# A NOTE ON $(k, l)$-KERNELS IN $B$-PRODUCTS OF GRAPHS 

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#### Abstract

$B$-products of graphs and their generalizations were introduced in [4]. We determined the parameters $k, l$ of $(k, l)$-kernels in generalized $B$-products of graphs. These results are generalizations of theorems from [2].


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## 1. Definitions and Notation

By $G$ we mean a finite connected graph without loops and multiple edges with the vertex set $V(G)$ and the edge set $E(G)$. The number $d_{G}(x, y)$ denotes the length of the shortest path connecting $x$ and $y$ in $G$. Note that $d_{G}(x, y)$ is finite and $d_{G}(x, y) \geq 1$ if $x \neq y$.

Let $k, l$ be integers, $k \geq 2$ and $l \geq 1 . J \subset V(G)$ is called a $(k, l)$-kernel of $G$ if and only if
(1) for distinct $x, y \in J, d_{G}(x, y) \geq k$ and
(2) for each $x \notin J$ there exists $y \in J$ such that $d_{G}(x, y) \leq l$.

For $k=2, l=1$ we obtain a kernel in Berge's sense.
The Cartesian product of two graphs $G_{1}, G_{2}$ is the graph $G_{1} \times G_{2}$ with the vertex set $V\left(G_{1}\right) \times V\left(G_{2}\right)$ and the edge set $E\left(G_{1} \times G_{2}\right)$, such that $\left[\left(x^{\prime}, y^{\prime}\right),(x, y)\right] \in E\left(G_{1} \times G_{2}\right)$ if and only if $\left[x^{\prime}, x\right] \in E\left(G_{1}\right)$ and $y=y^{\prime}$ or $\left[y, y^{\prime}\right] \in E\left(G_{2}\right)$ and $x=x^{\prime}$.

The normal product of two graphs $G_{1}, G_{2}$ is the graph $G_{1} \cdot G_{2}$, such that $V\left(G_{1} \cdot G_{2}\right)=V\left(G_{1}\right) \times V\left(G_{2}\right)$ and $\left[\left(x^{\prime}, y^{\prime}\right),(x, y)\right] \in E\left(G_{1} \cdot G_{2}\right)$ if and only if $\left[x^{\prime}, x\right] \in E\left(G_{1}\right)$ and $y=y^{\prime}$ or $\left[y^{\prime}, y\right] \in E\left(G_{2}\right)$ and $x=x^{\prime}$ or $\left[x^{\prime}, x\right] \in E\left(G_{1}\right)$ and $\left[y^{\prime}, y\right] \in E\left(G_{2}\right)$.

So-called $B$-products of graphs were defined in [4] as follows.
Let $B \subset N \times N-\{(0,0)\}$, where $N$ is the set of non-negative integers. Then the $B$-product of the graphs $G_{1}, G_{2}$ is the graph $B\left(G_{1}, G_{2}\right)$ with $V\left(B\left(G_{1}, G_{2}\right)\right)=V\left(G_{1}\right) \times V\left(G_{2}\right)$ and $E\left(B\left(G_{1}, G_{2}\right)\right)=\left\{\left[(i, j),\left(i^{\prime}, j^{\prime}\right)\right]:\right.$ $\left.\left(d_{G_{1}}\left(i, i^{\prime}\right), d_{G_{2}}\left(j, j^{\prime}\right)\right) \in B\right\}$. The set $B$ is called the basic set of the $B$-product.

The generalized Cartesian product $B_{m}^{n}\left(G_{1}, G_{2}\right)$ and the generalized normal product $B_{m n}\left(G_{1}, G_{2}\right)$ are defined by the basic sets $B_{m}^{n}=\{(i, 0): 1 \leq$ $i \leq m\} \cup\{(0, j): 1 \leq j \leq n\}, B_{m n}=\{(i, j): 0 \leq i \leq m$ and $1 \leq j \leq n$ or $1 \leq i \leq m$ and $0 \leq j \leq m\}$, respectively.

If $m=1$ and $n=1$, then $B_{1}^{1}\left(G_{1}, G_{2}\right)=G_{1} \times G_{2}$ and $B_{11}\left(G_{1}, G_{2}\right)=$ $G_{1} \cdot G_{2}$. For $r \geq 1$ the $r$-th power $G^{r}$ of a graph $G$ is defined as follows: $V\left(G^{r}\right)=V(G)$ and $E\left(G^{r}\right)=\left\{[x, y]: x, y \in V(G)\right.$ and $\left.1 \leq d_{G}(x, y) \leq r\right\}$.

In [4] the following dependences between the well-known products and their generalizations were proved.

Theorem 1 [4]. $\quad B_{m}^{n}\left(G_{1}, G_{2}\right)=G_{1}^{m} \times G_{2}^{n}, \quad B_{m n}\left(G_{1}, G_{2}\right)=G_{1}^{m} \cdot G_{2}^{n}$, $B_{n n}\left(G_{1}, G_{2}\right)=\left(G_{1} \cdot G_{2}\right)^{n}, \quad$ for $n, m \geq 1$.

For undefined terms, see [1].

## 2. Main Results

Theorem 2. If $J$ is a $(k, l)$-kernel of $G$, then $J$ is a $\left(k_{0}, l_{0}\right)$-kernel of $G^{r}$, for $k, k_{0} \geq 2$, and $l, l_{0} \geq 1, r \leq k-1$ where

$$
\begin{aligned}
& k_{0}= \begin{cases}\frac{k}{r}, & \text { if } \frac{k}{r} \text { is an integer } \\
{\left[\frac{k}{r}\right]+1,} & \text { otherwise }\end{cases} \\
& l_{0}= \begin{cases}\frac{l}{r}, & \text { if } \frac{l}{r} \text { is an integer } \\
{\left[\frac{l}{r}\right]+1,} & \text { otherwise }\end{cases}
\end{aligned}
$$

where $[p]$ denotes the largest integer less than or equal to $p$.
Proof. Suppose, that $J$ is a $(k, l)$-kernel of $G$. We shall show that $J$ is a $\left(k_{0}, l_{0}\right)$-kernel of $G^{r}$, for $k_{0}, l_{0}$ as described above. By the definition of $G^{r}$ it follows that if there exists a path of length $\leq r$ connecting $x_{i}$ to $x_{j}$ in $G$, then $\left[x_{i}, x_{j}\right] \in E\left(G^{r}\right)$. It is clear, that for distinct vertices $x_{i}, x_{j} \in J$
holds $d_{G}\left(x_{i}, x_{j}\right) \geq k$. This means that there is the shortest path of length $\geq k$, say ( $x_{i}, x_{i+1}, x_{i+2}, \ldots, x_{j}$ ), connecting vertices $x_{i}, x_{j}$ in $G$. Moreover, using the definition of $G^{r}$, we obtain that the shortest path between $x_{i}, x_{j}$ in $G^{r}$ is of the form: $\left(x_{i}, x_{i+r}, x_{i+2 r}, \ldots, x_{i+k}, \ldots, x_{j}\right)$, if $\frac{k}{r}$ is an integer, and $\left(x_{i}, x_{i+r}, x_{i+2 r}, \ldots, x_{i+\left[\frac{k}{r}\right] r}, \ldots, x_{j}\right)$, otherwise. Note, that if $d_{G}\left(x_{i}, x_{j}\right)=k$, then $i+k=j$, if $\frac{k}{r}$ is an integer, and $i+\left[\frac{k}{r}\right] r+1 \leq j$, if $\frac{k}{r}$ is not an integer.

Finally,

$$
d_{G^{r}}\left(x_{i}, x_{j}\right) \geq \begin{cases}\frac{k}{r}, & \text { if } \frac{k}{r} \text { is an integer }, \\ {\left[\frac{k}{r}\right]+1,} & \text { otherwise } .\end{cases}
$$

Let $x_{i} \notin J$. So it is clear that there exists $x_{j} \in J$ in $G$, such that $d_{G}\left(x_{i}, x_{j}\right) \leq l$. Moreover, using the definition of $G^{r}$ we have analogously that

$$
d_{G^{r}}\left(x_{i}, x_{j}\right) \leq \begin{cases}\frac{l}{r}, & \text { if } \frac{l}{r} \text { is an integer }, \\ {\left[\frac{l}{r}\right]+1,} & \text { otherwise }\end{cases}
$$

Thus, the theorem is proved.
For $r=k-1$ we obtain the result from [3].
Using Theorems 1, 2 and Theorems 3 and 4 given below we obtain immediately Theorems 5, 6 .

Theorem 3 [2]. If the subset $J_{i}$ is a $\left(k_{i}, l_{i}\right)$-kernel of $G_{i}$, where $k_{i} \geq 2$, $l_{i} \geq 1$, for $i=1,2$, then the set $J=J_{1} \times J_{2}$ is a $(k, l)$-kernel of the graph $G_{1} \times G_{2}$, where $k=\min \left\{k_{1}, k_{2}\right\}, l=l_{1}+l_{2}$.

Theorem 4 [2]. If the subset $J_{i}$ is a $\left(k_{i}, l_{i}\right)$-kernel of $G_{i}, k_{i} \geq 2, l_{i} \geq 1$, for $i=1,2$, then the set $J=J_{1} \times J_{2}$ is a $(k, l)$-kernel of the graph $G_{1} \cdot G_{2}$, where $k=\min \left\{k_{1}, k_{2}\right\}, l=\max \left\{l_{1}, l_{2}\right\}$.

Theorem 5. If $J_{i}$ is a $\left(k_{i}, l_{i}\right)$-kernel of $G_{i}$, for $k_{i} \geq 2, l_{i} \geq 1, i=1,2$, then the set $J=J_{1} \times J_{2}$ is a $(k, l)$-kernel of $B_{m}^{n}\left(G_{1}, G_{2}\right)$, for $m \leq k_{1}-1, n \leq$ $k_{2}-1$, where $k=\min \left\{\alpha_{1}, \alpha_{2}\right\}, l=\beta_{1}+\beta_{2}$ and

$$
\alpha_{1}= \begin{cases}\frac{k_{1}}{m}, & \text { if } \frac{k_{1}}{m} \text { is an integer }, \\ {\left[\frac{k_{1}}{m}\right]+1,} & \text { otherwise }\end{cases}
$$

$$
\begin{aligned}
& \alpha_{2}= \begin{cases}\frac{k_{2}}{n}, & \text { if } \frac{k_{2}}{n} \text { is an integer }, \\
{\left[\frac{k_{2}}{n}\right]+1,} & \text { otherwise, }\end{cases} \\
& \beta_{1}= \begin{cases}\frac{l_{1}}{m}, & \text { if } \frac{l_{1}}{m} \text { is an integer, } \\
{\left[\frac{l_{1}}{m}\right]+1,} & \text { otherwise },\end{cases} \\
& \beta_{2}= \begin{cases}\frac{l_{2}}{n}, & \text { if } \frac{l_{2}}{n} \text { is an integer, } \\
{\left[\frac{l_{2}}{n}\right]+1,} & \text { otherwise }\end{cases}
\end{aligned}
$$

Theorem 6. If $J_{i}$ is a $\left(k_{i}, l_{i}\right)$-kernel of $G_{i}$, for $k_{i} \geq 2, l_{i} \geq 1, i=1,2$, then the set $J=J_{1} \times J_{2}$ is a $(k, l)$-kernel of $B_{m n}\left(G_{1}, G_{2}\right)$, for $m \leq k_{1}-1, n \leq$ $k_{2}-1$, where $k=\min \left\{\alpha_{1}, \alpha_{2}\right\}, l=\max \left\{\beta_{1}, \beta_{2}\right\}$ and numbers $\alpha_{i}, \beta_{i}$ are defined as in Theorem 5.

If $m=1, n=1$, then from Theorem 5 we obtain Theorem 3 and from Theorem 6 it follows Theorem 4.

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