1 \times G_2$ and $B_{11}(G_1,G_2) = G_1 \cdot G_2$. For $r \geq 1$ the *r-th power* G^r of a graph G is defined as follows: $V(G^r) = V(G)$ and $E(G^r) = \{[x,y] : x, y \in V(G)$ and $1 \leq d_G(x,y) \leq r\}$.

In [4] the following dependences between the well-known products and their generalizations were proved.

Theorem 1 [4]. $B_m^n(G_1,G_2) = G_1^m \times G_2^n$, $B_{mn}(G_1,G_2) = G_1^m \cdot G_2^n$, $B_{nn}(G_1,G_2) = (G_1 \cdot G_2)^n$, for $n, m \geq 1$.

For undefined terms, see [1].

2. Main Results

Theorem 2. *If J is a (k,l)-kernel of G, then J is a (k_0,l_0)-kernel of G^r, for $k, k_0 \geq 2$, and $l, l_0 \geq 1$, $r \leq k-1$ where*

$$k_0 = \begin{cases} \dfrac{k}{r}, & if \ \dfrac{k}{r} \ is \ an \ integer, \\ \left[\dfrac{k}{r}\right] + 1, & otherwise, \end{cases}$$

$$l_0 = \begin{cases} \dfrac{l}{r}, & if \ \dfrac{l}{r} \ is \ an \ integer, \\ \left[\dfrac{l}{r}\right] + 1, & otherwise, \end{cases}$$

where $[p]$ denotes the largest integer less than or equal to p.

Proof. Suppose, that J is a (k,l)-kernel of G. We shall show that J is a (k_0,l_0)-kernel of G^r, for k_0, l_0 as described above. By the definition of G^r it follows that if there exists a path of length $\leq r$ connecting x_i to x_j in G, then $[x_i,x_j] \in E(G^r)$. It is clear, that for distinct vertices $x_i, x_j \in J$

holds $d_G(x_i,x_j) \geq k$. This means that there is the shortest path of length $\geq k$, say $(x_i, x_{i+1}, x_{i+2}, ..., x_j)$, connecting vertices x_i, x_j in G. Moreover, using the definition of G^r, we obtain that the shortest path between x_i, x_j in G^r is of the form: $(x_i, x_{i+r}, x_{i+2r}, ..., x_{i+k}, ..., x_j)$, if $\frac{k}{r}$ is an integer, and $(x_i, x_{i+r}, x_{i+2r}, ..., x_{i+[\frac{k}{r}]r}, ..., x_j)$, otherwise. Note, that if $d_G(x_i,x_j) = k$, then $i+k=j$, if $\frac{k}{r}$ is an integer, and $i+[\frac{k}{r}]r+1 \leq j$, if $\frac{k}{r}$ is not an integer.

Finally,

$$d_{G^r}(x_i,x_j) \geq \begin{cases} \dfrac{k}{r}\;, & if \;\dfrac{k}{r}\; is\ an\ integer, \\ \left[\dfrac{k}{r}\right]+1, & otherwise. \end{cases}$$

Let $x_i \notin J$. So it is clear that there exists $x_j \in J$ in G, such that $d_G(x_i,x_j) \leq\ l$. Moreover, using the definition of G^r we have analogously that

$$d_{G^r}(x_i,x_j) \leq \begin{cases} \dfrac{l}{r}\;, & if \;\dfrac{l}{r}\; is\ an\ integer, \\ \left[\dfrac{l}{r}\right]+1, & otherwise. \end{cases}$$

Thus, the theorem is proved. ■

For $r = k-1$ we obtain the result from [3].
Using Theorems 1, 2 and Theorems 3 and 4 given below we obtain immediately Theorems 5, 6.

Theorem 3 [2]. *If the subset J_i is a (k_i,l_i)-kernel of G_i, where $k_i \geq 2$, $l_i \geq 1$, for $i=1,2$, then the set $J = J_1 \times J_2$ is a (k,l)-kernel of the graph $G_1 \times G_2$, where $k = \min\{k_1,k_2\}, l = l_1 + l_2$.*

Theorem 4 [2]. *If the subset J_i is a (k_i,l_i)-kernel of $G_i, k_i \geq 2, l_i \geq 1$, for $i=1,2$, then the set $J = J_1 \times J_2$ is a (k,l)-kernel of the graph $G_1 \cdot G_2$, where $k = \min\{k_1,k_2\}, l = \max\{l_1,l_2\}$.*

Theorem 5. *If J_i is a (k_i,l_i)-kernel of G_i, for $k_i \geq 2,\ l_i \geq 1,\ i=1,2$, then the set $J = J_1 \times J_2$ is a (k,l)-kernel of $B^n_m(G_1,G_2)$, for $m \leq k_1 - 1,\ n \leq k_2 - 1$, where $k = \min\{\alpha_1,\alpha_2\}, l = \beta_1 + \beta_2$ and*

$$\alpha_1 = \begin{cases} \dfrac{k_1}{m}\;, & if \;\dfrac{k_1}{m}\; is\ an\ integer, \\ \left[\dfrac{k_1}{m}\right]+1, & otherwise, \end{cases}$$

$$\alpha_2 = \begin{cases} \dfrac{k_2}{n}, & if \ \dfrac{k_2}{n} \ is\ an\ integer, \\ \left[\dfrac{k_2}{n}\right] + 1, & otherwise, \end{cases}$$

$$\beta_1 = \begin{cases} \dfrac{l_1}{m}, & if \ \dfrac{l_1}{m} \ is\ an\ integer, \\ \left[\dfrac{l_1}{m}\right] + 1, & otherwise, \end{cases}$$

$$\beta_2 = \begin{cases} \dfrac{l_2}{n}, & if \ \dfrac{l_2}{n} \ is\ an\ integer, \\ \left[\dfrac{l_2}{n}\right] + 1, & otherwise. \end{cases}$$

■

Theorem 6. *If J_i is a (k_i, l_i)-kernel of G_i, for $k_i \geq 2,\ l_i \geq 1, i = 1, 2$, then the set $J = J_1 \times J_2$ is a (k, l)-kernel of $B_{mn}(G_1, G_2)$, for $m \leq k_1 - 1, n \leq k_2 - 1$, where $k = \min\{\alpha_1, \alpha_2\},\ l = \max\{\beta_1, \beta_2\}$ and numbers α_i, β_i are defined as in Theorem* 5.

If $m = 1, n = 1$, then from Theorem 5 we obtain Theorem 3 and from Theorem 6 it follows Theorem 4.

References

[1] F. Harary, Graph Theory (Addison-Wesley, Reading, Mass., 1969).

[2] M. Kwaśnik, (k, l)-kernels in graphs and in their products (Ph.D.Dissertation, Wrocław, 1980).

[3] M. Kwaśnik, A. Włoch and I. Włoch, *Some remarks about (k, l)-kernels in directed and undirected graphs*, Discuss. Math. **13** (1993) 29–37.

[4] G. Schaar, M. Sonntag and H.M. Teichert, Hamiltonian properties of products of graphs and digraphs (Teubner-Texte zur Mathematik, Leipzig, 1988).

Received 19 May 1995
Revised 27 March 1996