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PLACING BIPARTITE GRAPHS OF SMALL SIZE II

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Abstract

In this paper we give all pairs of non mutually placeable (p, q)-bipartite graphs G and H such that $2 \leq p \leq q$, $e(H) \leq p$ and $e(G) + e(H) \leq 2p + q - 1$.

Keywords: packing of graphs, bipartite graph.

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1. Definitions

For a bipartite graph $G = (L, R; E)$ with the vertex set $V(G) = L \cup R$ and the egde set $E(G) = E$ we denote by $L = L(G)$ and $R = R(G)$ the *left* and the *right* set of bipartition of the vertex set of G, while the cardinality of the set is denoted by $e(G)$. For example, the graphs $G = (\{a, b\}, \{c, d\}; \{ac, ad\})$ and $G' = (\{c, d\}, \{a, b\}; \{ac, ad\})$ shown in Figure 1 are different.

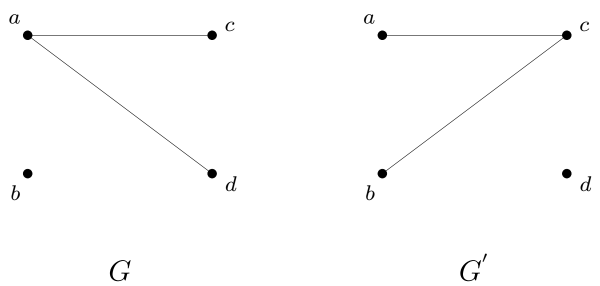

Figure 1

We denote by $N(x, G)$ the set of the *neighbors* of the vertex x in G. The *degree* $d(x, G)$ of the vertex x in G is the cardinality of the set $N(x, G)$; $\Delta_L(G)(\delta_L(G))$, $\Delta_R(G)(\delta_R(G))$ and $\Delta(G)(\delta(G))$ are the *maximum* (*minimum*) of the vertex degree in the sets $L(G)$, $R(G)$ and $V(G)$, respectively.

A vertex x of G is said to be *pendent* if $d(x, G) = 1$. K_{pq} stands for the *complete bipartite* graph with $|L(K_{pq})| = p$ and $|R(K_{pq})| = q$. A bipartite graph G is called (p, q)-*bipartite* if $|L(G)| = p$ and $|R(G)| = q$. If $p = q$, then G is called *balanced*.

Two graphs G and H of the same order are *packable* if G can be embedded in the complement $\bar{H}$ of H. If $G = (L, R; E)$ and $H = (L', R'; E')$ are two (p, q)-bipartite graphs, then we say that G and H are *mutually placeable* (or just m.p.) if there is a bijection $f\colon L \cup R \to L' \cup R'$ such that $f(L) = L'$ and $f(x)f(y)$ is not an of H whenever xy is an of G. The function f is called *bi-placement* of G and H. For example, the graphs $G = (\{a, b\}, \{c, d, e, f\}; \{ac, ad, be, bf\})$ and $H = (\{a', b'\}, \{c', d', e', f'\}; \{a'c', b'c'\})$ are not m.p. but G and H are packable. (See Figure 2).

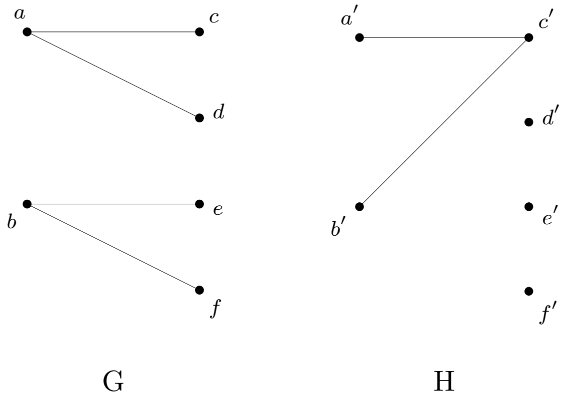

Figure 2

If a graph $G = (L, R; E)$ is a subgraph of a graph $F = (L, R'; E')$, then $L \subseteq L'$, $R \subseteq R'$, $E \subseteq E'$ and we write $G \leq F$.

2. Introduction

The classical "marriage theorem" of Frobenius [5] and Philip Hall's Theorem [6] may also be formulated as concerning the mutual placement of a matching and a bipartite graph. Richard Rado in [9] has proved a theorem in traversal theory, which may easily be transformed into a necessary and sufficient condition for two bipartite graphs to be mutually placeable (see [11]). So, even if the mutual placement of bipartite graphs has been introduced in [4], it is clear that the problem of mutual placeability of bipartite graphs is at least eighty years old.

The purpose of this paper is to characterize all pairs of (p, q)-bipartite graphs G and H such that $e(G) + e(H) \leq 2p + q - 1$, $e(H) \leq p$, $2 \leq p \leq q$

and G and H are not mutually placeable. For this reason we introduce now several graphs and families of graphs.

A (p, q)-bipartite graph of size q or p is said to be *left side bistar* $SL(p, q)$ or *right side bistar* $SR(p, q)$, respectively, if there is a vertex of the degree q or p, respectively, in its left or right, respectively, set of bipartition. If a (p, q)-bipartite graph G verifies $SL(p, q) \leq G$ or $SR(p, q) \leq G$, then G is an element of the set which we denote $\mathcal{S}'L(p, q)$ or $\mathcal{S}'R(p, q)$, respectively. $\mathcal{D}R(p, q)$ or $\mathcal{D}L(p, q))$ is the set of (p, q)-bipartite graphs G such that there is no isolated vertex in $R(G)$ or $L(G)$, respectively. $\mathcal{B}'L(p, p)$ or $\mathcal{B}'R(p, p)$ is the set of balanced bipartite graphs G of size $2p$ such that each vertex of $R(G)$ or $L(G)$, respectively has degree two. $BL(p, p)$ or $BR(p, p)$ is the (p, p)-bipartite graph of size $2p$ which has two vertices of degree p in its left or right, respectively, set of bipartition. Clearly, $BL(p, p) \in \mathcal{B}'L(p, p)$ and $BR(p, p) \in \mathcal{B}'R(p, p)$. $\mathcal{Z}'L(p, p)$ or $\mathcal{Z}'R(p, p)$ is the set of the (p, p)-bipartite graphs G of size $p-1$ such that for each vertex in $L(G)$ or $R(G)$, respectively, its degree is at most one. $ZL(p, p)$ or $ZR(p, p)$ is the (p, p)-bipartite graph of size $p - 1$ such that $ZL(p, p) \in \mathcal{Z}'L(p, p)$ or $ZR(p, p) \in \mathcal{Z}'R(p, p)$ and there is a vertex of degree $p - 1$ in its right or left, respectively, set of bipartition. We define $\mathcal{Z}_L(p, p)$ to be the set of pairs of (p, p)-bipartite graphs (G, H) such that either

$$G = BL(p, p) \text{ and } H \in \mathcal{Z}'L(p, p) \text{ (see Figure 3) or}$$
$$G \in \mathcal{B}'L(p, p) \text{ and } H = ZL(p, p) \text{ (see Figure 4).}$$

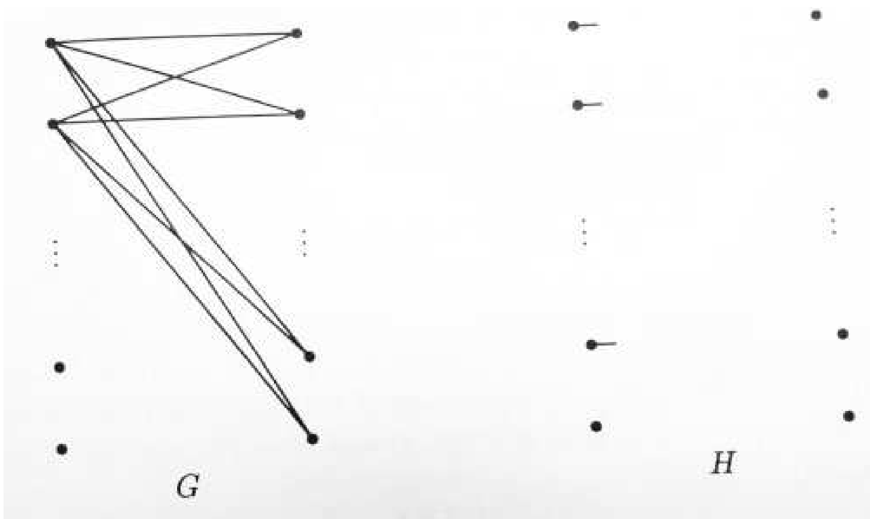

Figure 3

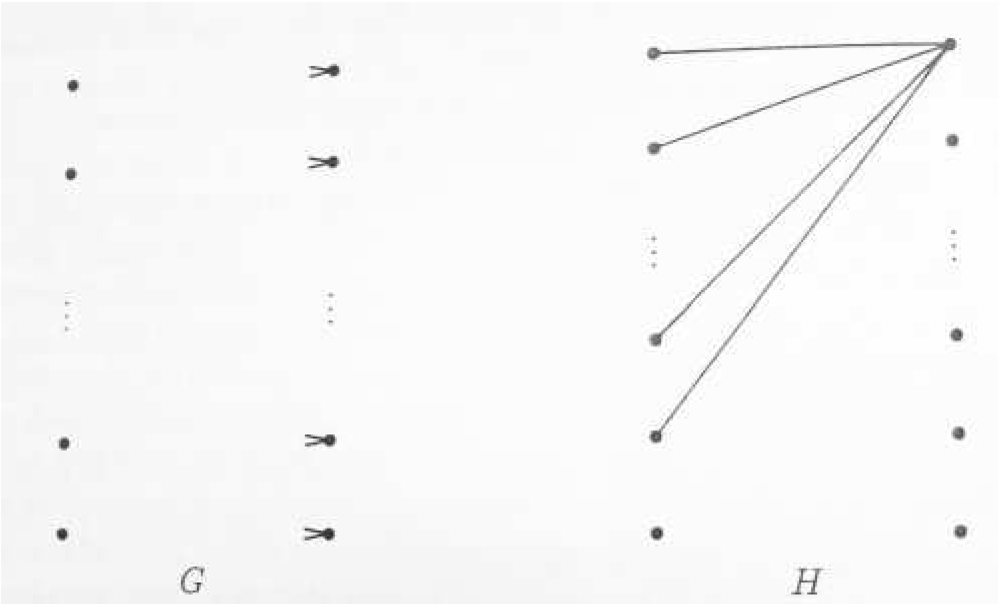

Figure 4

The set $\mathcal{Z}_R(p,p)$ we define analogically. By $\mathcal{Z}(p,p)$ we denote the set of graphs $\mathcal{Z}_L(p,p) \cup \mathcal{Z}_R(p,p)$. It is not difficult to see that $\mathcal{Z}(p,p)$ is a set of pairs of (p,p)-bipartite graphs (G,H) which are not bi-placeable, $e(G) = 2p$ and $e(H) = p-1$.

In a (p,q)-bipartite graph $G = (L,R;E)$ we denote by $P_k(R)$ or $P_k(L)$ the set of the k-element subsets of the set R or L, respectively. Let $A_k \in P_k(R)$. Define $N(A_k)$ to be the set of such vertices x that there is a vertex y in A_k such that xy is an edge in $E(G)$. By $T(A_k)$ we denote the set of vertices z which are adjacent to each vertex in the set A_k. We denote by $n(A_k)$ and $t(A_k)$ the cardinalities of the sets $N(A_k)$ and $T(A_k)$, respectively.

By $W(p,q)$ we denote the set of graphs such that

$$W(p,q) = \bigcup_{i=1}^{6} W_i(p,q), \quad \text{where}$$

$W_i(p,q)$ is the set of the pairs of (p,q)-bipartite graphs (G,H) such that $p \leq q$, $e(H) = p$, $e(G) \leq p+q-1$ and $W_1(p,q) = \{(G,H)\colon (G \in \mathcal{S}'L(p,q)$ and $H \in \mathcal{D}L(p,q))$ or $(p = q,\ G \in \mathcal{S}'R(p,p)$ and $H \in \mathcal{D}R(p,p))$ or $(G \in \mathcal{D}R(p,q)$ and $H = \mathcal{S}R(p,q))$ or $(p = q,\ G \in \mathcal{D}L(p,p)$ and $H = \mathcal{S}L(p,p))\}$ (see Figure 5a, 5b, 5c, 5d).

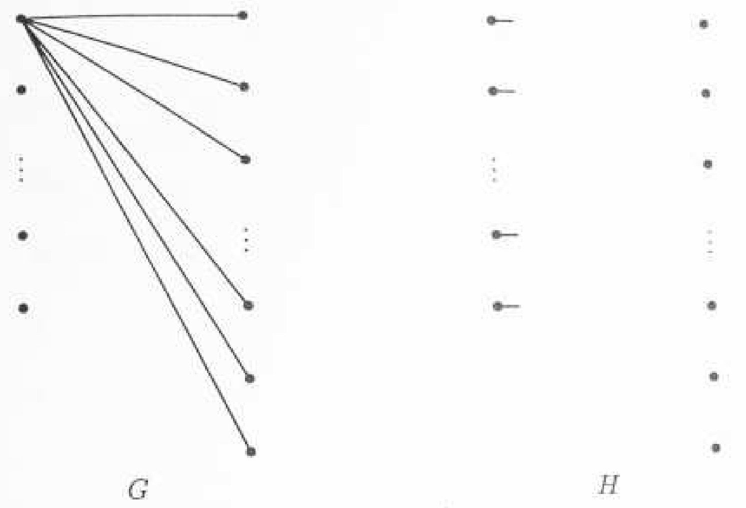

Figure 5a

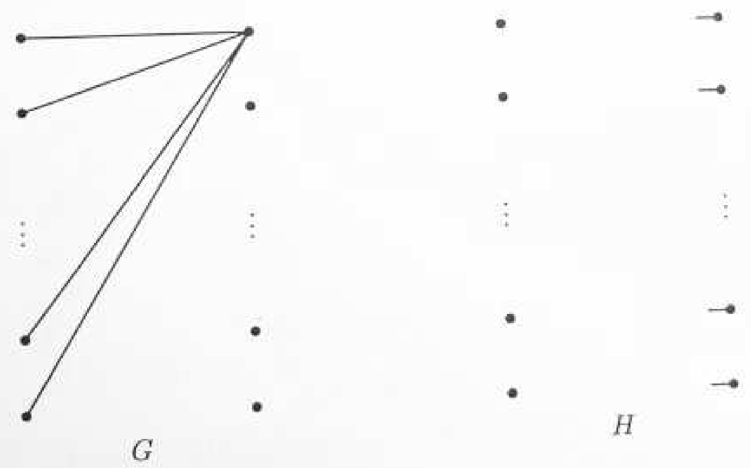

Figure 5b

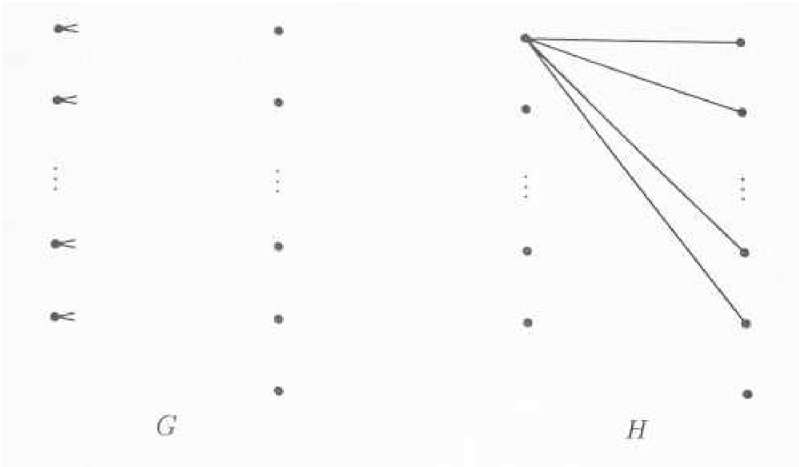

Figure 5c

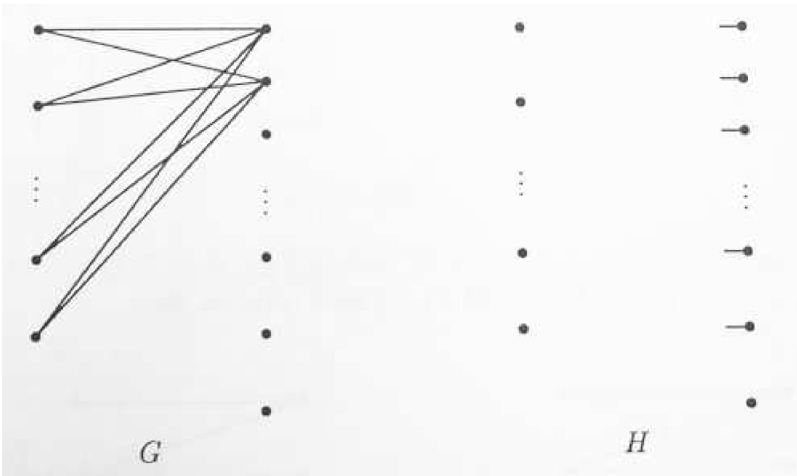

Figure 5d

$W_2(p,q) = \{(G,H)\colon q = p+1$, for each vertex in $L(G)$ its degree is two and there is a vertex of the degree p in $L(H)\}$ (see Figure 6),

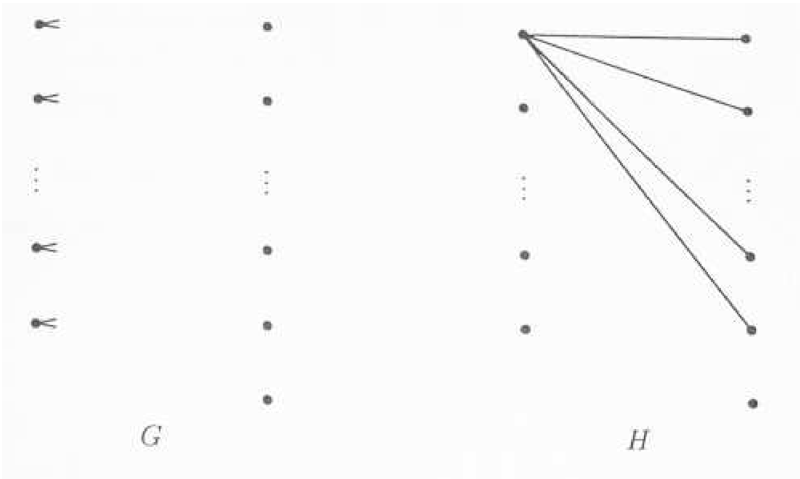

Figure 6

$W_3(p,q) = \{(G,H)\colon q = p+1$, there are two vertices y and y' in $R(G)$ such that $d(y,G) = d(y',G) = p$ and for each vertex in $R(H)$ its degree is at most one$\}$ (see Figure 7),

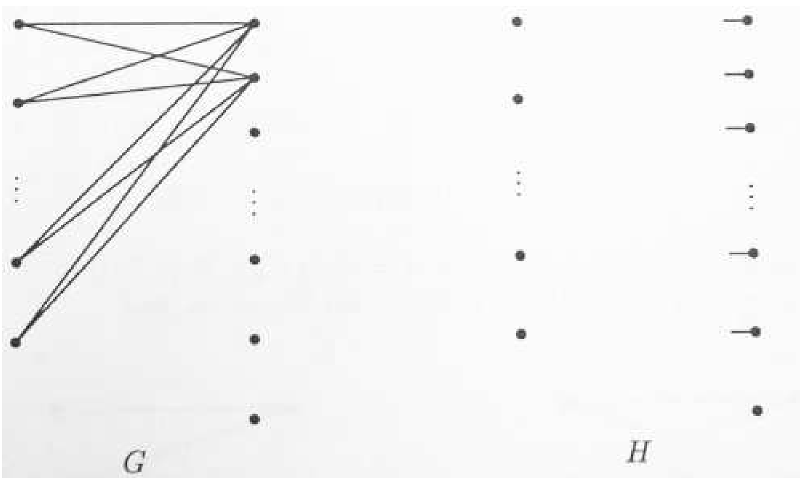

Figure 7

$W_4(p,q) = \{(G,H)$: either there is a vertex x in $L(G)$ such that its degree is $q-1$ and the degree of the vertex $y \in L$ non adjacent to x is at least two, each vertex in $L(H)$ is pendent, the degree of each non isolated vertex in $R(H)$ is at least $p - d(y,G) + 1$ (see Figure 8) or $p = q$, there is a vertex y in $R(G)$ such that its degree is $p-1$ and the degree of the vertex $x \in L$ non adjacent to y is at least two, each vertex in $R(H)$ is pendent, the degree of each non isolated vertex in $L(H)$ is at least $q - d(x, G) + 1\}$. In the latter case we say that $(G, H) \in W_{4'}$.

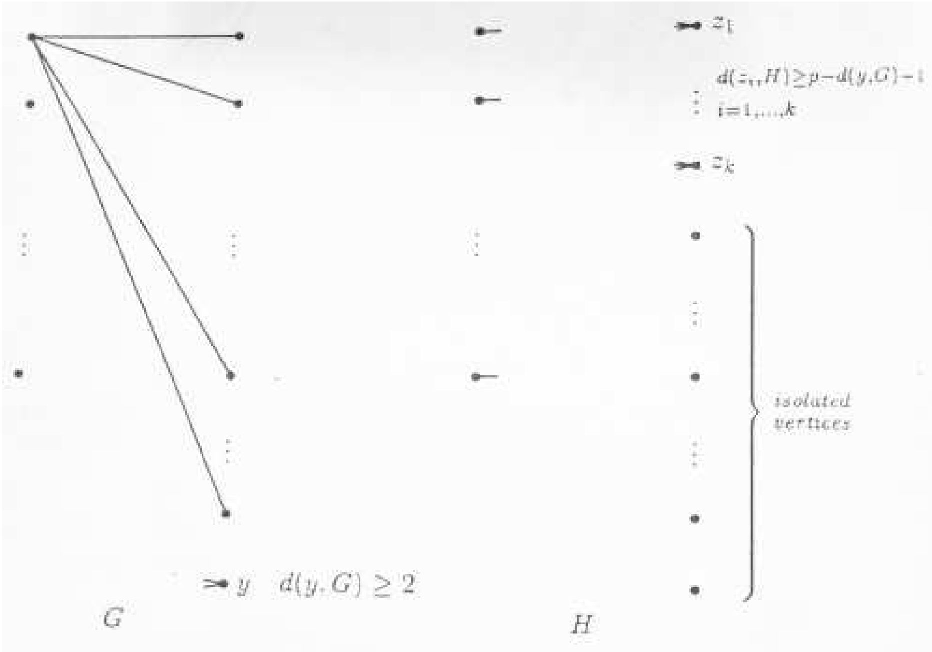

Figure 8

$W_5(p, q) = \{(G, H)\colon (p = 3,\ q = 3,\ G = C_4 \cup K_{1,1},\ H = P_3 \cup \overline{K_{1,1}}$ or $p = 4$, $q = 4$, $G = K_{1,1} \cup C_6$, $H = C_4 \cup \overline{K_{2,2}}\}$ (see Figure 9a, 9b),

Figure 9a

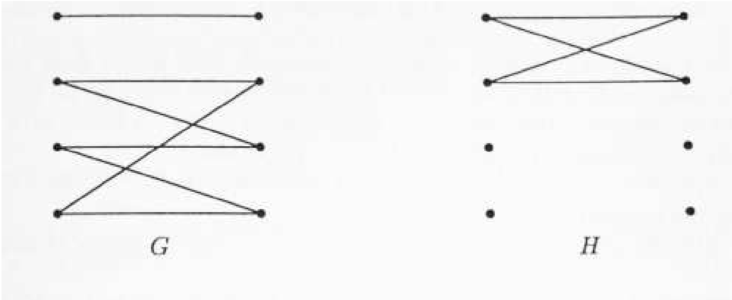

Figure 9b

$W_6(p,q) = \{(G,H)\colon pK_{1,1} \le H$ and there is an integer k in the set $\{1,\dots,p\}$ and a set $A_k \in P_k(L)$ such that $q - t(A_k) < k\}$ (see Figure 10).

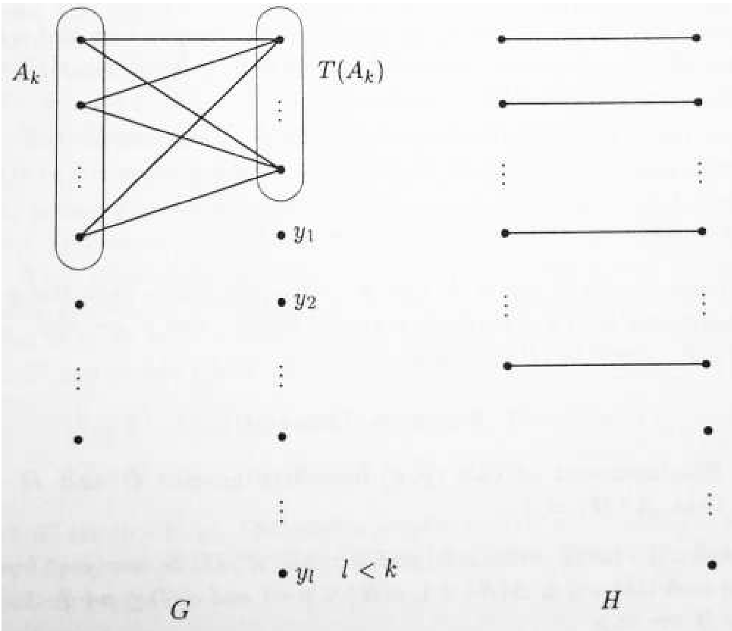

Figure 10

We denote by $W'(p,q)$ the subset $\bigcup_{i=1}^{5} W_i(p,q)$ of the set $W(p,q)$ and by $V(p,q)$ the set $\mathcal{Z}(p,p) \cup W(p,q)$.

3. Results

Consider graphs G and G' with n vertices such that $\Delta(G)$, $\Delta(G') < n-1$. The main packing theorem proved by Bollobás and Eldridge [2] shows that if we impose the extra condition: $e(G)+e(G') \leq 2n-3$ then, with finitely many exceptions (7 pairs), G and G' are packable.

S.T. Teo and H.P. Yap [10] proved that if $e(G)+e(G') \leq 2n-2$, $n \geq 5$, then the number of forbidden pairs is 49.

J.-L. Fouquet and A.P. Wojda proved in [4] the following theorem.

Theorem A. *Let $G = (L, R; E)$ and $H = (L', R'; E')$ be two (p, q)-bipartite graphs, $p, q \geq 2$, such that $e(G) + e(H) \leq p + q$. Then G and H are m.p. unless $\{G, H\} = \{F_1, F_2\}$, where either $F_1 = SL(p, q)$ and $F_2 \in \mathcal{D}L(p, q)$ or $F_1 = SR(p, q)$ and $F_2 \in \mathcal{D}R(p, q)$.* ■

Theorem B was proved in [8].

Theorem B. *Let G and H be two (p, q)-bipartite graphs such that $e(G) \leq p+q$, $e(H) \leq p-1$, $p \leq q$. Then G and H are m.p. unless $p = q$, $e(G) = 2p$, $e(H) = p-1$ and $(G, H) \in \mathcal{Z}(p, p)$.* ■

In this paper we give a necessary and sufficient condition for two (p, q)-bipartite graphs G and H such that $2 \leq p \leq q$, $e(H) \leq p$ and $e(G)+e(H) \leq 2p+q-1$ to be m.p.

The main result of this paper is Theorem 1.

Theorem 1. *Let G and H be two (p, q)-bipartite graphs such that $p \leq q$, $e(H) \leq p$ and $e(G) + e(H) \leq 2p + q - 1$. Then G and H are m.p. unless $e(H) \geq p-1$ and $(G, H) \in V(p, q)$.*

4. Proof of Theorem 1

4.1. Bi-placement of two (p, q)-bipartite graphs G and H such that $\Delta\ (H) = 1$

Remark 1.1. *Let $G = (L, G; E)$ and $H = (L', R'; E')$ be two (p, q)-bipartite graphs such that $p \leq q$, $\Delta(H) = 1$, $e(H) \leq p-1$ and $e(G) \leq p+q-1$. Then G and H are m.p.*

Remark 1.2. *If G and H are two (p, q)-bipartite graphs such that $e(H) = p$, $pK_{1,1} \leq H$, then G and H are m.p. if and only if there is a matching of the cardinality p in the graph $G' = K_{p,q} - G$.*

Theorem of Philip Hall. *Let $G = (L, G; E)$ be a (p, q)-bipartite graph. Then G has matching of cardinality p if and only if for each integer $k \in \{1, \dots, p\}$, for each set $A_k \in P_k(L)$*

$$k \leq n(A_k).$$ ■

The following theorem is a corollary from Remark 1.2.

Theorem 1.3. *Let G and H be two (p, q)-bipartite graphs such that $p \leq q$, $e(H) = p$ and $\Delta(H) = 1$. Then G and H are m.p. unless $(G, H) \in W_6(p, q)$.*

4.2. Placing two (p, q)-bipartite graphs G and H such that $\Delta\ (H) \geq 2$

We shall prove two lemmas first.

Lemma 2.1. *Let $G = (L, R; E)$ and $H = (L', R'; E')$ be two (p, q)-bipartite graphs such that $2 \leq p \leq q$, $\Delta(H) \geq 2$, $e(G) \leq p + q - 1$, $e(H) \leq p$, there is a vertex x in L such that $d(x, G) = q - 1$, the degree of the vertex $y \in R - N(x, G)$ is at least two and $\Delta_L(H) = 1$. Then G and H are mutually placeable unless either $(G, H) \in W_1(p, q)$ or $(G, H) \in W_4(p, q)$.*

Proof. Let $G = (L, R; E)$ and $H = (L', R'; E')$ be two (p, q)-bipartite graphs verifying the assumptions of the lemma. If $e(H) \leq p-1$ then H and G are m.p. by Theorem B. Now we suppose that $e(H) = p$ and $e(G) \leq p+q-1$. If $(G, H) \in W_1(p, q)$ or $(G, H) \in W_4(p, q)$, then clearly G and H are not mutually placeable. Let $d(y, G) = k$. We may assume that there exists a vertex w in R' such that $0 < d(w, H) = k' \leq p - k$. Let us define sets Z and Z' and graphs G', H' in the following way:

$$Z \subseteq L - N(y, G),\ x \in Z \text{ and } |Z| = k';\ Z' = N(w, H),$$
$$G' = G - Z - \{y\},\ H' = H - Z' - \{w\}.$$

G' and H' are $(p - k', q - 1)$-bipartite graphs, $e(G') \leq p - k$, $e(H') = p - k'$. By the Theorem A, G' and H' are mutually placeable. Hence a bi-placement of G and H is evident. ■

Lemma 2.2. *Let $G = (L, R; E)$ and $H = (L', R'; E')$ be two (p, q)-bipartite graphs such that $2 \leq p \leq q$, $e(G) \leq p + q - 1$, $e(H) \leq p$, $\Delta(H) \geq 2$, $\Delta_L(H) = 1$ and there is no isolated vertex in R. Then G and H are m.p. unless either $(G, H) \in W_1(p, q)$ or $(G, H) \in W_4(p, q)$.*

Proof. Let $G = (L, R; E)$ and $H = (L', R'; E')$ be two (p, q)-bipartite graphs verifying the assumptions of the lemma. We may suppose that $e(G) \leq p + q - 1$, $e(H) = p$ and each vertex in L' is pendent (if $e(H) < p$ we may use Theorem B). If there is a vertex of degree q in L, then $(G, H) \in W_1(p, q)$. It is clear that there is a pendent vertex, say y, in R. Let us take a vertex w in R' such that $d(w, H) = \Delta_R(H) \geq 2$. If $d(w, H) = p$ then $(G, H) \in W_1(p, q)$. If $d(w, H) < p$ then, by Theorem B, there is a bi-placement $f\colon L' \cup R' - \{w\} \to L \cup R - \{y\}$ of the graphs $H' = H - \{w\}$ and $G' = G - \{y\}$. If $f[N(w, H)] \subseteq L - N(y, G)$, then a bi-placement of H and G is evident. So now, we may assume that $N(y, G) \cap f[N(w, H)] \neq \emptyset$. Put $\{x\} = N(y, G)$. Let us denote by z the vertex in L' such that $z \in N(w, H)$ and $f(z) = x$. If there is a non isolated vertex w' in $R' - \{w\}$ such that $f(w') = y' \in R - N(x, G)$, we take a vertex z' in L' such that $z' \in N(w', H)$ and the vertex $x' = f(z') \in L$. Now g defined by

$$g(s) = f(s) \text{ if } s \in V(H) - \{w, z, z'\},$$
$$g(z') = x,\ g(z) = x',\ g(w) = y$$

is a bi-placement of H and G. So we assume that $f(w') \in N(x, G)$ for each non isolated vertex w' in $R' - \{w\}$. Since $d(w, H) = \Delta_R(H)$, then $d(w', H) \leq d(w, H)$. Hence $d(w', H) \leq p/2$. We may assume that there is a vertex $y' \in R - N(x, G)$ such that $d(y', G) + d(w', H) \leq p$. The above condition is true unless $d(x, G) = q$ (then $(G, H) \in W_1(p, q)$) or $d(x, G) = q - 1$ and $d(y'', G) > p/2$ for $y'' \notin N(x, G)$ (in this case we may use Lemma 2.1). Let us denote by w'' such a vertex of R' that $f(w'') = y'$. Let w' be a non isolated vertex in $R' - \{w\}$, $y'' = f(w')$, $z' \in N(w', H)$ and $f(z') = x'$. If $f[N(w', H)] \cap N(y', G) = \emptyset$, then we define by g a new bi-placement of $H - \{w\}$ and $G - \{y\}$ in the following way:

$$g(s) = f(s) \text{ if } s \in V(H) - \{w, w', w'', z, z'\},$$
$$g(w') = y', g(w'') = y'', g(z) = x', g(z') = x.$$

Now a bi-placement of H and G is evident. If $f[N(w', H)] \cap N(y', G) = A \neq \emptyset$ let us denote elements of the set A by x'_i, $i = 1, 2, \ldots, l$; by z'_i such elements of the set L' that $f(z'_i) = x'_i$. Let $B = L - (N(y', G) \cup f[N(w', H)]) \cup \{x\}$. Let us denote elements of the set B by x_i, $i = 1, 2, \ldots, k$, $x_1 = x$ and by z_i such elements of the set L' that $f(z_i) = x_i$. Since $d(y', G) + d(w', H) \leq p$,

$l \le k$. So now we define a bi-placement g of $H - \{w\}$ and $G - \{y\}$ such that

$$g(s) = f(s) \text{ if } s \in V(H) - \{\{z_1, z_2, \ldots, z_l, z'_1, \ldots z'_l\} \cup \{w, w', w''\}\}$$
$$g(z_i) = x'_i, g(z'_i) = x_i, i = 1, 2, \ldots, l,$$
$$g(w') = y', g(w'') = y''.$$

It is clear now that H and G are m.p. ■

Theorem 2.3. *Let $G = (L, R; E)$ and $H = (L', R'; E')$ be two (p, q)-bipartite graphs such that $2 \le p \le q$, $\Delta(H) \ge 2$, $e(G) \le p+q-1$, $e(H) \le p$. Then G and H are m.p. unless $e(H) = p$ and $(G, H) \in W'(p, q)$.*

Proof. The proof is by induction on $p + q$. It is not difficult to check that the theorem is true for $p = 2$ and arbitrary $q \ge p$ and for $p + q \le 8$. Let us assume that $p \ge 3$, $p \le q$, $p + q \ge 9$ and every two (p', q')-bipartite graphs G' and H', with $p' + q' < p + q$ verifying the assumptions of the theorem, are m.p. Let $G = (L, R; E)$ and $H = (L', R'; E')$ be two (p, q)-bipartite graphs such that $p \ge 3$, $p \le q$ and $\Delta(H) \ge 2$. By Theorem B we may assume that $e(H) = p$ and $e(G) \le p + q - 1$. To prove the theorem we shall distinguish two cases.

Case 1. There is an isolated vertex, say y, in R. If there is no isolated vertex in L', then for each vertex in L' its degree is one. Then there is a vertex w in R' such that its degree is at least two, otherwise $\Delta(H) = 1$. Let $x \in L$ be such that $d(x, G) = \Delta_L(G)$ and let $z \in N(w, H)$. $G - \{x, y\}$ and $H - \{w, z\}$ are m.p. by the induction hypothesis when $\Delta(H - \{z, w\}) > 1$ or by Remark 1.1 when $\Delta(H - \{z, w\}) = 1$. Hence also G and H are m.p. So now, let z be an isolated vertex in L', $w \in R'$ such that $d(w, H) = \Delta_R(H)$, and $x \in L$ such that $d(x, G) = \Delta_L(G)$. If $d(w, H) \ge 2$, then the graphs $G' = G - \{x, y\}$ and $H' = H - \{w, z\}$ are $(p - 1, q - 1)$-bipartite and $e(G') \le (p - 1) + (q - 1) - 1$, $e(H') \le p - 2$. Hence, by the induction hypothesis or Remark 1.1, G' and H' are m.p. A bi-placement of G and H is evident. If $d(w, H) = 1$, then $e(H') = p - 1$ and if $\Delta(H') \ge 2$, then by the induction hypothesis, G' and H' are m. p. unless $(G', H') \in W'(p-1, q-1)$. (If $(p - 1)K_{1,1} \le H'$, then there are two pendent adjacent vertices w' and z' in H'. We may take $H' = H - \{w', z'\}$ and $\Delta(H') \ge 2$.) Observe that $(G', H') \notin W_5(p - 1, q - 1)$. Hence we may assume that $(G', H') \in \bigcup_{i=1}^{4} W_i(p - 1, q - 1)$. Below we consider all possible cases.

1. $(G', H') \in W_1(p - 1, q - 1)$
   (a) $G' \in \mathcal{S}'R(p - 1, q - 1)$ and $H' \in \mathcal{D}R(p - 1, q - 1)$. Then $p = q$ and we may use Lemma 2.2.

(b) $G' \in \mathcal{S}'L(p-1,q-1)$ and $H' \in \mathcal{D}L(p-1,q-1)$. There is a vertex x' in $L-\{x\}$ such that $d(x',G)=q-1$. Since $d(x,G)=\Delta_L(G)$, then $d(x,G)=q-1$ and $e(G)\geq 2(q-1)$. Hence $p=q$ or $p=q-1$. We may choose a pendent vertex z' in L' and a vertex w' in $N(z',H)$. If $p=q$ let us define the graphs $G''=G-\{x,x',y\}$ and $H''=H-\{z,w',z'\}$, where $z'\in L'\setminus\{z\}$ and $w'\in N(z')$. By the Theorem A G'' and H'' are m.p. Hence G and H are m.p. too. If $p=q-1$, then any function f such that $f(x)=z$, $f(x')=z'$, $f(y)=w'$ is a bi-placement of G and H.

(c) $G' \in \mathcal{D}R(p-1,q-1)$ and $H'=SR(p-1,q-1)$. We can check that in this case $p=2$.

(d) $G' \in \mathcal{D}L(p-1,q-1)$ and $H'=SL(p-1,q-1)$. Now $p=q$ and if $w\in N(z',H)$, where $d(z',H')=p-1$, then $(G,H)\in W'(p,p)$. If $w\notin N(z',H)$, then a bi-placement of G and H is evident unless $G\in\mathcal{S}'R(p,p)$ and $(G,H)\in W'(p,p)$.

2. $(G',H')\in W_2(p-1,q-1)$. Now we have: $p=q-1$, $d(x,G)=2$ and for each vertex in L its degree is two. Let z_1 be a vertex in L' of degree $p-1$ in H'. If $d(z_1,H)=p$, then $(G,H)\in W_2(p,p+1)$. Now we suppose that $w\notin N(z_1,H)$. If there are two vertices of degree p in R, then $(G,H)\in W_3(p,p+1)$. In other case there is a non isolated vertex in R, say y_1, of degree less than p. Let $x_1\in N(y_1,G)$, $y_2\in N(x_1,G)-\{y_1\}$, $z_2\in N(w,H)$. Any bijection f such that: $f(z_1)=x_1$, $f(w)=y_1$, $f(z_2)\in L-N(y_1,G)$, $f[N(z_1,H)]=R-\{y_1,y_2\}$ is a bi-placement of G and H.
3. $(G',H')\in W_3(p-1,q-1)$. Let y_1, y_2 be vertices in $R-\{y\}$ of degree $p-1$ in G'. If $d(y_1,G)=d(y_2,G)=p$ or $\Delta_L(H)=p$, then $(G,H)\in W'(p,q)$. Now we assume that $x\notin N(y_1,G)$ and $\Delta_L(H)<p$. Let z_1 be a vertex in L' such that $2\leq d(z_1,H)\leq p-1$. Let $x_1\in L-\{x\}$. Then y_1, $y_2\in N(x_1,G)$ and $d(y_1,G)\leq d(y_2,G)$. Let w_1 be isolated in R'. The graphs $G''=G-\{x,x_1,y,y_1\}$ and $H''=H-\{z,z_1,w,w_1\}$ are m.p. A function, say f, defined by $f(s)=g(s)$, $s\in V(H'')$, where g — a bi-placement of H'' and G'', $f(z_1)=x$, $f(z)=x_1$, $f(w)=y$, $f(w_1)=y_1$ is a bi-placement of H and G.
4. $(G',H')\in W_4(p-1,q-1)$ Let $x'\in L-\{x\}$ be such that $d(x',G')=q/-2$, hbox$y'\in(R-N(x',G')-\{y\})$. Notice that $d(y',G')\leq p-2$. For each non isolated vertex w' in $R'-\{w\}$ $d(w',H')+d(y',G')>p-1$. But $\Delta_R(H')=1$. Hence $d(y',G')\geq p-1$ a contradiction. If $(G',H')\in W_{4'}(p-1,p-1)$, then we may use Lemma 2.2.

Case 2. There is no isolated vertex in R. Notice that there is a pendent vertex, say y, in R. If each vertex in L' is pendent then we may use Lemma 2.2. So we assume that there is an isolated vertex z in L'. Let $w \in R'$ be such that $d(w, H) = \Delta_R(H)$. Now we consider two subcases.

Subcase 2.1. The degree of the vertex w is at least two. If $d(x, G) \geq 2$ for $x \in N(y, G)$ then, by the induction hypothesis, there is a bi-placement of $H - \{z, w\}$ and $G - \{x, y\}$. A bi-placement of H and G is evident. If $d(x, H) = 1$, then we define the pair of graphs (G', H') in the following way:

— if $d(w, H) \geq 3$ put $G' = G - \{x, y, y'\}$, $H' = H - \{z, w, w'\}$, where y' is a vertex in R such that $e(G - \{x, x', y\}) \leq p + q - 4$, and w' is isolated in R';

— if $d(w, H) = 2$ and there is a non isolated vertex z' in $L' - N(w, H)$ then put $G' = G - \{x, x', y\}$, $H' = H - \{w, z, z'\}$, where $x' \in L$ such that $e(G - \{x, x', y\}) \leq p + q - 4$;

— if $d(w, H) = 2$ and each vertex in $L' - N(w, H)$ is isolated, then we take a vertex $z'' \in L'$ such that $d(z'', H) \geq p/2$ and an isolated vertex w' in R.

For $p \geq 5$ let $G' = G - \{x, x'', y\}$, $H' = H - \{z, z'', w'\}$, where $x'' \in L$ such that $d(x'', G) \geq 2$. By the induction hypothesis or Remark 1.1, G' and H' are m.p. Hence G and H are m.p., too. For $p \leq 4$ it is easy to check that either (G, H) are m.p. or $(G, H) \in W_5(p, q)$.

Subcase 2.2. For each vertex in R' its degree is at most one. Let $z' \in L'$ and $d(z', H) = \Delta_L(H) = k$. Hence $k \geq 2$. Let us suppose first that $d(x, G) > q - k$, for each vertex $x \in L$. Then we have

$$(*) \qquad p + q - 1 \geq e(G) \geq p(q - k + 1)$$

and, since $k \leq p$, we have $q - 1 \geq p(q - p)$ and we see that $q \leq p + 1$. Therefore, and by the unequality $(*)$, $p \geq q - 1 \geq p(q - k)$ and $k \geq q - 1$. Thus $k = p$ and now we may easily deduce that either $(G, H) \in W_1(p, q)$, or else $(G, H) \in W_2(p, q)$. So from now on we may assume that there is a vertex $x \in L$ such that $d(x, G) \leq q - k$. If $d(x, G) = 0$, then G and H are m.p. by Theorem B. If $d(x, G) \geq 1$, then we define the sets Z and Z' such that $Z' = N(z', H)$, $|Z'| = k$, $Z \subseteq R - N(x, G)$ and $|Z| = k$. Let $G' = G - \{x\} - Z$, $H' = H - \{z'\} - Z$. G' and H' are $(p-1, q-k)$-bipartite graphs and $e(G') \leq (p - 1) + (q - k) - 1$, $e(H') = p - k$ and there is a bi-placement of G' and H'. Hence for $p < q$ we have $p - k < \min\{p - 1, q - k\}$ and there is a bi-placement of G' and H'. A bi-placement of G and H is evident. For $p = q$ each vertex in R' is pendent and there are no isolated vertices in L. Hence we may use Lemma 2.2. ■

Theorem 2.4. *Let $G = (L, R; E)$ and $H = (L', R'; E')$ be two (p, q)-bipartite graphs such that $2 \le p \le q$, $e(G) \le p + q + k - 1$, $e(H) \le p - k$, where $2 \le k \le p$. Then G and H are m.p.*

Proof. Let k be an integer such that $k \in \{2, \dots, p\}$. The proof is by induction on $p+q$. The theorem is easy to check when $p \le 3$ and $p \le q$. So let us suppose $p \ge 4$, $q \ge p$ and the theorem is true for all positive integers p', q' such that $p'+q' < p+q$. Let $H = (L, R; E)$ and $G = (L', R'; E')$ be two (p, q)-bipartite graphs verifying the assumptions of the theorem. We may assumme that $e(G) = p + q + k - 1$ and $e(H) = p - k$. The theorem is easy to check for $k = p$ or $k = p-1$. Now we suppose that $2 \le k \le p-2$. Notice that $\delta_R(G) \le 2$ and let z_0 be a vertex in R such that $d(z_0, G) = \delta_R(G)$. To prove that G and H are m.p. we shall distinguish four cases.

Case 1. The vertex z_0 is isolated. Let $w \in L$ be such a vertex that $d(w, G) \ge 2$, y be non isolated in R' and x be an isolated in L'. By the induction hypothesis, there is a bi-placement of $G - \{w, z_0\}$ and $H - \{x, y\}$. A bi-placement of G and H is evident.

Case 2. The vertex z_0 is pendent and the degree of its neighbor w is at least two. Now choose a vertex $y \in R'$ such that $d(y, H) \ge 1$ and an isolated vertex, say x, in L' and proceed like in the preceding case.

Case 3. There is no isolated vertex in R, there are pendent vertices in R and for each vertex in R also its neighbor is pendent. We may choose vertices $\{w_0, w_1, z_0, z_1\}$ of the graph G and vertices $\{x_0, x_1, y_0, y_1\}$ of the graph H in the following way: w_0 is pendent in L, $z_0 \in N(w_0, G)$, $w_1 \in L$, $z_1 \in R$ such that $d(w_1, G) \ge 2$, $d(z_1, G) \ge 2$ and x_0, $x_1 \in L'$, y_0, $y_1 \in R'$ such that $d(x_1, H) = d(y_1, H) = 0$, $(x_0 y_0) \notin E'$ and $|N(x_0, H) \cup N(y_0, H)| \ge 2$. The graphs $G' = G - \{w_0, w_1, z_0, z_1\}$ and $H' = H - \{x_0, x_1, y_0, y_1\}$ are $(p-2, q-2)$-bipartite and, for $k \le p-2$, verify the induction hypothesis. Hence there is a bi-placement, say g, of G' and H'. The bijection f defined by

$$\begin{aligned} &f(w) = g(w), \text{ for } w \in V(G'), \\ &f(w_i) = x_i, \\ &f(z_i) = y_i \ i = 0, 1 \end{aligned}$$

is a bi-placement of G and H.

Case 4. The degree of the vertex z_0 is two.

a) If there is a vertex, say y_0, in R' such that $d(y_0, H) \geq 2$, then we choose vertices x, x_1 and y_1 of the graph H in the following way: x, x_1 are isolated in L', y is isolated in R'. Let w_1, $w_2 \in N(z_0, G)$ and $z_1 \in R - \{z_0\}$. Now we define the graphs $G' = G - \{z_0, z_1, w_1, w_2\}$ and $H' = H - \{y_1, y_0, x, x_1\}$ and construct a bi-placement of G and H.

b) Now for each vertex in R' let its degree is at most one. Let x be a vertex in L' such that $d(x, H) = \Delta_L(H)$. Hence $d(x, H) \in \{1, \ldots, p-k\}$. There is a vertex $w \in L$ such that $d(w, G) + d(x, H) \leq q$. Otherwise the degree of each vertex in L would be at least

$$(**) \qquad \begin{aligned} &q - d(x, H) + 1 \;\text{ and} \\ &e(G) \geq p(q - p + k + 1) \end{aligned}$$

But, for $p \leq q$ and $2 \leq k \leq p - 2$ the unequality $(**)$ cannot hold. If $d(w, G) = 0$, then we choose a vertex z_0, an isolated vertex in R', say y, and construct a bi-placement of G and H like in Case 1. If $d(w, G) > 0$, we define the sets Z and Z' and graphs G' and H' in the following way:

$$\begin{aligned} &Z' = N(x, H),\; Z \subseteq R - N(w, G) \text{ and } |Z| = d(x, H), \\ &G' = G - \{w\} - Z,\; H' = H - \{x\} - Z'. \end{aligned}$$

G' and H' are $(p-1, q-d(x, H))$-bipartite graphs such that

$$\begin{aligned} &e(G') \leq p + q + k - 1 - 2d(x, H) \leq (p - 1) + (q - d(x, H)) + k - 1 \\ &e(H') \leq p - k - d(x, H) \leq \min\{p - 1 - k, q - k - d(x, H)\}. \end{aligned}$$

Hence G' and H' are m.p. A bi-placement of G and H is evident. ∎

***Proof of Theorem* 1.** Theorem 1 is a consequence of Theorem B, Remark 1.1, Theorem 1.3 and Theorem 2.4.

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