

## AN INEQUALITY CONCERNING EDGES OF MINOR WEIGHT IN CONVEX 3-POLYTOPES

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*Dedicated to Professor E. Jucovič on the occasion of his 70th birthday.*

### Abstract

Let  $e_{ij}$  be the number of edges in a convex 3-polytope joining the vertices of degree  $i$  with the vertices of degree  $j$ . We prove that for every convex 3-polytope there is  $20e_{3,3} + 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + 6\frac{2}{3}e_{3,7} + 5e_{3,8} + 2\frac{1}{2}e_{3,9} + 2e_{3,10} + 16\frac{2}{3}e_{4,4} + 11e_{4,5} + 5e_{4,6} + 1\frac{2}{3}e_{4,7} + 5\frac{1}{3}e_{5,5} + 2e_{5,6} \geq 120$ ; moreover, each coefficient is the best possible. This result brings a final answer to the conjecture raised by B. Grünbaum in 1973.

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### 1. INTRODUCTION AND STATEMENT OF RESULTS

This note deals with connected planar maps. We use standard terminology and notation of graph theory, see e.g. Ore [16]. We recall, however, more specialized notions. A plane map is called *normal* if it contains neither vertices nor faces incident with less than 3 edges. Notice, however, that both loops and multiple edges can appear in a normal plane map. By the

Steinitz's theorem (see e.g. Grünbaum [11], Jucovič [13]) convex 3-polytopes are distinguished among all planar maps by the property that their graphs are 3-connected. The degree of a face  $\omega$  is the number of edges incident to  $\omega$  where each cut-edge is counted twice. Similarly, each loop contributes 2 to the degree of the incident vertex. Vertices and faces of degree  $i$  are called  $i$ -vertices and  $i$ -faces, respectively. Let  $e_{i,j}(M) = e_{i,j}$  be the number of edges in a planar map  $M$  which join  $i$ -vertices and  $j$ -vertices. Recall that a convex 3-polytope is called simplicial if all its faces are 3-gons.

An excellent theorem of Kotzig [14] (see also [1,3,6,7,8,13,15,]) states that every convex 3-polytope contains an edge of the weight (i.e., the sum of degrees of its endvertices) at most 13; in other words  $\sum_{i+j \leq 13} e_{i,j} > 0$ . This Kotzig's result was further developed in various directions, see e.g. Borodin [1,2,3], Grünbaum [6,7,8], Grünbaum and Shephard [9], Ivančo [10], Ivančo and Jendrol' [11], Jucovič [12,13], Zaks [17].

Grünbaum [8] has brought an idea that a relation of the type  $\sum_{i+j \leq 13} \alpha_{i,j} e_{i,j} \geq 1$  should hold for each convex 3-polytope ( $\alpha_{i,j}$  denotes the coefficient at  $e_{i,j}$ ) and has conjectured that the following holds for every simplicial convex 3-polytope

$$\begin{aligned} 20e_{3,3} + 15e_{3,4} + 12e_{3,5} + 10e_{3,6} + 6\frac{2}{3}e_{3,7} + 5e_{3,8} + 3\frac{1}{3}e_{3,9} + 2e_{3,10} \\ + 12e_{4,4} + 7e_{4,5} + 5e_{4,6} + 4e_{4,7} + 2\frac{2}{3}e_{4,8} + \frac{2}{3}e_{4,9} \\ + 4e_{5,5} + 2e_{5,6} + \frac{1}{3}e_{5,7} \\ + 12e_{6,6} \geq 120. \end{aligned}$$

Jucovič [12] proved that for each simplicial convex 3-polytope there is

$$\begin{aligned} 20e_{3,3} + 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + 6\frac{2}{3}e_{3,7} + 5e_{3,8} + 2\frac{1}{2}e_{3,9} + 2e_{3,10} \\ + 20e_{4,4} + 11e_{4,5} + 5e_{4,6} + 6e_{4,7} + 5e_{4,8} + 3e_{4,9} \\ + 8e_{5,5} + 2e_{5,6} + 2e_{5,7} + 2e_{5,8} \geq 120. \end{aligned}$$

Later on Jucovič, in [13], proved that this inequality holds for all convex 3-polytopes.

For a wider class of planar maps which also includes convex 3-polytopes Borodin [3] has obtained.

**Theorem 1.** *For each normal planar map there holds*

$$\begin{aligned} (1) \quad 40e_{3,3} + 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + 6\frac{2}{3}e_{3,7} + 5e_{3,8} + 2\frac{1}{2}e_{3,9} + 2e_{3,10} \\ + 16\frac{2}{3}e_{4,4} + 11e_{4,5} + 5e_{4,6} + 1\frac{2}{3}e_{4,7} \\ + 5\frac{1}{3}e_{5,5} + 2e_{5,6} \geq 120; \end{aligned}$$

*moreover, each coefficient of this inequality is the best possible.*

In the same paper Borodin [3] proves that for simplicial convex 3-polytopes (1) is the best possible if we put  $\alpha_{3,3} = 20$  instead of  $\alpha_{3,3} = 40$ . For other results of this type see Borodin [1,2,3], Borodin and Sanders [5], Jucovič [13].

The main purpose of the present note is to give a final answer to the above mentioned conjecture by Grünbaum [8]. We prove the following

**Theorem 2.** *For each convex 3-polytopes there holds*

$$(2) \quad \begin{aligned} & 20e_{3,3} + 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + 6\frac{2}{3}e_{3,7} + 5e_{3,8} + 2\frac{1}{2}e_{3,9} + 2e_{3,10} \\ & + 16\frac{2}{3}e_{4,4} + 11e_{4,5} + 5e_{4,6} + 1\frac{2}{3}e_{4,7} \\ & + 5\frac{1}{3}e_{5,5} + 2e_{5,6} \geq 120; \end{aligned}$$

moreover, each coefficient of this inequality is the best possible.

## 2. PROOF OF THEOREM 2

We prove our Theorem 2 in a dual form. It is well known that the dual of a 3-connected planar map is also 3-connected, see e.g. Ore [16, Chapter 3] and, due to Steinitz's theorem, it is also true for convex 3-polytopes. It is easy to check that the dual of a normal map is again normal.

For the purposes of this proof an edge  $h$  is called an  $(i, j)$ -edge when it is incident with an  $i$ -gon and a  $j$ -gon. Let  $g_{i,j}(M) = g_{ij}$  denote the number of  $(i, j)$ -edges in a map  $M$ . If  $M^d$  is the dual to a normal map  $M$ , then clearly  $e_{ij}(M^d) = g_{ij}(M)$ . Let  $V(M)$ ,  $E(M)$  and  $F(M)$  denote the set of vertices, edges and faces of the map  $M$ , respectively.

The proof is by contradiction. Replace  $e_{ij}$  with  $g_{ij}$  in the left part of (2) and denote it by  $\sum$ . We want to prove that for every 3-connected planar  $M$  there is  $\sum(M) \geq 120$ . Suppose  $M$  be a counterexample having a minimum number of faces.

To obtain a contradiction we are going to look for a suitable configuration in  $M$  which will be changed locally to obtain a new 3-connected plane map  $M^*$  with  $\sum(M^*) \leq \sum(M) < 120$  and with a fewer number of faces than in  $M$ . During this transformation of the map  $M$  into  $M^*$  some edges and vertices of  $M$  are deleted, some edges change their types (an edge is of the type  $(i, j)$  if it is an  $(i, j)$ -edge) and some new edges and vertices can appear in  $M^*$ .

Associate with an  $(i, j)$ -edge  $h$  of the map  $M$  the charge  $\alpha(h, M) = \alpha_{ij}$ , where  $\alpha_{ij}$  is as in (2) or  $\alpha_{i,j} = 0$  for  $i = 3, j \geq 11$  or  $i = 4, j \geq 8$  or

$i = 5, j \geq 7$  or  $i \geq 6, j \geq 6$ . Hence  $\sum(M) = \sum_{h \in E(M)} \alpha(h, M)$ . Let  $\Delta(h) = \alpha(h, M) - \alpha(h, M^*)$ .

Since every 3-connected plane map is also normal Theorem 1 yields  $g_{3,3}(M) \neq 0$ , i.e.,  $M$  contains a  $(3,3)$ -edge  $h_0 = uv$ . Denote by  $s$  and  $t$  the vertices incident to triangles incident with  $h_0$  and different from  $u$  and  $v$ , see Figure 1. Let  $h_1 = us, h_2 = sv, h_3 = vt$  and  $h_4 = ut$  be edges of  $M$ .

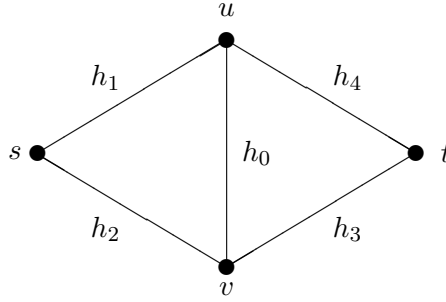


Figure 1

To finish our proof several cases have to be considered

*Case 1.*  $\deg u \geq 4$  and  $\deg v \geq 4$ .

1.1. Let  $\deg s = 3$  or  $\deg t = 3$ . The required map  $M^*$  is obtained by deleting the edge  $h_0$  from  $M$ , i.e.,  $M^* = M - h_0$ . Because  $M$  is 3-connected and at least one of the vertices  $s$  and  $t$  is a 3-vertex also  $M^*$  is 3-connected. We can easily see that  $|F(M^*)| = |F(M)| - 1$  and  $\Delta(\sum) = \sum(M) - \sum(M^*) = \alpha(h_0, M) + \sum_{i=1}^4 (\alpha(h_i, M) - \alpha(h_i, M^*)) = \alpha_{3,3} + \sum_{i=1}^4 \Delta(h_i) \geq 20 + 4 \cdot (-5) = 0$ . The last inequality is due to the fact that if a  $(3, k)$ -edge  $h$  is transformed into a  $(4, k)$ -edge, its charge always decreases or is the same except of the case  $k = 3$ . We also refer to the fact that  $\Delta(h_i) \geq -5$  for any edge  $h_i \in E(M)$ .

1.2.  $\deg s \geq 4$  and  $\deg t \geq 4$ . In this case we transform  $M$  into  $M^*$  as shown in Figure 2. We delete the edge  $h_0$  from  $M$  and split the vertex  $t$  of  $M$  into two new vertices  $t_1$  and  $t_2$  such that we obtain, in  $M^*$ ,  $\deg t_1 = 3$  and  $\deg t_2 = \deg t - 1$ . (The reason for this transformation of  $M$  into  $M^*$  is to preserve 3-connectivity also in  $M^*$ .) Let  $h', h_1, h_2, h_3$  and  $h_4$  be edges and  $\omega_1$  and  $\omega_2$  be faces of  $M^*$  as depicted in Figure 2. Without loss of generality we can assume that  $4 \leq \deg \omega_1 \leq \deg \omega_2$ . Put  $\Delta^* = \sum_{x \in E(M) - \{h_0, h_3, h_4\}} \Delta(x)$ . Then we have  $|F(M^*)| = |F(M)| - 1$  and  $\Delta(\sum) = \alpha(h_0, M) + \Delta(h_3) +$

$\Delta(h_4) + \Delta^* - \alpha(h', M^*) \geq 0$ . To check it use  $\alpha(h_0, M) = \alpha_{3,3} = 20$  and for the values  $\Delta(h_3), \Delta(h_4), \alpha(h', M^*)$  and  $\Delta^*$  see Table 1 below. To count  $\Delta^*$  we also refer to the fact that  $g_{3,3}(M) \leq 5$  (because  $M$  is a counterexample) and consider the "worst" case.

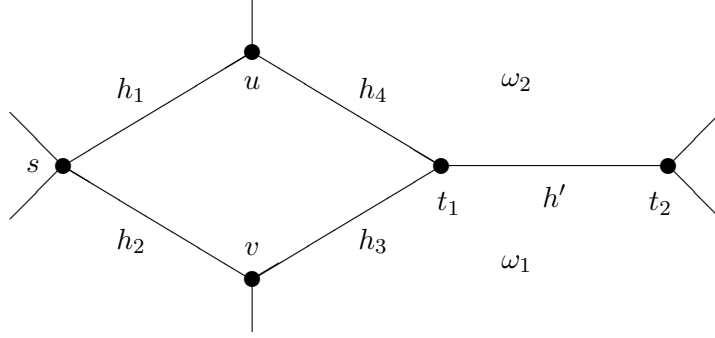


Figure 2

*Case 2.*  $\deg u = 3$  and  $\deg v \geq 4$ .

Let  $w$  be a face incident to the edges  $h_1$  and  $h_4$ , see Figure 1.

2.1. If  $\deg w = 3$  then  $M^*$  is obtained by removing the vertex  $u$  from  $M$ , i.e.  $M^* = M - \{u\}$ . We have  $\Delta(\Sigma) = \alpha(h_0, M) + \alpha(h_1, M) + \alpha(h_4, M) = 60 > 0$  and  $|F(M^*)| = |F(M)| - 2$ .

2.2. Let  $\deg w = k \geq 4$ . If we delete the vertex  $u$  from  $M$  and then insert a new edge  $h^* = st$  we obtain a required map  $M^*$ ,  $M^* = M - \{u\} + \{h^*\}$ . In this case  $|F(M^*)| = |F(M)| - 1$  and we can check that  $\Delta(\Sigma) = \alpha(h_0, M) + \alpha(h_1, M) + \alpha(h_4, M) - \alpha(h^*, M^*) + \tilde{\Delta} \geq 0$ . To see it, take  $\alpha(h_0, M) = \alpha_{3,3} = 20$  and the values  $\alpha(h_1, M), \alpha(h_4, M), \alpha(h^*, M^*)$  and  $\tilde{\Delta}$  from the Table 2 below; here  $\tilde{\Delta} = \sum \Delta(g)$ , where the sum is taken over all edges  $g$  incident to the face  $\omega, g \neq h_1, h_4$ . Note that during this transformation the edge  $g$  changes its type  $(n, k)$  into the type  $(n, k - 1)$  and in the counting we consider the worst case, that is  $\tilde{\Delta} \geq (k - 2)(\alpha_{3,k} - \alpha_{3,k-1})$ .

*Case 3.*  $\deg u = \deg v = 3$ .

This assumption leads immediately to the graph of the tetrahedron or to a 2-connected planar map. In both cases we get a contradiction.

The proof that a 3-connectivity of  $M$  implies a 3-connectivity of  $M^*$  is easy and is left to the reader.

The coefficient  $\alpha_{3,3} = 20$  cannot be improved as we can see from the tetrahedron. The above mentioned examples by Borodin [3] also show the impossibility to improve the other coefficient  $\alpha_{i,j}$  in Theorem 2.

Table 1

$\deg \omega_1$	$\deg \omega_2$	$\Delta(h_3)$	$\Delta(h_4)$	$\alpha(h', M^*)$	$\Delta^* \geq$
4	4	$3\frac{1}{3}$	$3\frac{1}{3}$	$16\frac{2}{3}$	-10
4	5	$3\frac{1}{3}$	14	11	-20
4	6	$3\frac{1}{3}$	11	5	-20
4	7	$3\frac{1}{3}$	$8\frac{1}{3}$	$1\frac{2}{3}$	-20
4	$\geq 8$	$3\frac{1}{3}$	$\geq 0$	0	-20
5	5	14	14	$5\frac{1}{3}$	-10
5	6	14	11	2	-10
5	$\geq 7$	14	$\geq 0$	0	-10
$\geq 6$	$\geq 6$	$\geq 0$	$\geq 0$	0	-10

Table 2

$\deg \omega$	$d(h_1, M)$	$d(h_4, M)$	$d(h^*, M^*)$	$\tilde{\Delta} \geq$
4	25	25	20	$2 \cdot (-8\frac{1}{3})$
5	16	16	25	$3 \cdot (-9)$
6	10	10	16	$4 \cdot (-6)$
7	$6\frac{2}{3}$	$6\frac{2}{3}$	10	$5 \cdot (-3\frac{1}{3})$
8	5	5	$6\frac{2}{3}$	$6 \cdot (-1\frac{2}{3})$
9	$2\frac{1}{2}$	$2\frac{1}{2}$	5	$7 \cdot (-2\frac{1}{2})$
10	2	2	$2\frac{1}{2}$	$8 \cdot (-\frac{1}{2})$
11	0	0	2	$9 \cdot (-2)$
$\geq 12$	0	0	0	0

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