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ON k-FACTOR-CRITICAL GRAPHS

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Abstract

A graph is said to be k-factor-critical if the removal of any set of k vertices results in a graph with a perfect matching. We study some properties of k-factor-critical graphs and show that many results on q-extendable graphs can be improved using this concept.

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1. Introduction

The graphs $G = (V, E)$ we consider here are undirected, simple and finite of order $|V| = n$. A graph is *even* if its order is even and *odd* if its order is odd. The *neighborhood* of a vertex x is the set $N(x) = \{y : y \in V \text{and } xy \in E\}$ and the *degree* of x is the integer $d(x) = |N(x)|$. The integer $\delta = \min\{d(x) : x \in V\}$ is called the *minimum degree* of G. For any set $A \subseteq V$, $\langle A \rangle$ denotes the subgraph of G induced by A, $G - A$ stands for $\langle V \setminus A \rangle$, and $c(G - A)$ denotes the number of connected components of $G - A$. A set A such that $c(G - A) > 1$ is called a *cutset* of G. The *connectivity* of G is the integer $\kappa(G) = \min\{|A| : A \text{ is a cutset of } G\}$. A *claw* of G is an induced subgraph isomorphic to the star $K_{1,3}$, and the *claw center* is the center of the star. A *matching* F of G is a set of independent edges. The *maximum matching number* of G is the integer $\nu(G) = \max\{|F| : F \text{ is a matching of } G\}$. A *perfect matching*, or a 1-*factor*, is a matching covering all the vertices of G. For convenience, we will say that a graph of order 0 has a perfect matching. A graph without perfect matching is called *prime*. Clearly, every odd graph is prime. For details concerning the Matching Theory, the reader is referred to [7] by Lovász and Plummer.

The concepts of factor-critical and bicritical graphs were introduced by Gallai [5] and by Lovász [6], respectively. A graph G is *factor-critical* if $G - x$ has a perfect matching for every vertex x of G. A graph G is *bicritical* if $G - x - y$ has a perfect matching for every pair of distinct vertices of G. Motivated by the similitude and the interest of these two concepts, which lead to powerful results, the author extended them to k-factor-critical graphs in [3].

Definition. For a given integer k with $0 \leq k \leq n$, a graph G of order n is *k-factor-critical*, in brief k-fc, if $G - X$ has a perfect matching for every set X of k vertices of G.

Equivalently, G is k-factor-critical if $\langle Y \rangle$ has a perfect matching for every set Y of $n - k$ vertices of G.

Remarks. 1. If a graph of order n is k-fc, then n and k have the same parity, i.e., $n + k$ is even.

2. 0-fc, 1-fc and 2-fc graphs are respectively graphs with a perfect matching, factor-critical graphs and bicritical graphs. The only $(n-2)$-fc graph of order n is the clique K_n.

3. With the convention that the graph of order 0 has a perfect matching, every graph of order n is n-fc. The concept of k-factor-criticality for $k = n$ is thus not very interesting. We still admit the possibility for k to be equal to n to get a definition more consistent with that of factor-critical or bicritical graphs. In particular, the factor-critical components appearing in the Edmonds-Gallai structure theorem can be effectively reduced to one vertex, which corresponds to the case $k = n = 1$. But most properties of k-fc graphs will be proved for $n > k$.

Examples of k-fc graphs. For a given integer k, $0 \leq k \leq n-3$, a graph G of order n is said to be k-hamiltonian [2] if the removal of any set of at most k vertices of G results in a hamiltonian graph. Since any even cycle has a perfect matching, every k-hamiltonian graph of order n with $n + k$ even is k-fc. Similarly, if we remove one vertex from an odd cycle we obtain a path with a perfect matching. Hence every k-hamiltonian graph of order n with $n + k$ odd is $(k + 1)$-fc. The most famous examples of k-hamiltonian graphs are powers of graphs whose hamiltonian properties have been extensively studied. The q^{th} power G^q of a connected graph G has as its vertices those of G, and two distinct vertices u and v are adjacent in G^q if their distance in G is at most q. For instance, it is known that if G is connected of order n, then G^{k+2} is k-hamiltonian for any k, $1 \leq k \leq n-3$ [1]. On the other hand, a graph G_k that is obtained by taking an arbitrary nonhamiltonian graph having a perfect matching and by joining each of its vertices with every vertex of a clique K_k is an example of a k-fc graph with $n+k$ even that is not k-hamiltonian.

In Section 2 of this paper, we study some simple properties of k-fc graphs. In Section 3, we extend to k-fc graphs the characterization of 0-fc, 1-fc and 2-fc graphs in terms of the number of odd or prime components of induced subgraphs of G. In Section 4, we discuss the relationship between the concepts of k-factor-criticality and q-extendability.

2. Basic Properties of k-FC Graphs

We begin with some easy observations.

Theorem 2.1. *For $k \geq 2$, any k-fc graph of order $n > k$ is $(k-2)$-fc.*

Proof. Let G be a k-fc graph of order $n > k$ (i.e., by parity, $n \geq k+2$) and Y a set of $n-(k-2) \geq 4$ vertices of G. The set Y is not independent since $\langle Y \setminus \{z,t\}\rangle$ has a perfect matching for any pair $\{z,t\}$ of vertices of Y. Let xy be an edge of $\langle Y\rangle$. Since G is k-fc, $\langle Y \setminus \{x,y\}\rangle$ has a perfect matching F, and $F \cup xy$ is a perfect matching of $\langle Y\rangle$. ■

As a first consequence we get that every k-fc graph of order $n > k$ is 0-fc or 1-fc, depending on its parity. In particular

Corollary 2.2. *For $k \geq 1$, no k-fc graph G of order $n > k$ is bipartite.*

Proof. By theorem 2.1, G is 1-fc or 2-fc, and it is known, and easy to check, that 1-fc or 2-fc graphs of order greater than 2 are not bipartite. ■

Theorem 2.3. *If a graph G is k-fc, then $G-Y$ is $(k-p)$-fc for every integer p with $0 \leq p \leq k$ and every set Y of p vertices of G.*

Proof. If Z is a set of $k-p$ vertices of $G-Y$, then $X = Y \cup Z$ is a set of k vertices of G and thus $G-X = (G-Y)-Z$ has a perfect matching. ■

The next property extends Gallai's result saying that a connected graph G is 1-fc if and only if $\nu(G-x) = \nu(G)$ for every vertex x of G [5], to k-fc graphs with $k > 1$.

Theorem 2.4. *Let G be a graph of order $n > 2$ and maximum matching number ν. For $2 \leq k < n$, the following three properties are equivalent.*

(i) *The graph G is k-fc.*
(ii) *The integer $n+k$ is even and $\nu(G-X) = \frac{n-k}{2} = \nu - \lfloor \frac{k}{2} \rfloor$ for every set X of k vertices of G.*
(iii) *The graph G contains at least one edge, $n+k$ is even, and $\nu(G-X)$ has the same value for any set X of k vertices of G.*

Proof. (i) $\Longrightarrow$ (ii). For any k-fc graph G of order $n > k$, $n+k$ is even and for any subset X of V with $|X| = k$, $\nu(G-X) = \frac{n-k}{2}$. If n and k are even, then G is 0-fc and $\nu = \frac{n}{2}$. If n and k are odd, G is 1-fc and $\nu = \frac{n-1}{2}$. In both cases, $\frac{n-k}{2} = \nu - \lfloor \frac{k}{2} \rfloor$.
(ii) $\Longrightarrow$ (iii). Obvious.
(iii) $\Longrightarrow$ (i). Let G be a graph satisfying (iii), xy an edge of G, X a set of k vertices containing x and y, and M a maximum matching of $G-X$. If at least two vertices u and v of $G-X$ are not covered by M, then for $X' = (X - \{x,y\}) \cup \{u,v\}$, which is another set of k vertices of G, $M \cup xy$ is a matching of $G - X'$ greater than M, in contradiction to the hypothesis. Therefore, by parity, M is a perfect matching of $G-X$ and thus $G-Y$ has also a perfect matching for any other set Y of k vertices of G. ■

For $k = 1$, the implication (iii) $\Longrightarrow$ (i) fails, as shown for instance by the graph formed by one odd clique and two isolated vertices. This explains the necessity of the hypothesis "G is connected" in Gallai's statement.

We finish this section with two connectivity results.

Theorem 2.5. *Every k-fc graph of order $n > k$ is k-connected and this result is sharp.*

Proof. If $k = 0$, the result is obvious, and obviously sharp since a graph with a perfect matching is not necessarily connected. Suppose now that the k-fc graph G with $k \geq 1$ admits a cutset S of $k-1$ vertices. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two components of $G-S$, and a_i a vertex of $\mathcal{C}_i$ for $i \in \{1,2\}$. Since both $G-(S\cup\{a_1\})$ and $G-(S\cup\{a_2\})$ have a perfect matching, every component $\mathcal{C}_i$ must be even and odd, a contradiction. Therefore each cutset of G has at least k vertices. The graph G obtained by joining all the vertices of a clique K_k to all the vertices of two disjoint even cliques K_{2q} is k-fc and its connectivity is exactly k, which proves the sharpness of the result. ■

Theorem 2.6. *For $k \geq 1$, every k-fc graph G of order $n > k$ is $(k+1)$-edge-connected.*

Proof. By Theorem 2.5, G is at least k-edge-connected. Suppose G is not $(k+1)$-edge-connected and let $F = \{a_i b_i\}$ be a set of k edges such that $G - F$ consists of two connected components $\langle X \rangle$ and $\langle Y \rangle$, with $a_i \in X$ and $b_i \in Y$ for $1 \leq i \leq k$. Let $A = \{a_i; 1 \leq i \leq k\}$ and $B = \{b_i; 1 \leq i \leq k\}$. If $|A| = k$, i.e., if the k vertices a_i are distinct, then $A' = (A \setminus \{a_k\}) \cup \{b_k\}$ is a cutset of G, and, since $|A| = k$, the component $\langle (X \setminus A) \cup \{a_k\} \rangle$ of $G - A'$ has a perfect matching. Hence $|X| - k + 1$ is even, $X \neq A$, A is another cutset of k vertices of G, and the component $\langle X \setminus A \rangle$ of $G - A$ has also a perfect matching. This leads to a contradiction. Hence $|A| < k$ and $X = A$, for otherwise A is a cutset of G smaller than k. Since G is k- connected, every vertex has degree at least k. Every vertex of A has at most $|A| - 1$ neighbors in A, and thus at least $k - |A| + 1$ neighbors in B. Therefore, $k = |F| \geq |A|(k - |A| + 1)$, from which $(|A| - 1)(k - |A|) \leq 0$, which implies $|A| = 1$. Similarly, $Y = B$ and $|B| = 1$, which gives a final contradiction since $n \geq 3$. Hence G is $(k+1)$-edge-connected. ■

Theorem 2.7. *Every k-fc graph of order $n > k$ has at least $\frac{(k+1)n}{2}$ edges and this bound is sharp.*

Proof. In a k-fc graph, the minimum degree is at least $k + 1$ since no vertex can be isolated after the removal of k vertices (for $k \geq 1$, this is also a consequence of Theorem 2.6), and thus the number of edges is at least $\frac{(k+1)n}{2}$. It is known that every $(k-1)$-hamiltonian graph of order n has minimum degree at least $k + 1$, and thus at least $\frac{(k+1)n}{2}$ edges [2]. In [15] and [8], the authors give examples of $(k-1)$-hamiltonian graphs having exactly this number of edges. These graphs are, if k is odd, powers of cycles, and if k is even, powers of cycles plus all or part of the diameters where one pair of consecutive diameters is untwisted in some particular cases. As seen in the previous section, when $n + k$ is even these graphs are k-fc. Hence the result on the number of edges, and thus that on the edge-connectivity, of k-fc graphs is sharp. ■

3. Tutte and Gallai Type Properties

Let B be a set of vertices of the graph G. We denote by $c_o(G - B)$ the number of odd components of $G - B$ and by $c_p(G - B)$ the number of its prime components. Clearly $c_o(G - B) \leq c_p(G - B)$ since a component of odd order has no perfect matching. Tutte and Gallai respectively characterized 0-fc and 1-fc graphs in terms of $c_o(G - B)$ and $c_p(G - B)$ where B is any subset of V. In order to compare their results and to extend them to k-fc graphs, we first unify the notation.

Definition. For a graph $G = (V, E)$, the properties $\mathbf{Q_k}$, $\mathbf{Q'_k}$ and $\mathbf{Q''_k}$ are defined as follows:

$\mathbf{Q_k}$: $c_o(G - B) \le |B| - k$ for any $B \subseteq V$ with $|B| \ge k$,

$\mathbf{Q'_k}$: $c_p(G - B) \le |B| - k$ for any $B \subseteq V$ with $|B| \ge k$,

$\mathbf{Q''_k}$: $c_p(G - B) \le |B| - k + 1$ for any $B \subseteq V$ with $|B| \ge k$.

Note that Property $\mathbf{Q'_k}$ is stronger than $\mathbf{Q_k}$ and than $\mathbf{Q''_k}$.

The first result in the domain was Tutte's Theorem.

Theorem 3.1 (Tutte [14]). *The following two properties are equivalent.*

(i) *The graph G is* 0*-fc.*

(ii) *The graph G satisfies Property* $\mathbf{Q_0}$.

There exist many proofs of Tutte's Theorem. In one of them, Gallai implicitly gave another characterization of 0-fc graphs, and a characterization of 1-fc graphs.

Theorem 3.2 (Gallai [5]). 1. *The following two properties are equivalent.*

(i) *The graph G is* 0*-fc.*

(ii) *The graph G satisfies Property* $\mathbf{Q'_0}$.

2. *If G is not* 0*-fc, there exists a subset B of V with $c_p(G - B) > |B|$ such that every prime component of $G - B$ is* 1*-fc.*

Theorem 3.3 (Gallai [5]). *The following two properties are equivalent.*

(i) *The graph G is* 1*-fc.*

(ii) *The graph G is connected, not* 0*-fc, and satisfies Property* $\mathbf{Q''_1}$.

Lovász gave a similar characterization of 2-fc graphs.

Theorem 3.4 (Lovász [6]). *The following two properties are equivalent.*

(i) *The graph G is* 2*-fc.*

(ii) *The graph G satisfies Property* $\mathbf{Q_2}$.

Starting from Tutte's and Gallai's results, we extend Theorems 3.1, 3.2 and 3.4 (related to $\mathbf{Q_k}$ and $\mathbf{Q'_k}$) to k-fc graphs in Theorem 3.5 and we extend Theorem 3.3 (related to $\mathbf{Q''_k}$) to k-fc graphs in Corollary 3.6.

Theorem 3.5. 1. *For a graph G of order $n \geq k$, the following three properties are equivalent.*

(i) *The graph G is k-fc.*

(ii) *The graph G satisfies Property $\mathbf{Q_k}$.*

(iii) *The graph G satisfies Property $\mathbf{Q'_k}$.*

2. *If G is not k-fc, then, for every subset S of V of maximum order among all the subsets B such that $|B| \geq k$ and $c_p(G - B) > |B| - k$, every prime component of $G - S$ is 1-fc. If moreover $n + k$ is even, then S is a cutset and $c_o(G - S) > |S| - k + 1$.*

Proof. 1. For $k = 0$, the equivalence follows from Theorems 3.1 and 3.2. For $k = n$, the three properties are always satisfied. We suppose henceforth $1 \leq k < n$.

(i) $\Longrightarrow$ (iii). Let G be a k-fc graph, B any set of at least k vertices of G, and X an arbitrary set of k vertices of B. Put $B' = B \setminus X$ and $V' = V \setminus X$. Hence $V' \setminus B' = V \setminus B$. By (i), $\langle V' \rangle$ has a perfect matching. By Theorem 3.2, $c_p(V' \setminus B') \leq |B'|$ and thus $c_p(G - B) \leq |B| - k$. Therefore G satisfies $\mathbf{Q'_k}$.

(iii) $\Longrightarrow$ (ii). Obvious since $\mathbf{Q'_k}$ implies $\mathbf{Q_k}$.

(ii) $\Longrightarrow$ (i). Suppose G satisfies $\mathbf{Q_k}$. Let X be any set of k vertices of G, Y any subset of $V \setminus X$ and $B = X \cup Y$. By (ii), $c_o(G - B) \leq |B| - k$, or, equivalently, $c_o((G - X) - Y) \leq |Y|$ for any $Y \subseteq V \setminus X$. By Theorem 3.1, $G - X$ admits a perfect matching and thus G is k-fc.

2. Let G be a graph of order n that is not k-fc. By the first part of the theorem, there is a subset B of V such that $|B| \geq k$ and $c_p(G-B) > |B|-k$. Among all such sets B, choose a set S of maximum order and suppose that some prime component Γ of $G-S$ is not 1-fc. By Theorem 3.3, Γ contains a subset C with $|C| \geq 1$ and $c_p(\Gamma-C) \geq |C|+1$. If we put $D = S \cup C$, the prime components of $G-D$ are those of $G-S$ except Γ, and those of $\Gamma-C$. Hence $c_p(G-D) = c_p(G-S)-1+c_p(\Gamma-C) > |S|-k-1+|C|+1 = |D|-k$, which contradicts the maximality of S. Therefore, every prime component of $G-S$ is 1-fc and thus odd, which implies $c_p(G - S) = c_o(G - S)$. Moreover, it is easy to observe that if $n+k$ is even, then the three integers $|V \setminus S|$, $c_o(G-S)$ and $|S| - k$ have the same parity. Therefore $c_o(G - S) > |S| - k + 1 \geq 1$ and thus S is a cutset. ∎

Corollary 3.6. *For a graph G of order $n > k \geq 1$, the following three properties are equivalent.*

(i) *The graph G is k-fc.*

(ii) *The graph G satisfies Property $\mathbf{Q''_k}$, is k-connected, and is not $(k-1)$-fc.*

(iii) *The graph G satisfies Property $\mathbf{Q''_k}$ and $n+k$ is even.*

Proof. (i) $\Longrightarrow$ (ii). Any k-fc graph G of order $n > k$ is k-connected by Theorem 2.5, satisfies $\mathbf{Q''_k}$ by Theorem 3.5 (since $\mathbf{Q'_k}$ implies $\mathbf{Q''_k}$), and is not $(k-1)$-fc for parity reasons.

(ii) $\Longrightarrow$ (iii). Let G satisfy (ii). By $\mathbf{Q''_k}$, $c_p(G-D) \leq |D| - k + 1$ and thus $c_o(G-D) \leq |D|-k+1$, for every subset D of at least k vertices of V. Since G is not $(k-1)$-fc, it does not satisfy $\mathbf{Q_{k-1}}$ by Theorem 3.5, and hence there exists a set B of at least $k-1$ vertices of V for which $c_o(G-B) > |B|-k+1$. From what precedes, $|B| = k-1$. Since G is k-connected, $G-B$ is connected and $c_o(G-B) \leq 1$. Therefore $c_o(G-B) = 1$, $|V|$ and $|B|$ have different parities and, since $|B| = k-1$, $n+k$ is even.

(iii) $\Longrightarrow$ (i). If G is not k-fc and $n+k$ is even, then, by Theorem 3.5, there exists a set S of at least k vertices for which $c_o(G-S) > |S| - k + 1$, in contradiction to $\mathbf{Q''_k}$. ■

4. Factor-Criticality and Matching Extension

In 1980, Plummer introduced the concept of q-extendability [9]. An even graph G is *q-extendable* if G is connected, contains a set of q independent edges, and every set of q independent edges extends to (i.e., is a subset of) a perfect matching. Clearly, for n and k even, every k-fc graph is $\frac{k}{2}$-extendable and hence the class of k-fc graphs is intermediate between the class of k-hamiltonian graphs and that of $\frac{k}{2}$-extendable graphs. There are many results on matching extension that have been obtained recently (see e.g. [12]). Some of these results, saying that "if an even graph G satisfies Property $\mathcal{P}$, then G is q-extendable", can be improved to "if an even graph G satisfies $\mathcal{P}$, then G is $2q$-fc" (and an analogous statement when G is odd). This is the case when, in the proof of the q-extendability, we delete the set X of the $2q$ endvertices of q independent edges, and show that $G-X$ has a perfect matching without using the property that $\langle X \rangle$ itself has a perfect matching. The first example of such a proof can be found in [4]. Two other examples of simple adaptations of proofs and results on matching extension (see [10] and [11]) to k-factor-criticality are given below.

The *toughness* of a noncomplete graph G is the number $\mathrm{tough}(G) = \min\{\frac{|S|}{c(G-S)} : S \text{ is a cutset of } G\}$. If G is a clique, we put $\mathrm{tough}(G) = +\infty$.

Theorem 4.1. *Let G be a graph of order n, and let k be an integer such that $2 \le k < n$ and $n+k$ is even. If* $\mathrm{tough}(G) > \frac{k}{2}$*, then G is k-fc, and the value $\frac{k}{2}$ is sharp.*

Proof. Suppose G is not k-fc and let X be a set of k vertices of G such that $G' = G - X$ has no perfect matching. By Tutte's Theorem 3.1, there exists a set S of vertices of G' such that $c_0(G' - S) > |S|$. By parity, $c_o(G' - S) \ge |S| + 2$, and thus $c(G' - S) \ge s + 2$, where $s = |S|$. The set $S \cup X$ is a cutset of G, and $c(G - (S \cup X)) = c(G' - S)$. By the definition of toughness, $\mathrm{tough}(G) \le \frac{|S \cup X|}{c(G-(S\cup X))} \le \frac{s+k}{s+2} \le \frac{k}{2}$ since $k \ge 2$. Therefore, if $\mathrm{tough}(G) > \frac{k}{2}$, then G is k-fc. The toughness of the graph G obtained by joining all the vertices of a clique K_k to all the vertices of two disjoint odd cliques K_{2q+1} is equal to $\frac{k}{2}$, and G is not k-fc, which proves the sharpness of the bound $\frac{k}{2}$. ■

The *t-degree sum* and the *t-generalized independent minimum degree* of G are respectively $\sigma_t(G) = \min\{\Sigma_{w_i \in W} d(w_i) : W$ is an independent set of t vertices of $G\}$ and $U_t = \min\{|\bigcup_{w_i \in W} N(w_i)| : W$ is an independent set of t vertices of $G\}$. These two parameters are defined for t at most equal to the independence number of G. For $t = 1$, $\sigma_1 = U_1 = \delta$.

Theorem 4.2. *Let G be a graph of order n and connectivity κ, and let k be an integer such that $0 \le k \le \kappa$ and $n + k$ is even. If for some integer t with $1 \le t \le \kappa - k + 2$, $\sigma_t(G) \ge t(\frac{n+k}{2} - 1) + 1$ or $U_t(G) \ge n - \kappa + k - 1$, then G is k-fc.*

Proof. Let G be a graph of order n and connectivity κ, which is not k-fc for some integer $k \le \kappa$ such that $n + k$ is even. Let X be a set of k vertices of G such that $G' = G - X$ has no perfect matching. As in Theorem 4.1, by Tutte's theorem, there is a set S of vertices of G' such that the number c of components $\mathcal{C}_i$ of $G' - S$ is at least $|S| + 2$. Let $s = |S|$. Since G is κ-connected, G' is $(\kappa - k)$-connected and thus $s \ge \kappa - k$. On the other hand, the sets X, S and $\mathcal{C}_i$ are all disjoint and thus $|X| + |S| + c \le n$, which implies, since $c \ge s + 2$, $s \le \frac{n-k}{2} - 1$. For $1 \le i \le c$, let w_i be a vertex of $\mathcal{C}_i$. For any integer t with $1 \le t \le \kappa - k + 2 \le c$, the set $\{w_i; 1 \le i \le t\}$ is independent.

1. The degree in G of each w_i satisfies $d(w_i) \le |X| + |S| + |\mathcal{C}_i - \{w_i\}| = k + s + |\mathcal{C}_i| - 1$. Therefore $\sigma_t \le \sum_{i=1}^{t} d(w_i) \le t(k + s - 1) + \sum_{i=1}^{t} |\mathcal{C}_i|$. But

$$\begin{aligned}\sum_{i=1}^{t}|\mathcal{C}_i| &= |V \setminus S \setminus X \setminus \bigcup_{i=t+1}^{c} \mathcal{C}_i| \\ &= n-s-k-\sum_{i=t+1}^{c}|\mathcal{C}_i| \le n-s-k-(c-t) \le n-k+t-2s-2\end{aligned}$$

and thus $\sigma_t \le tk + (t-2)s + n - k - 2$.

If $t \ge 2$, we get $\sigma_k \le tk + (t-2)(\frac{n-k}{2} - 1) + n - k - 2 = t(\frac{n+k}{2} - 1)$. Hence if for some t between 2 and $\kappa - k + 2$, $\sigma_t > t(\frac{n+k}{2} - 1)$, then G is k-fc.

For $t = 1$ the condition $\sigma_t(G) \ge t(\frac{n+k}{2} - 1) + 1$ reduces to $\delta \ge \frac{n+k}{2}$ and it is known [2] that this implies that G is k-hamiltonian and thus k-fc.

2. The neighborhood in G of each w_i satisfies $N(w_i) \subseteq X \cup S \cup \mathcal{C}_i$. Therefore, $U_t \le |\bigcup_{i=1}^{t} N(w_i)| \le |X| + |S| + \sum_{i=1}^{t}(|\mathcal{C}_i| - 1) \le k + s + (n - k - 2s - 2 + t) - t = n - s - 2 \le n - \kappa + k - 2$. Hence if for some t between 1 and $\kappa - k + 2$, $U_t \ge n - \kappa + k - 1$, then G is k-fc. ■

We finish with an example related to a property of the same kind as in [4] but for which the conclusion "G is k-extendable" cannot be replaced by "G is $2k$-fc". Ryjáček proved in [13] that every even $(2k+1)$- connected $K_{1,k+3}$-free graph such that the set of claw centers is independent, is k-extendable. The hypotheses do not imply that the graph is $2k$-fc as shown, for $k = 1$, by the following construction. The graph G consists of four copies H_i of cliques K_p of odd order $p \ge 3$, and four extra vertices x_i, $1 \le i \le 4$. In each H_i, we select three vertices y_{ij} with $1 \le j \le 4$ and $j \ne i$. Each vertex x_i is adjacent to the three vertices y_{ji} with $1 \le j \le 4$ and $j \ne i$. The graph G is 3-connected, $K_{1,4}$-free and the claw centers, which are the vertices x_i, are independent. It is 1-extendable but not 2-fc since $G - \{x_1, x_2\}$ has no perfect matching.

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