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THE COBONDAGE NUMBER OF A GRAPH

V.R. Kulli and B. Janakiram

Department of Mathematics, Gulbarga University
Gulbarga-585 106, India

Abstract

A set D of vertices in a graph $G = (V, E)$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. We define the cobondage number $b_c(G)$ of G to be the minimum cardinality among the sets of edges $X \subseteq P_2(V) - E$, where $P_2(V) = \{X \subseteq V : |X| = 2\}$ such that $\gamma(G + X) < \gamma(G)$. In this paper, the exact values of $b_c(G)$ for some standard graphs are found and some bounds are obtained. Also, a Nordhaus-Gaddum type result is established.

Keywords: graph, domination number, cobondage number.

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1. Introduction

The graphs considered here are finite, undirected without loops and multiple edges having p vertices and q edges. Any undefined term in this paper may be found in Harary [3]. $\lceil X \rceil$ is a least integer not less than X. Graphs considered in this paper have maximum degree at most $p - 2$.

A set D of vertices in a graph $G = (V, E)$ is a *dominating set* of G if every vertex in $V - D$ is adjacent to some vertex in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set. For a survey of results on domination (see [1]).

The communication network is an arrangement establishing a link between two or more locations come under some region. The problem is to set up the transmitting stations at some selected locations so that other locations should receive communication from at least one location, where the transmitter is to be established. The question is to find out the minimum number of transmitting stations required so that all the locations without transmitter will receive the message. The problem in the communication network can be reduced to the problem of finding a minimum dominating set in graph G, where the locations correspond to the vertices of a graph G and the communication links between the locations correspond to the edges in the graph G.

Suppose one wants to reduce the number of transmitting stations. Then additional communication links should be added. What is the fewest number of such links should be added to accomplish this task? It is the cobondage number of a graph.

The *cobondage number* $b_c(G)$ of a graph G is the minimum cardinality among the sets of edges $X \subseteq P_2(V) - E$, where $P_2(V) = \{X \subseteq V : |X| = 2\}$ such that $\gamma(G + X) < \gamma(G)$.

A γ-set is a minimum dominating set. Similarly, a b_c-set of edges can be defined.

2. Results

The following result is easy to prove, hence we omit its proof.

Theorem 1. *For any graph G,*

$$b_c(\overline{G}) \leq \delta(G) \tag{1}$$

where $\overline{G}$ and $\delta(G)$ are the complement and minimum degree of G, respectively. ■

Corollary 1.1. *For any graph G,*

$$b_c(G) \leq p - 1 - \Delta(G) \tag{2}$$

where $\Delta(G)$ is the maximum degree of G. ■

Now we obtain the exact values of $b_c(G)$ for some standard graphs.

Proposition 2. *If $G = K_{n_1, n_2, \ldots, n_t}$, where $n_1 \leq n_2 \leq \ldots \leq n_t$, then,*

$$b_c(G) = n_1 - 1. \tag{3}$$

Proof. Let $V = V_{n_1} \cup V_{n_2} \cup \ldots \cup V_{n_t}$. Then for any two vertices $v \in V_{n_1}$ and $w \in V_{n_j}$ for $2 \leq n_j \leq n_t$, $\{v, w\}$ is a γ-set for G. Since each V_{n_i} for $1 \leq n_i \leq n_t$, is independent with $|V_{n_i}| \geq 2$, by joining each vertex in $V_{n_i} - \{v\}$ to v we obtain a graph which has $\{v\}$ as a γ-set. This proves (3). ■

Proposition 3. *For any cycle C_p with $p \geq 4$ vertices,*

$$
\begin{aligned}
b_c(C_p) &= 1, \text{ if } p = 1 \pmod 3; && (4)\\
&= 2, \text{ if } p = 2 \pmod 3; && (5)\\
&= 3, \text{ otherwise.} && (6)
\end{aligned}
$$

Proof. Let $C_p : v_1v_2...v_pv_1$ denote a cycle on $p \geq 4$ vertices. We consider the following cases.

Case 1. If $p \equiv 1 \pmod 3$, then by joining the vertex v_{p-1} to v_1, we obtain a graph G which is a cycle $C_{p-1} : v_1v_2...v_{p-1}v_1$ together with a path $v_{p-1}v_pv_1$. This implies that,

$$
\begin{aligned}
\gamma(G) &= \gamma(C_{p-1})\\
&= \lceil (p-1)/3 \rceil < \lceil p/3 \rceil = \gamma(C_p).
\end{aligned}
$$

This proves (4).

Case 2. If $p \equiv 2 \pmod 3$, then by joining the vertices v_1 and v_p to v_{p-2} the resulting graph G is a cycle $C_{p-2} : v_1v_2...v_{p-2}v_1$ together with a path $v_{p-2}\ v_{p-1}v_pv_1$ such that v_{p-2} is adjacent to v_p. Thus

$$
\begin{aligned}
\gamma(G) &= \gamma(C_{p-2})\\
&= \lceil (p-2)/3 \rceil < \lceil p/3 \rceil = \gamma(C_p).
\end{aligned}
$$

Hence (5) holds.

Case 3. If $p \equiv 3 \pmod 3$, then by adding the edges v_1v_{p-3}, v_pv_{p-3}, $v_{p-1}v_{p-3}$, the resulting graph G is a cycle $C_{p-3} : v_1v_2...v_{p-3}v_1$ together with a path $v_{p-3}v_{p-2}v_{p-1}v_pv_1$ such that v_{p-3} is adjacent to both v_{p-1} and v_p. Hence,

$$
\begin{aligned}
\gamma(G) &= \gamma(C_{p-3})\\
&= \lceil (p-3)/3 \rceil < \lceil p/3 \rceil = \gamma(C_p).
\end{aligned}
$$

Thus (6) holds. ∎

Proposition 4. *For any path P_p with $p \geq 4$ vertices,*

$$
\begin{aligned}
b_c(P_p) &= 1, \text{ if } p \equiv 1 \pmod 3; && (7)\\
&= 2, \text{ if } p \equiv 2 \pmod 3; && (8)\\
&= 3, \text{ if } p \equiv 3 \pmod 3. && (9)
\end{aligned}
$$

Proof. Proofs (7), (8) and (9) are similar to that of proofs of (4), (5) and (6), respectively. ■

Theorem 5. *Let T be a tree with at least two cutvertices such that each cutvertex is adjacent to an endvertex. Then,*

$$b_c(T) = r \tag{10}$$

where r is the minimum number of endvertices adjacent to a cutvertex.

Proof. Let S be the set of all cutvertices of T. Then S is a γ-set for T. Let $u \in S$ be a cutvertex which is adjacent to minimum number of endvertices $u_1, u_2, ..., u_r$. Since there exists a cutvertex $v \in S$ such that v is adjacent to u, by joining $u_1, u_2, ..., u_r$ to v the graph obtained has $S - \{u\}$ as a γ-set. This proves (10). ■

Now we obtain some more upper bounds on $b_c(G)$.

Theorem 6. *For any graph G,*

$$b_c(G) \leq \Delta(G) + 1. \tag{11}$$

Furthermore, the bound is attained if and only if every γ-set D of G satisfying the following conditions:

(i) *D is independent;*

(ii) *every vertex in D is of maximum degree;*

(iii) *every vertex in $V - D$ is adjacent to exactly one vertex in D.*

Proof. Let D be a γ-set of G. We consider the following cases.

Case 1. Suppose D is not independent. Then there exist two adjacent vertices $u, v \in D$. Let $S \subset V - D$ such that for each vertex $w \in S$, $N(w) \cap D = \{v\}$. Then by joining each vertex in S to u, we see that $D - \{v\}$ is a γ-set of the resulting graph.

Thus,

$$b_c(G) \leq |S| \leq \Delta(G) - 1.$$

Case 2. Suppose D is independent. Then each vertex $v \in D$ is an isolated vertex in $< D >$. Let S be a set defined in Case 1. Since D has at least two vertices, by joining each vertex in $S \cup \{v\}$ to some vertex $w \in D - \{v\}$, we obtain a graph which has $D - \{v\}$ as a γ-set. Hence,

$$b_c(G) \leq |S \cup \{v\}| \leq \Delta(G) + 1.$$

The second part of the theorem directly follows from Cases 1 and 2. ■

Corollary 6.1. *For any graph G,*

$$b_c(G) \leq \min\{p - \Delta(G) - 1,\ \Delta(G) + 1\}. \tag{12}$$

Theorem 7. *For any graph G,*

$$b_c(G) \leq p - 1. \tag{13}$$

Further, the bound is attained if and only if $G = \overline{K}_2$.

Proof. Since $\Delta(G) \leq p - 2$, (13) follows from (11).

Suppose the bound is attained. Then by (1), it follows that $\overline{G} = K_p$. Suppose G has at least three vertices. Then $b_c(G) = 1 < p - 1$, a contradiction. This implies that $\overline{G} = K_2$ and hence $G = \overline{K}_2$. Converse is immediate. ∎

The next result improves the inequality (13).

Theorem 8. *For any graph G with $p \geq 3$ vertices,*

$$b_c(G) \leq p - 2. \tag{14}$$

Further, the bound is attained if and only if $G = 2K_2$ or $\overline{K}_3$ or $K_2 \cup K_1$.

Proof. (14) follows from (13).

Suppose the bound is attained. Then $\Delta(G) = 1$. Suppose $p \geq 5$. Then, $b_c(G) \leq p - 3$, a contradiction. This implies that $p = 3$ or 4. For $p = 3$, obviously $G = \overline{K}_3$ or $K_2 \cup K_1$. If $p = 4$ and G contains an isolate, then, $b_c(G) = 1$, a contradiction. This proves that $G = 2K_2$. Converse is easy to prove. ∎

The *bondage number* $b(G)$ of G is the minimum cardinality among the sets of edges $X \subseteq E$ such that $\gamma(G - X) > \gamma(G)$.

Theorem A [2]. *For any nontrivial tree T,*

$$b(T) \leq 2.$$

As a consequence of Theorem 6 and Theorem A, we have

Theorem 9. *Let T be a tree with* $\operatorname{diam}(T) = 5$ *and has exactly two cutvertices which are adjacent to endvertices and further they have same degree. Then,*

$$b_c(T) \geq b(T) + 1 \tag{15}$$

where $\operatorname{diam}(T)$ *is the diameter of T.* ■

Theorem 10. *For any tree T,*

$$b_c(T) \leq 1 + \min\{\deg u\} \tag{16}$$

where u is a cutvertex adjacent to an endvertex.

Proof. Since there exists a γ-set containing u, by applying same technique as we used in proving (11) we get (16). ■

The next result relates to $b_c(G)$ and $b_c(T)$.

Theorem 11. *Let T be a spanning tree of G such that $\gamma(T) = \gamma(G)$. Then,*

$$b_c(G) \leq b_c(T). \tag{17}$$

Proof. Let X be a b_c-set of T. Then there exists a set $X' \subseteq X$ such that $\gamma(G + X') < \gamma(G)$. This proves (17). ■

Now we obtain a relationship between $b_c(G)$ and $\gamma(G)$.

Theorem 12. *For any graph G,*

$$b_c(G) + \gamma(G) \leq p + 1. \tag{18}$$

Further, the equality holds if and only if $G = \overline{K}_p$.

Proof. Let D be a γ-set of G. Let $v \in V - D$. Then there exists a vertex $u \in D$ such that v is adjacent to u. Since there exists a vertex $w \in D - \{u\}$, by joining the vertices of $((V - D) - \{v\}) \cup \{w\}$ to u, we see that $D - \{w\}$ is a γ-set of the resulting graph. This proves (18). Now we prove the second part.

Suppose the equality holds. On the contrary, $G \neq \overline{K}_p$. Then by above, $b_c(G) \leq p - \gamma(G)$, a contradiction. This proves that $G = \overline{K}_p$. Converse is obvious. ■

The next result sharpnes the inequality (18).

Theorem 13. *Let D be a γ-set of G. If There exists a vertex $v \in D$ which is adjacent to every other vertex in D, then,*

$$b_c(G) \leq p - \gamma(G) - 1. \tag{19}$$

Proof. This follows from (2), since $\Delta(G) \geq \deg v \geq \gamma(G)$. ■

Lastly we obtain a Nordhaus-Gaddum type result.

Theorem 14. *Let G be a graph with $p \geq 4$ vertices such that neither G nor $\overline{G}$ is $2K_2$. Then,*

$$b_c(G) + b_c(\overline{G}) \leq 2\,(p-3). \tag{20}$$

The equality holds if and only if $G = P_4$ or C_5.

Proof. (20) follows from Theorem 8.

Suppose the equality holds. Then, $\Delta(G)$, $\Delta(\overline{G}) \leq 2$.
Suppose $\Delta(G)$ or $\Delta(\overline{G}) = 1$, say $\Delta(G) = 1$. Then, $\Delta(\overline{G}) \geq 3$, a contradiction. Hence, $\Delta(G) = \Delta(\overline{G}) = 2$. If $p \geq 6$, then, $\Delta(\overline{G}) \geq 3$, a contradiction. Thus, $p = 4$ or 5. This implies that $G = P_4$ or C_5. Converse is easy to prove. ■

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