

ON k -FACTOR-CRITICAL GRAPHS

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Abstract

A graph is said to be k -factor-critical if the removal of any set of k vertices results in a graph with a perfect matching. We study some properties of k -factor-critical graphs and show that many results on q -extendable graphs can be improved using this concept.

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1. INTRODUCTION

The graphs $G = (V, E)$ we consider here are undirected, simple and finite of order $|V| = n$. A graph is *even* if its order is even and *odd* if its order is odd. The *neighborhood* of a vertex x is the set $N(x) = \{y : y \in V \text{ and } xy \in E\}$ and the *degree* of x is the integer $d(x) = |N(x)|$. The integer $\delta = \min\{d(x) : x \in V\}$ is called the *minimum degree* of G . For any set $A \subseteq V$, $\langle A \rangle$ denotes the subgraph of G induced by A , $G - A$ stands for $\langle V \setminus A \rangle$, and $c(G - A)$ denotes the number of connected components of $G - A$. A set A such that $c(G - A) > 1$ is called a *cutset* of G . The *connectivity* of G is the integer $\kappa(G) = \min\{|A| : A \text{ is a cutset of } G\}$. A *claw* of G is an induced subgraph isomorphic to the star $K_{1,3}$, and the *claw center* is the center of the star. A *matching* F of G is a set of independent edges. The *maximum matching number* of G is the integer $\nu(G) = \max\{|F| : F \text{ is a matching of } G\}$. A *perfect matching*, or a *1-factor*, is a matching covering all the vertices of G . For convenience, we will say that a graph of order 0 has a perfect matching. A graph without perfect matching is called *prime*. Clearly, every odd graph is prime. For details concerning the Matching Theory, the reader is referred to [7] by Lovász and Plummer.

The concepts of factor-critical and bicritical graphs were introduced by Gallai [5] and by Lovász [6], respectively. A graph G is *factor-critical* if $G - x$ has a perfect matching for every vertex x of G . A graph G is *bicritical* if $G - x - y$ has a perfect matching for every pair of distinct vertices of G . Motivated by the similitude and the interest of these two concepts, which lead to powerful results, the author extended them to k -factor-critical graphs in [3].

Definition. For a given integer k with $0 \leq k \leq n$, a graph G of order n is *k -factor-critical*, in brief *k -fc*, if $G - X$ has a perfect matching for every set X of k vertices of G .

Equivalently, G is k -factor-critical if $\langle Y \rangle$ has a perfect matching for every set Y of $n - k$ vertices of G .

Remarks. 1. If a graph of order n is k -fc, then n and k have the same parity, i.e., $n + k$ is even.

2. 0-fc, 1-fc and 2-fc graphs are respectively graphs with a perfect matching, factor-critical graphs and bicritical graphs. The only $(n-2)$ -fc graph of order n is the clique K_n .

3. With the convention that the graph of order 0 has a perfect matching, every graph of order n is n -fc. The concept of k -factor-criticality for $k = n$ is thus not very interesting. We still admit the possibility for k to be equal to n to get a definition more consistent with that of factor-critical or bicritical graphs. In particular, the factor-critical components appearing in the Edmonds-Gallai structure theorem can be effectively reduced to one vertex, which corresponds to the case $k = n = 1$. But most properties of k -fc graphs will be proved for $n > k$.

Examples of k -fc graphs. For a given integer k , $0 \leq k \leq n - 3$, a graph G of order n is said to be *k -hamiltonian* [2] if the removal of any set of at most k vertices of G results in a hamiltonian graph. Since any even cycle has a perfect matching, every k -hamiltonian graph of order n with $n + k$ even is k -fc. Similarly, if we remove one vertex from an odd cycle we obtain a path with a perfect matching. Hence every k -hamiltonian graph of order n with $n + k$ odd is $(k + 1)$ -fc. The most famous examples of k -hamiltonian graphs are powers of graphs whose hamiltonian properties have been extensively studied. The q^{th} power G^q of a connected graph G has as its vertices those of G , and two distinct vertices u and v are adjacent in G^q if their distance in G is at most q . For instance, it is known that if G is connected of order n ,

then G^{k+2} is k -hamiltonian for any k , $1 \leq k \leq n - 3$ [1]. On the other hand, a graph G_k that is obtained by taking an arbitrary nonhamiltonian graph having a perfect matching and by joining each of its vertices with every vertex of a clique K_k is an example of a k -fc graph with $n + k$ even that is not k -hamiltonian.

In Section 2 of this paper, we study some simple properties of k -fc graphs. In Section 3, we extend to k -fc graphs the characterization of 0-fc, 1-fc and 2-fc graphs in terms of the number of odd or prime components of induced subgraphs of G . In Section 4, we discuss the relationship between the concepts of k -factor-criticality and q -extendability.

2. BASIC PROPERTIES OF k -FC GRAPHS

We begin with some easy observations.

Theorem 2.1. *For $k \geq 2$, any k -fc graph of order $n > k$ is $(k - 2)$ -fc.*

Proof. Let G be a k -fc graph of order $n > k$ (i.e., by parity, $n \geq k + 2$) and Y a set of $n - (k - 2) \geq 4$ vertices of G . The set Y is not independent since $\langle Y \setminus \{z, t\} \rangle$ has a perfect matching for any pair $\{z, t\}$ of vertices of Y . Let xy be an edge of $\langle Y \rangle$. Since G is k -fc, $\langle Y \setminus \{x, y\} \rangle$ has a perfect matching F , and $F \cup xy$ is a perfect matching of $\langle Y \rangle$. ■

As a first consequence we get that every k -fc graph of order $n > k$ is 0-fc or 1-fc, depending on its parity. In particular

Corollary 2.2. *For $k \geq 1$, no k -fc graph G of order $n > k$ is bipartite.*

Proof. By theorem 2.1, G is 1-fc or 2-fc, and it is known, and easy to check, that 1-fc or 2-fc graphs of order greater than 2 are not bipartite. ■

Theorem 2.3. *If a graph G is k -fc, then $G - Y$ is $(k - p)$ -fc for every integer p with $0 \leq p \leq k$ and every set Y of p vertices of G .*

Proof. If Z is a set of $k - p$ vertices of $G - Y$, then $X = Y \cup Z$ is a set of k vertices of G and thus $G - X = (G - Y) - Z$ has a perfect matching. ■

The next property extends Gallai's result saying that a connected graph G is 1-fc if and only if $\nu(G - x) = \nu(G)$ for every vertex x of G [5], to k -fc graphs with $k > 1$.

Theorem 2.4. *Let G be a graph of order $n > 2$ and maximum matching number ν . For $2 \leq k < n$, the following three properties are equivalent.*

- (i) *The graph G is k -fc.*
- (ii) *The integer $n + k$ is even and $\nu(G - X) = \frac{n-k}{2} = \nu - \lfloor \frac{k}{2} \rfloor$ for every set X of k vertices of G .*
- (iii) *The graph G contains at least one edge, $n + k$ is even, and $\nu(G - X)$ has the same value for any set X of k vertices of G .*

Proof. (i) $\implies$ (ii). For any k -fc graph G of order $n > k$, $n + k$ is even and for any subset X of V with $|X| = k$, $\nu(G - X) = \frac{n-k}{2}$. If n and k are even, then G is 0-fc and $\nu = \frac{n}{2}$. If n and k are odd, G is 1-fc and $\nu = \frac{n-1}{2}$. In both cases, $\frac{n-k}{2} = \nu - \lfloor \frac{k}{2} \rfloor$.

(ii) $\implies$ (iii). Obvious.

(iii) $\implies$ (i). Let G be a graph satisfying (iii), xy an edge of G , X a set of k vertices containing x and y , and M a maximum matching of $G - X$. If at least two vertices u and v of $G - X$ are not covered by M , then for $X' = (X - \{x, y\}) \cup \{u, v\}$, which is another set of k vertices of G , $M \cup xy$ is a matching of $G - X'$ greater than M , in contradiction to the hypothesis. Therefore, by parity, M is a perfect matching of $G - X$ and thus $G - Y$ has also a perfect matching for any other set Y of k vertices of G . ■

For $k = 1$, the implication (iii) $\implies$ (i) fails, as shown for instance by the graph formed by one odd clique and two isolated vertices. This explains the necessity of the hypothesis “ G is connected” in Gallai’s statement.

We finish this section with two connectivity results.

Theorem 2.5. *Every k -fc graph of order $n > k$ is k -connected and this result is sharp.*

Proof. If $k = 0$, the result is obvious, and obviously sharp since a graph with a perfect matching is not necessarily connected. Suppose now that the k -fc graph G with $k \geq 1$ admits a cutset S of $k - 1$ vertices. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two components of $G - S$, and a_i a vertex of $\mathcal{C}_i$ for $i \in \{1, 2\}$. Since both $G - (S \cup \{a_1\})$ and $G - (S \cup \{a_2\})$ have a perfect matching, every component $\mathcal{C}_i$ must be even and odd, a contradiction. Therefore each cutset of G has at least k vertices. The graph G obtained by joining all the vertices of a clique K_k to all the vertices of two disjoint even cliques K_{2q} is k -fc and its connectivity is exactly k , which proves the sharpness of the result. ■

Theorem 2.6. *For $k \geq 1$, every k -fc graph G of order $n > k$ is $(k + 1)$ -edge-connected.*

Proof. By Theorem 2.5, G is at least k -edge-connected. Suppose G is not $(k + 1)$ -edge-connected and let $F = \{a_i b_i\}$ be a set of k edges such that $G - F$ consists of two connected components $\langle X \rangle$ and $\langle Y \rangle$, with $a_i \in X$ and $b_i \in Y$ for $1 \leq i \leq k$. Let $A = \{a_i; 1 \leq i \leq k\}$ and $B = \{b_i; 1 \leq i \leq k\}$. If $|A| = k$, i.e., if the k vertices a_i are distinct, then $A' = (A \setminus \{a_k\}) \cup \{b_k\}$ is a cutset of G , and, since $|A| = k$, the component $\langle (X \setminus A) \cup \{a_k\} \rangle$ of $G - A'$ has a perfect matching. Hence $|X| - k + 1$ is even, $X \neq A$, A is another cutset of k vertices of G , and the component $\langle X \setminus A \rangle$ of $G - A$ has also a perfect matching. This leads to a contradiction. Hence $|A| < k$ and $X = A$, for otherwise A is a cutset of G smaller than k . Since G is k -connected, every vertex has degree at least k . Every vertex of A has at most $|A| - 1$ neighbors in A , and thus at least $k - |A| + 1$ neighbors in B . Therefore, $k = |F| \geq |A|(k - |A| + 1)$, from which $(|A| - 1)(k - |A|) \leq 0$, which implies $|A| = 1$. Similarly, $Y = B$ and $|B| = 1$, which gives a final contradiction since $n \geq 3$. Hence G is $(k + 1)$ -edge-connected. ■

Theorem 2.7. *Every k -fc graph of order $n > k$ has at least $\frac{(k+1)n}{2}$ edges and this bound is sharp.*

Proof. In a k -fc graph, the minimum degree is at least $k + 1$ since no vertex can be isolated after the removal of k vertices (for $k \geq 1$, this is also a consequence of Theorem 2.6), and thus the number of edges is at least $\frac{(k+1)n}{2}$. It is known that every $(k - 1)$ -hamiltonian graph of order n has minimum degree at least $k + 1$, and thus at least $\frac{(k+1)n}{2}$ edges [2]. In [15] and [8], the authors give examples of $(k - 1)$ -hamiltonian graphs having exactly this number of edges. These graphs are, if k is odd, powers of cycles, and if k is even, powers of cycles plus all or part of the diameters where one pair of consecutive diameters is untwisted in some particular cases. As seen in the previous section, when $n + k$ is even these graphs are k -fc. Hence the result on the number of edges, and thus that on the edge-connectivity, of k -fc graphs is sharp. ■

3. TUTTE AND GALLAI TYPE PROPERTIES

Let B be a set of vertices of the graph G . We denote by $c_o(G - B)$ the number of odd components of $G - B$ and by $c_p(G - B)$ the number of its prime components. Clearly $c_o(G - B) \leq c_p(G - B)$ since a component of odd order has no perfect matching. Tutte and Gallai respectively characterized 0-fc and 1-fc graphs in terms of $c_o(G - B)$ and $c_p(G - B)$ where B is any

subset of V . In order to compare their results and to extend them to k -fc graphs, we first unify the notation.

Definition. For a graph $G = (V, E)$, the properties $\mathbf{Q}_k$, $\mathbf{Q}'_k$ and $\mathbf{Q}''_k$ are defined as follows:

$$\mathbf{Q}_k : \quad c_o(G - B) \leq |B| - k \quad \text{for any } B \subseteq V \text{ with } |B| \geq k,$$

$$\mathbf{Q}'_k : \quad c_p(G - B) \leq |B| - k \quad \text{for any } B \subseteq V \text{ with } |B| \geq k,$$

$$\mathbf{Q}''_k : \quad c_p(G - B) \leq |B| - k + 1 \quad \text{for any } B \subseteq V \text{ with } |B| \geq k.$$

Note that Property $\mathbf{Q}'_k$ is stronger than $\mathbf{Q}_k$ and than $\mathbf{Q}''_k$.

The first result in the domain was Tutte's Theorem.

Theorem 3.1 (Tutte [14]). *The following two properties are equivalent.*

- (i) *The graph G is 0-fc.*
- (ii) *The graph G satisfies Property $\mathbf{Q}_0$.*

There exist many proofs of Tutte's Theorem. In one of them, Gallai implicitly gave another characterization of 0-fc graphs, and a characterization of 1-fc graphs.

Theorem 3.2 (Gallai [5]). 1. *The following two properties are equivalent.*

- (i) *The graph G is 0-fc.*
- (ii) *The graph G satisfies Property $\mathbf{Q}'_0$.*

2. *If G is not 0-fc, there exists a subset B of V with $c_p(G - B) > |B|$ such that every prime component of $G - B$ is 1-fc.*

Theorem 3.3 (Gallai [5]). *The following two properties are equivalent.*

- (i) *The graph G is 1-fc.*
- (ii) *The graph G is connected, not 0-fc, and satisfies Property $\mathbf{Q}''_1$.*

Lovász gave a similar characterization of 2-fc graphs.

Theorem 3.4 (Lovász [6]). *The following two properties are equivalent.*

- (i) *The graph G is 2-fc.*
- (ii) *The graph G satisfies Property $\mathbf{Q}_2$.*

Starting from Tutte's and Gallai's results, we extend Theorems 3.1, 3.2 and 3.4 (related to $\mathbf{Q}_k$ and $\mathbf{Q}'_k$) to k -fc graphs in Theorem 3.5 and we extend Theorem 3.3 (related to $\mathbf{Q}''_k$) to k -fc graphs in Corollary 3.6.

Theorem 3.5. 1. For a graph G of order $n \geq k$, the following three properties are equivalent.

- (i) The graph G is k -fc.
- (ii) The graph G satisfies Property $\mathbf{Q}_k$.
- (iii) The graph G satisfies Property $\mathbf{Q}'_k$.

2. If G is not k -fc, then, for every subset S of V of maximum order among all the subsets B such that $|B| \geq k$ and $c_p(G - B) > |B| - k$, every prime component of $G - S$ is 1-fc. If moreover $n + k$ is even, then S is a cutset and $c_o(G - S) > |S| - k + 1$.

Proof. 1. For $k = 0$, the equivalence follows from Theorems 3.1 and 3.2. For $k = n$, the three properties are always satisfied. We suppose henceforth $1 \leq k < n$.

(i) $\implies$ (iii). Let G be a k -fc graph, B any set of at least k vertices of G , and X an arbitrary set of k vertices of B . Put $B' = B \setminus X$ and $V' = V \setminus X$. Hence $V' \setminus B' = V \setminus B$. By (i), $\langle V' \rangle$ has a perfect matching. By Theorem 3.2, $c_p(V' \setminus B') \leq |B'|$ and thus $c_p(G - B) \leq |B| - k$. Therefore G satisfies $\mathbf{Q}'_k$.

(iii) $\implies$ (ii). Obvious since $\mathbf{Q}'_k$ implies $\mathbf{Q}_k$.

(ii) $\implies$ (i). Suppose G satisfies $\mathbf{Q}_k$. Let X be any set of k vertices of G , Y any subset of $V \setminus X$ and $B = X \cup Y$. By (ii), $c_o(G - B) \leq |B| - k$, or, equivalently, $c_o((G - X) - Y) \leq |Y|$ for any $Y \subseteq V \setminus X$. By Theorem 3.1, $G - X$ admits a perfect matching and thus G is k -fc.

2. Let G be a graph of order n that is not k -fc. By the first part of the theorem, there is a subset B of V such that $|B| \geq k$ and $c_p(G - B) > |B| - k$. Among all such sets B , choose a set S of maximum order and suppose that some prime component Γ of $G - S$ is not 1-fc. By Theorem 3.3, Γ contains a subset C with $|C| \geq 1$ and $c_p(\Gamma - C) \geq |C| + 1$. If we put $D = S \cup C$, the prime components of $G - D$ are those of $G - S$ except Γ , and those of $\Gamma - C$. Hence $c_p(G - D) = c_p(G - S) - 1 + c_p(\Gamma - C) > |S| - k - 1 + |C| + 1 = |D| - k$, which contradicts the maximality of S . Therefore, every prime component of $G - S$ is 1-fc and thus odd, which implies $c_p(G - S) = c_o(G - S)$. Moreover, it is easy to observe that if $n + k$ is even, then the three integers $|V \setminus S|$, $c_o(G - S)$ and $|S| - k$ have the same parity. Therefore $c_o(G - S) > |S| - k + 1 \geq 1$ and thus S is a cutset. ■

Corollary 3.6. For a graph G of order $n > k \geq 1$, the following three properties are equivalent.

- (i) *The graph G is k -fc.*
- (ii) *The graph G satisfies Property $\mathbf{Q}_k''$, is k -connected, and is not $(k-1)$ -fc.*
- (iii) *The graph G satisfies Property $\mathbf{Q}_k''$ and $n+k$ is even.*

Proof. (i) $\implies$ (ii). Any k -fc graph G of order $n > k$ is k -connected by Theorem 2.5, satisfies $\mathbf{Q}_k''$ by Theorem 3.5 (since $\mathbf{Q}_k'$ implies $\mathbf{Q}_k''$), and is not $(k-1)$ -fc for parity reasons.

(ii) $\implies$ (iii). Let G satisfy (ii). By $\mathbf{Q}_k''$, $c_p(G-D) \leq |D| - k + 1$ and thus $c_o(G-D) \leq |D| - k + 1$, for every subset D of at least k vertices of V . Since G is not $(k-1)$ -fc, it does not satisfy $\mathbf{Q}_{k-1}$ by Theorem 3.5, and hence there exists a set B of at least $k-1$ vertices of V for which $c_o(G-B) > |B| - k + 1$. From what precedes, $|B| = k-1$. Since G is k -connected, $G-B$ is connected and $c_o(G-B) \leq 1$. Therefore $c_o(G-B) = 1$, $|V|$ and $|B|$ have different parities and, since $|B| = k-1$, $n+k$ is even.

(iii) $\implies$ (i). If G is not k -fc and $n+k$ is even, then, by Theorem 3.5, there exists a set S of at least k vertices for which $c_o(G-S) > |S| - k + 1$, in contradiction to $\mathbf{Q}_k''$. $\blacksquare$

4. FACTOR-CRITICALITY AND MATCHING EXTENSION

In 1980, Plummer introduced the concept of q -extendability [9]. An even graph G is q -extendable if G is connected, contains a set of q independent edges, and every set of q independent edges extends to (i.e., is a subset of) a perfect matching. Clearly, for n and k even, every k -fc graph is $\frac{k}{2}$ -extendable and hence the class of k -fc graphs is intermediate between the class of k -hamiltonian graphs and that of $\frac{k}{2}$ -extendable graphs. There are many results on matching extension that have been obtained recently (see e.g. [12]). Some of these results, saying that “if an even graph G satisfies Property $\mathcal{P}$, then G is q -extendable”, can be improved to “if an even graph G satisfies $\mathcal{P}$, then G is $2q$ -fc” (and an analogous statement when G is odd). This is the case when, in the proof of the q -extendability, we delete the set X of the $2q$ endvertices of q independent edges, and show that $G-X$ has a perfect matching without using the property that $\langle X \rangle$ itself has a perfect matching. The first example of such a proof can be found in [4]. Two other examples of simple adaptations of proofs and results on matching extension (see [10] and [11]) to k -factor-criticality are given below.

The *toughness* of a noncomplete graph G is the number $\text{tough}(G) = \min\{\frac{|S|}{c(G-S)} : S \text{ is a cutset of } G\}$. If G is a clique, we put $\text{tough}(G) = +\infty$.

Theorem 4.1. *Let G be a graph of order n , and let k be an integer such that $2 \leq k < n$ and $n + k$ is even. If $\text{tough}(G) > \frac{k}{2}$, then G is k -fc, and the value $\frac{k}{2}$ is sharp.*

Proof. Suppose G is not k -fc and let X be a set of k vertices of G such that $G' = G - X$ has no perfect matching. By Tutte's Theorem 3.1, there exists a set S of vertices of G' such that $c_0(G' - S) > |S|$. By parity, $c_0(G' - S) \geq |S| + 2$, and thus $c(G' - S) \geq s + 2$, where $s = |S|$. The set $S \cup X$ is a cutset of G , and $c(G - (S \cup X)) = c(G' - S)$. By the definition of toughness, $\text{tough}(G) \leq \frac{|S \cup X|}{c(G - (S \cup X))} \leq \frac{s+k}{s+2} \leq \frac{k}{2}$ since $k \geq 2$. Therefore, if $\text{tough}(G) > \frac{k}{2}$, then G is k -fc. The toughness of the graph G obtained by joining all the vertices of a clique K_k to all the vertices of two disjoint odd cliques K_{2q+1} is equal to $\frac{k}{2}$, and G is not k -fc, which proves the sharpness of the bound $\frac{k}{2}$. ■

The t -degree sum and the t -generalized independent minimum degree of G are respectively $\sigma_t(G) = \min\{\sum_{w_i \in W} d(w_i) : W \text{ is an independent set of } t \text{ vertices of } G\}$ and $U_t = \min\{|\bigcup_{w_i \in W} N(w_i)| : W \text{ is an independent set of } t \text{ vertices of } G\}$. These two parameters are defined for t at most equal to the independence number of G . For $t = 1$, $\sigma_1 = U_1 = \delta$.

Theorem 4.2. *Let G be a graph of order n and connectivity κ , and let k be an integer such that $0 \leq k \leq \kappa$ and $n + k$ is even. If for some integer t with $1 \leq t \leq \kappa - k + 2$, $\sigma_t(G) \geq t(\frac{n+k}{2} - 1) + 1$ or $U_t(G) \geq n - \kappa + k - 1$, then G is k -fc.*

Proof. Let G be a graph of order n and connectivity κ , which is not k -fc for some integer $k \leq \kappa$ such that $n + k$ is even. Let X be a set of k vertices of G such that $G' = G - X$ has no perfect matching. As in Theorem 4.1, by Tutte's theorem, there is a set S of vertices of G' such that the number c of components $\mathcal{C}_i$ of $G' - S$ is at least $|S| + 2$. Let $s = |S|$. Since G is κ -connected, G' is $(\kappa - k)$ -connected and thus $s \geq \kappa - k$. On the other hand, the sets X , S and $\mathcal{C}_i$ are all disjoint and thus $|X| + |S| + c \leq n$, which implies, since $c \geq s + 2$, $s \leq \frac{n-k}{2} - 1$. For $1 \leq i \leq c$, let w_i be a vertex of $\mathcal{C}_i$. For any integer t with $1 \leq t \leq \kappa - k + 2 \leq c$, the set $\{w_i; 1 \leq i \leq t\}$ is independent.

1. The degree in G of each w_i satisfies $d(w_i) \leq |X| + |S| + |\mathcal{C}_i - \{w_i\}| = k + s + |\mathcal{C}_i| - 1$. Therefore $\sigma_t \leq \sum_{i=1}^t d(w_i) \leq t(k + s - 1) + \sum_{i=1}^t |\mathcal{C}_i|$. But

$$\begin{aligned}
\sum_{i=1}^t |\mathcal{C}_i| &= |V \setminus S \setminus X \setminus \bigcup_{i=t+1}^c \mathcal{C}_i| \\
&= n - s - k - \sum_{i=t+1}^c |\mathcal{C}_i| \leq n - s - k - (c - t) \leq n - k + t - 2s - 2
\end{aligned}$$

and thus $\sigma_t \leq tk + (t - 2)s + n - k - 2$.

If $t \geq 2$, we get $\sigma_k \leq tk + (t - 2)(\frac{n-k}{2} - 1) + n - k - 2 = t(\frac{n+k}{2} - 1)$. Hence if for some t between 2 and $\kappa - k + 2$, $\sigma_t > t(\frac{n+k}{2} - 1)$, then G is k -fc.

For $t = 1$ the condition $\sigma_t(G) \geq t(\frac{n+k}{2} - 1) + 1$ reduces to $\delta \geq \frac{n+k}{2}$ and it is known [2] that this implies that G is k -hamiltonian and thus k -fc.

2. The neighborhood in G of each w_i satisfies $N(w_i) \subseteq X \cup S \cup \mathcal{C}_i$. Therefore,

$$\begin{aligned}
U_t &\leq |\bigcup_{i=1}^t N(w_i)| \leq |X| + |S| + \sum_{i=1}^t (|\mathcal{C}_i| - 1) \leq k + s + (n - k - 2s - 2 + t) - t = \\
&n - s - 2 \leq n - \kappa + k - 2. \text{ Hence if for some } t \text{ between 1 and } \kappa - k + 2, \\
&U_t \geq n - \kappa + k - 1, \text{ then } G \text{ is } k\text{-fc.} \quad \blacksquare
\end{aligned}$$

We finish with an example related to a property of the same kind as in [4] but for which the conclusion “ G is k -extendable” cannot be replaced by “ G is $2k$ -fc”. Ryjáček proved in [13] that every even $(2k+1)$ -connected $K_{1,k+3}$ -free graph such that the set of claw centers is independent, is k -extendable. The hypotheses do not imply that the graph is $2k$ -fc as shown, for $k = 1$, by the following construction. The graph G consists of four copies H_i of cliques K_p of odd order $p \geq 3$, and four extra vertices x_i , $1 \leq i \leq 4$. In each H_i , we select three vertices y_{ij} with $1 \leq j \leq 4$ and $j \neq i$. Each vertex x_i is adjacent to the three vertices y_{ji} with $1 \leq j \leq 4$ and $j \neq i$. The graph G is 3-connected, $K_{1,4}$ -free and the claw centers, which are the vertices x_i , are independent. It is 1-extendable but not 2-fc since $G - \{x_1, x_2\}$ has no perfect matching.

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