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PANCYCLISM AND SMALL CYCLES IN GRAPHS

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Abstract

We first show that if a graph G of order n contains a hamiltonian path connecting two nonadjacent vertices u and v such that $d(u) + d(v) \geq n$, then G is pancyclic. By using this result, we prove that if G is hamiltonian with order $n \geq 20$ and if G has two nonadjacent vertices u and v such that $d(u) + d(v) \geq n + z$, where $z = 0$ when n is odd and $z = 1$ otherwise, then G contains a cycle of length m for each $3 \leq m \leq \max\,(d_C(u,v) + 1,\ \frac{n+19}{13})$, $d_C(u,v)$ being the distance of u and v on a hamiltonian cycle of G.

Keywords: cycle, hamiltonian, pancyclic.

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1. Notation, Terminology and Introduction

We will consider only finite, undirected graphs, without loops or multiple edges. If G is a graph, we denote by $V(G)$ the vertex set of G and by $E(G)$ the edge set of G. If A is a subgraph or a subset of vertices, $|A|$ is the number

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of vertices in A. For any $a \in V(G)$, $A \subseteq V(G)$, $B \subseteq V(G) - A$ and a subgraph H of G, we let

$$N_H(a) = \{v \in V(H) : av \in E(G)\},$$

$$d_H(a) = |N_H(a)|,$$

$$E_H(A, B) = \{uv \in E(H) : u \in A \text{ and } v \in B\},$$

and

$$e_H(A, B) = |E_H(A, B)|.$$

If $C = c_1c_2...c_pc_1$ is a cycle, we let $C[c_i, c_j]$, for each $i \neq j$, be the path $c_ic_{i+1}...c_j$, where the indices are taken modulo p. If $|C[c_i, c_j]| \leq |C[c_j, c_i]|$, then the distance of c_i and c_j on C, denoted by $d_C(c_i, c_j)$, is equal to $j - i$ modulo p. Similarly, if $P = p_1p_2...p_q$ is a path, let $P[p_i, p_j] = p_ip_{i+1}...p_j$. The length of P is $|P|-1$ which is the number of edges on P. For two vertices u and v, a (u, v)-path is a path connecting u and v and a hamiltonian (u, v)-path is a hamiltonian path connecting u and v. For any integer m, denote by C_m a cycle of length m. A graph of order n is said to be pancyclic if it contains cycles of all lengths from 3 to n.

Other notation and terminology can be found in [4].

Bondy suggested the interesting "metaconjecture" in [3] that almost any nontrivial condition on graphs which implies that the graph is hamiltonian also implies that the graph is pancyclic (there may be a family of exceptional graphs). Various sufficient conditions for a graph to be hamiltonian have been given in term of the vertex degrees and many of them have been shown to imply pancyclism. For example, we have the following :

Theorem 1. (a) (Ore's condition, [3]). *If a graph G satisfies Ore's condition that the degree sum of any pair of nonadjacent vertices is at least the order of G, then G is pancyclic or isomorphic to $K_{n/2,n/2}$.*

(b) (Chvátal's condition, [6]). *Let G be a graph on $n \geq 3$ vertices with vertex degree sequence $d_1 \leq d_2 \leq ... \leq d_n$. If $d_k \leq k < \frac{n}{2}$ implies $d_{n-k} \geq n - k$, then G is pancyclic or bipartite.*

(c) (Fan's condition, [2]). *Let G be a 2-connected graph on n vertices. If for all vertices x and y, distance $(x, y) = 2$ implies max $\{d(x), d(y)\} \geq \frac{n}{2}$, then G is either pancyclic, $K_{\frac{n}{2},\frac{n}{2}}, K_{\frac{n}{2},\frac{n}{2}} - e$, or the graph shown in Figure 1.*

(d) (Bondy's condition, [7]). *Let G be a 2-connected graph on n vertices. If for all independent vertices x, y and z, we have $d(x)+d(y)+d(z) \geq \frac{3n}{2} - 1$, then G is either pancyclic, $K_{\frac{n}{2},\frac{n}{2}}, K_{\frac{n}{2},\frac{n}{2}} - e$, or C_5.*

(e) [1]. *If G is a hamiltonian, non bipartite graph of order n with minimum degree at least $\frac{2n+1}{5}$, then G is pancyclic.*

(f) [8]. *If G is a hamiltonian graph of order $n \geq 40$ such that $d(x) + d(y) \geq \frac{4n}{5}$ for any pair of nonadjacent vertices x and y, then G is pancyclic or bipartite. The bound is sharp.*

(g) [7]. *If G is a hamiltonian graph of order n with a hamiltonian cycle $x_1, x_2, ..., x_n, x_1$ such that $d(x_1) + d(x_n) \geq n$, then G is either*

(1) *pancyclic,*

(2) *bipartite, or*

(3) *missing only an $(n-1)$-cycle and in this case, $x_1x_4 \in E(G)$.*

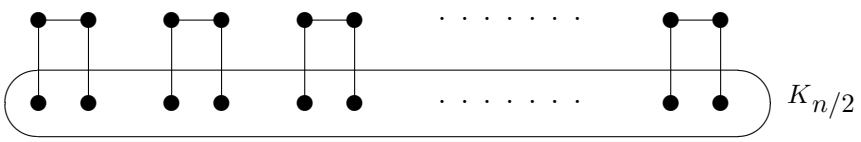

Figure 1

We note that most of the previous results are proved by starting with a hamiltonian cycle and by considering two consecutive vertices on the hamiltonian cycle, as we will see in Lemma 1 below. However with the exception of (g), these results depend on the degree conditions of almost all vertices. Similar to Lemma 1, our new results verify that degree conditions on only two special vertices are sufficient to insure pancyclism or small cycles.

We will first consider graphs with a hamiltonian path. We will show that if the degree sum of the two end vertices of the hamiltonian path is at least n (the order of the graph) then the graph is pancyclic. It will be seen that this result is interesting not only by itself, but also for applications to finding cycles of small lengths. For example, using this result, we obtain several useful lemmas and show that if G is hamiltonian with order $n \geq 20$ with two nonadjacent vertices u and v such that $d(u) + d(v) \geq n + z$, where $z = 0$ when n is odd and $z = 1$ otherwise, then G contains cycles of lengths m for all $3 \leq m \leq \max\ (d_C(u,v) + 1,\ \frac{n+19}{13})$, $d_C(u,v)$ being the distance of u and v on a hamiltonian cycle of G.

2. MAIN RESULTS

First let us mention the following result which is implicit in the proof of the main Theorem in [3].

Lemma 1. *Let $C = c_1c_2...c_nc_1$ be a hamiltonian cycle in a graph G. If $d(c_1) + d(c_n) \geq n + 1$, then for any k, $3 \leq k \leq n$, G contains a cycle C_k of one of the following forms* (see Figure 2):

(1) $c_nc_pc_{p-1}...c_{p-k+3}c_1c_n$, *for some p, $k-1 \leq p \leq n-1$,*

(2) $c_nc_pc_{p-1}...c_1c_{p-k+n+1}c_{p-k+n+2}...c_n$, *for some p, $1 \leq p \leq k-2$.*

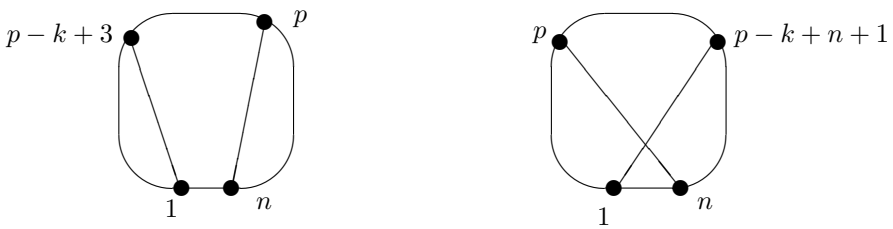

Figure 2

Theorem 2. *Let G be a graph of order n. If G has a hamiltonian (u, v)-path for a pair of nonadjacent vertices u and v such that $d(u) + d(v) \geq n$, then G is pancyclic. Moreover, if u (or v) has degree at least $\frac{n}{2}$, it is contained in a triangle and for any m, $4 \leq m \leq n$, there exists some C_m in G that contains both u and v.*

Proof. Let $P = c_1c_2...c_n$ be the hamiltonian path in G with $c_1 = u$ and $c_n = v$. First of all, since $d(u) + d(v) \geq n$, there exists some i such that $uc_{i+1}, vc_i \in E(G)$, and some j such that both of c_j and c_{j+1} are adjacent to either u (or v) that has degree at least $\frac{n}{2}$. So G contains C_n and a triangle that contains either u or v, whichever has degree at least $\frac{n}{2}$. We then consider cycles of all m, $4 \leq m \leq n-1$.

By the degree condition, u and v have at least two common adjacencies. Let $d = max\{i : c_i \in N(u) \cap N(v)\}$ and $W = P[c_{d+1}, c_n]$. By the symmetry of u and v, without loss of generality, we can assume $d \geq \frac{n}{2} + 1$ if n is even and $d \geq \frac{n+1}{2} + 1$ if n is odd. Let us define a graph H by putting $V(H) = \{c_1, c_2, ..., c_d\}$ and $E(H) = \{c_lc_{l+1} : 1 \leq l \leq d\} \cup \{c_1c_i : 3 \leq i \leq d, uc_i \in E(G)\} \cup \{c_dc_j : 2 \leq j \leq d-2, vc_j \in E(G)\}$.

The indices in H will be taken modulo d so that $c_{d+1} = c_1$.
For any k, $3 \leq k \leq d-1$, we define an integer t_k and a graph H_k as follows. If $vc_{d-1} \in E(G)$, then $t_k = 0$ and $H_k = H$ and if $vc_{d-1} \notin E(G)$, then $t_k = 1$ and $H_k = H$ when $c_1c_{d-k+2} \notin E(G)$ or $H_k = H - \{c_1c_{d-k+2}\}$ when $c_1c_{d-k+2} \in E(G)$.

Thus,

$$d_{H_k}(c_1) \geq d_G(u) - d_W(u) - t_k,$$

$$d_{H_k}(c_d) \geq d_G(v) - d_W(v) + t_k.$$

Hence,

$$d_{H_k}(c_1) + d_{H_k}(c_d) \geq d_G(u) + d_G(v) - d_W(u) - d_W(v).$$

The definition of d gives $d_W(u) + d_W(v) \leq n - d - 1$. It follows that

$$d_{H_k}(c_1) + d_{H_k}(c_d) \geq d + 1.$$

By Lemma 1, H_k is pancyclic and has a C_k of the form (1) or (2). We note from the definition of H_k that one of the following cases occur:

(a) the C_k contains an edge $c_d c_j$ with $j \neq 1, d-1$.

In this case, in G we put

$$C'_{k+1} = [C_k - \{c_j c_d\}] \cup \{c_n c_j, c_d c_n\}$$

and

$$C'_{k+n-d} = [C_k - \{c_j c_d\}] \cup \{c_n c_j\} \cup P[c_d, c_n].$$

(b) the C_k contains $c_1 c_d$ and $c_{d-1} c_d$.

In this case $C_k = c_1 c_d c_{d-1} ... c_{d-k+2} c_1$ and $c_n c_{d-1} \in E(G)$ by the definition of H_k. So in G we let

$$C''_{k+1} = [C_k - \{c_{d-1} c_d\}] \cup \{c_n c_{d-1}, c_n c_d\},$$

and

$$C''_{k+n-d} = [C_k - \{c_{d-1} c_d\}] \cup \{c_n c_{d-1}\} \cup P[c_d, c_n].$$

Therefore G has all cycles of length m for $4 \leq m \leq d$, and all cycles of length m for $n - d + 3 \leq m \leq n - 1$. These cycles contain both u and v. Since $n - d + 3 \leq d + 1$, we have cycles of all lengths in G satisfying the requirement. The proof of Theorem 2 is complete. ■

The bound of the degree sum in the theorem is sharp for odd n. This can easily be seen from a complete bipartite graph $K_{\frac{n-1}{2}, \frac{n+1}{2}}$ for any odd n.

Corollary 1 (see [5], Theorem 4.1). *Let G be a graph of order n. If the $(n+1)$-closure of G is complete, then G is pancyclic.*

Proof. (a) If the $(n+1)$-closure of G is complete, then G is hamiltonian connected. So if G is not complete, suppose that the first edge added to G to get the $(n+1)$-closure is xy. Then, there exists a hamiltonian path in G between x and y and $d(x)+d(y) \geq n+1$. By Theorem 2, G is pancyclic. ■

Corollary 2 (see [3], Theorem 1 (a)). *If n is odd and if the degree sum of any pair of nonadjacent vertices is at least n, then G is pancyclic.*

Proof. If the $(n+1)$-closure of G is complete, then from Corollary 1, G is pancyclic. Otherwise, by a theorem in [5], $G = K_2 + (K_r \cup K_{n-2-r})$ or $G = \bar{K}_2 + (K_r \cup K_{n-2-r})$ for some r. Clearly then G is pancyclic. ■

Corollary 3. *Let G be a hamiltonian graph of order n. If there exist two vertices u and v at distance 2 on a hamiltonian cycle C of G such that $d(u)+d(v) \geq n+1$, then G is pancyclic.*

Proof. If u and v are nonadjacent, it is an immediate consequence of Theorem 2 (by considering the subgraph $G-\{w\}$ where w is the vertex between u and v on C). Otherwise, if u and v are adjacent, let $C = x_1, x_2, ..., x_n$ with $u = x_1$ and $v = x_{n-1}$. By Theorem 1(g), we know that the hamiltonian graph G'=$G[x_1, x_2, ..., x_{n-1}]$ is in one of the following three cases: (1) pancyclic, (2) bipartite, (3) missing only an $(n-2)$-cycle and containing the edge x_1x_4. If G' is bipartite then G contains cycles of all even lengths passing through the edge x_1x_{n-1}. Replacing x_1x_{n-1} by the path $x_1x_{n-1}x_n$, we get cycles of all odd lengths between 5 and n and G is pancyclic since it also contains a triangle. If G' misses only a $(n-2)$-cycle, then G is also pancyclic since $x_1, x_4, x_5, ..., x_{n-1}, x_n, x_1$ is a cycle of G of length $n-2$. ■

From Theorem 2, we know that if a graph $G = (V, E)$ satisfies $d_P(u) + d_P(v) \geq p$ where $P = x_1x_2...x_p$ is a path of G such that $u = x_1$ and v_p, with u and v nonadjacent, then G contains cycles of all lengths between 3 and and p. We now are interested in what can be said if we assume $d_P(u) + d_P(v) \geq p-1$ instead of $d_P(u) + d_P(v) \geq p$. In this case, we make an additional hypothesis on the neighbors of u and v in $G-P$ and get the following result which will be useful in the proof of Theorem 4.

Theorem 3. *Suppose that the endvertices $u = x_1$ and $v = x_p$ of a path $P = x_1x_2...x_p$ of a graph $G = (V, E)$ are nonadjacent and satisfy $d_P(u) + d_P(v) \geq p-1$, and that there exist three vertices y, z, t not in P, with $z \neq t$ (and that y could be one of z and t), such that uy, yv, uz, zt and tv are edges of G. Then for each integer k, $4 \leq k \leq p+2$, there is a cycle of length k passing through at least one of the vertices u and v.*

Proof. The path $P' = x_2x_3...x_{p-1}$ satisfies $d_{P'}(u) + d_{P'}(v) \geq |P'| + 1$. Therefore u and v admit at least one common neighbor x_i on P' and the cycles ux_ivyu and ux_ivtzu have respective lengths 4 and 5. Similarly, the cycles $x_1...x_pyx_1$ and $x_1...x_ptzx_1$ have respective lengths $p+1$ and $p+2$.

Let us suppose that for a given value of m with $6 \leq m \leq p$, G contains no cycle C_m through u or v. The path $Q = x_2x_3...x_{p-2}$ satisfies

$$d_Q(u) + d_Q(v) \geq \begin{cases} p-3 = |Q|, & \text{if } ux_{p-1} \in E, \\ p-2 = |Q|+1, & \text{if } ux_{p-1} \notin E. \end{cases} \tag{1}$$

The bijection $f: \{2,3,...,p-2\} \longrightarrow \{2,3,...,p-2\}$ defined by

$$f(j) = \begin{cases} j+m-4, & \text{if } 2 \leq j \leq p-m+2 \quad (\text{then } m-2 \leq f(j) \leq p-2), \\ m-p+j-1, & \text{if } p-m+3 \leq j \leq p-2 \quad (\text{then } 2 \leq f(j) \leq m-3), \end{cases}$$

induces a bijection of $V(Q)$ onto $V(Q)$.

For $2 \leq j \leq p-2$, at most one of the two edges ux_j and $vx_{f(j)}$ can exist for otherwise the cycles $ux_jP[x_j, x_{f(j)}]x_{f(j)}vyu$ for $j \leq p-m+2$ and $ux_jP[x_j, v]vx_{f(j)}P[x_{f(j)}, u]u$ for $j \geq p-m+3$ would have length m. Therefore $d_Q(u) + d_Q(v) \leq |Q|$ and $d_Q(u) + d_Q(v) = |Q|$ if and only if, for every j between 2 and $p-2$, exactly one of the two edges ux_j and $vx_{f(j)}$ does exist. By (1), we are necessarily in this last case. This implies $ux_{p-1} \in E$ and thus $m \neq p-1$. By the symmetry between u and v, vx_2 is also an edge of G.

If $m = p$, then $f(4) = 3$. Since the edge ux_4 does not exist (because of $ux_4P[x_4, x_p]x_ptzu$), vx_3 is an edge and the cycle $vx_3P[x_3, x_{p-1}]x_{p-1}uyv$ of length m leads to a contradiction.

Suppose henceforth $m \leq p-2$. If there exists some index $k \leq p-m+2$ such that $ux_k \notin E$, then $vx_{f(k)} \in E$ with $f(k) = k+m-4$ and thus $ux_{k+1} \notin E$ for otherwise $ux_{k+1}P[x_{k+1}, x_{k+m-4}]x_{k+m-4}vtzu$ is a cycle of length m. Therefore there exists an index i with $2 \leq i \leq p-m+2$ such that $ux_j \in E$ for all j between 2 and i and $ux_j \notin E$ for all j between $i+1$ and $p-m+3$. Note that $i \leq p-m$ because of $ux_{p-m+1}P[x_{p-m+1}, x_{p-1}]x_{p-1}u$, and that $i \leq m-4$ because of $vx_2P[x_2, x_{m-3}]x_{m-3}uztv$. When $2 \leq j \leq i$, then $m-2 \leq f(j) = j+m-4 \leq i+m-4$ and when $i+1 \leq j \leq p-m+2$, then $m+i-3 \leq f(j) \leq p-2$. Looking at $vx_{f(j)}$, we see that $vx_k \in E$ for all k with $m+i-3 \leq k \leq p-2$ and $vx_k \notin E$ for all k with $m-2 \leq k \leq m+i-4$. The edge $vx_{p-m+i+1}$ does not exist for otherwise $ux_{p-1}P[x_{p-1}, x_{p-m+i+1}]x_{p-m+i+1}vx_2P[x_2, x_i]x_iu$ is a cycle of length m. Therefore $p-m+i+1 \leq m+i-4$; that is, $2m \geq p+5$.

Finally, the existence of the edge ux_i implies that ux_{i+m-2} is not in E (because of $ux_iP[x_i, x_{i+m-2}]x_{i+m-2}u$) where $i + m - 2 \geq p - m + 3$, and thus $vx_{f(i+m-2)} = vx_{2m+i-p-3}$ is an edge of G. The cycle $uP[u, x_{2m+i-p-3}]\, x_{2m+i-p-3}vx_{m+i-2}P[x_{m+i-2}, x_{p-1}]x_{p-1}u$ of length m leads to a contradiction, which completes the proof. ■

Corollary 4. *Let G be a hamiltonian graph of order n. If there exist two nonadjacent vertices u and v at distance $d \geq 3$ on a hamiltonian cycle of G such that $d(u) + d(v) \geq n + d - 2$, then G contains cycles of all lengths between 3 and $n - d + 1$.*

Proof. Let $C = x_1, x_2, ..., x_n, x_1$ be a hamiltonian cycle of G, $u = x_1$, $v = x_{n-d+1}$ $(3 \leq d \leq \frac{n}{2})$ and $P = x_1, x_2, ..., x_{n-d+1}$. If all the edges exist between $\{u, v\}$ and $\{x_{n-d+2}, x_{n-d+1}, ..., x_n\}$, then $d_P(u) + d_P(v) \geq (n+d-2)-2(d-1) = n-d$ and G contains a triangle and cycles of all lengths between 4 and $n - d + 3$ by Theorem 3. If at least one of the edges between $\{u, v\}$ and $\{x_{n-d+2}, x_{n-d+1}, ..., x_n\}$ is missing, then $d_P(u)+d_P(v) \geq n-d+1$ and $G[x_1, x_2, ..., x_{n-d+1}]$ is pancyclic by Theorem 2.

Considering the case when u and v are adjacent, as a corollary of Theorem 1(g) with a proof quite analogous to the proof of Corollary 3, we have that if G is a hamiltonian graph of order n and if there exist two adjacent vertices u and v at distance $d \geq 3$ on a hamiltonian cycle of G such that $d(u) + d(v) \geq n + d - 1$, then G contains cycles of all lengths between 3 and $n - d + 1$. ■

We now show some lemmas that will be useful in the proof of the next main result that gives a condition insuring the existence of small cycles.

Lemma 2. *Let $P = v_1v_2...v_q$ be a path and u_1 and u_2 two vertices not in P.*
(1) If $d_P(u_1) + d_P(u_2) \geq q + 1$, there exists some i such that $v_i \in N(u_1) \cap N(u_2)$.
(2) If $d_P(u_1) + d_P(u_2) \geq q + 1$ and if there does not exist an index i such that u_1 is adjacent to one of v_i and v_{i+1} and u_2 is adjacent to the other one, then $d_P(u_1) + d_P(u_2) = q + 1$, q is odd and $N_P(u_1) = N_P(u_2) = \{v_i : i = 1, 3, 5, ..., q\}$.

Proof. (1) is trivial.
Suppose the hypotheses of (2) satisfied. Then for any $v_i \in N(u_1)$ we have $v_{i-1}, v_{i+1} \notin N(u_2)$ and hence for any odd integer t with $1 \leq t \leq q$,

$$d_{P[v_1,v_t]}(u_2) \leq t - d_{P[v_1,v_t]}(u_1) + 1,$$

and

$$d_{P[v_t,v_q]}(u_2) \leq (q-t+1) - d_{P[v_t,v_q]}(u_1) + 1.$$

This implies that the two previous inequalities are in fact equalities. We can easily deduce that q must be odd and

$$N_P(u_1) = N_P(u_2) = \{v_i : i = 1, 3, ..., q\}.$$

■

Lemma 3. *If the graph G of order n has a two-path partition $P' = v_1v_2...v_f$ and $P'' = v_{f+1}...v_n$ satisfying the two conditions*

(i) $v_1v_n \notin E(G)$

and

(ii) $d(v_1) + d(v_n) \geq n + 1$

then it contains a cycle C_m for each integer m with $3 \leq m \leq min(\frac{f+6}{2}, \frac{n-f+6}{2})$.

Proof. Note that the upper bound on m can be written $f \geq 2m - 6$ and $n - f \geq 2m - 6$. By the symmetry between f and $n - f$, assume without loss of generality that

$(*)$ $d_{P'}(v_1) + d_{P'}(v_n) \geq f + 1$.

Hence $|N(v_n) \cap P'| \geq 2$ and $f \geq 3$.

If we put $s = max\{i \leq f : v_i \in N(v_n)\}$, $s' = min\{i \leq f : v_i \in N(v_n)\}$ (note that $s' \geq 2$ by (i)) and $x = max\{i \leq f : v_i \in N(v_1)\}$, we have $d_{P'}(v_1) \leq x - 1$, $d_{P'}(v_n) \leq s - s' + 1$ and thus by $(*)$,

$(**)$ $s - s' + x \geq f + 1$.

If $3 \leq m \leq s + 1$, the graph $G' = G[v_1, v_2, ..., v_s, v_n]$ of order $s + 1$ satisfies by $(*)$ the condition $d_{G'}(v_1) + d_{G'}(v_n) \geq f + 1 - (f - s) = s + 1$. By Theorem 2, G' contains a cycle C_m.

Suppose now $m \geq s + 2$ which implies $s \leq \frac{f+6}{2} - 2 = \frac{f+2}{2}$. By $(**)$, $x \geq s' + \frac{f}{2} \geq \frac{f+4}{2} > s$ and $x - s' + 3 \geq f + 4 - s \geq \frac{f+6}{2} \geq m$. The graph $G'' = G[v_n, v_{s'}, v_{s'+1}, ..., v_s, ..., v_x, v_1]$ of order $x - s' + 3$ satisfies by $(*)$ the condition $d_{G''}(v_1) + d_{G''}(v_n) \geq f + 1 - (s' - 2) \geq x - s' + 3$. By Theorem 2, G'' contains a cycle C_m. ■

Lemma 4. *Let G contain a hamiltonian path $P = v_1v_2...v_n$ such that $v_1v_n \notin E(G)$ and $d(v_1) + d(v_n) \geq n + d$ for some integer d, $0 \leq d \leq n - 4$. Then for any l, $2 \leq l \leq d + 3$, there exists a (v_1, v_n)-path of length l.*

Proof. Suppose that G has no (v_1, v_n)-path of length l for some l between 2 and $d + 3$. Then for any $v_i \in N(v_1)$ with $2 \leq i \leq n - l$, $v_nv_{i+l-2} \notin E(G)$.

Thus

$$\{v_j : v_jv_n \in E(G)\} \subseteq V - \{v_1, v_n\} - \{v_{i+l-2} : v_i \in N(v_1),\ 2 \le i \le n-l\}$$

and $v_{n-l+1}v_1 \notin E(G)$. Therefore $d(v_n) \le n-2-d(v_1)+l-2$ which leads to $d(v_1)+d(v_n) \le n+l-4$, and so $l \ge d+4$, a contradiction. ■

Lemma 5. *Let $P = v_1v_2...v_q$ be a path and u_1 and u_2 two vertices not on P without any common adjacency on P. For $m \ge 5$, if $d_P(u_1)+d_P(u_2) \ge q-t+1$ for some integer t with either*
$t = 2$ and $q \ge max\{16, m+2\}$ or
$t \ge 3$ and $\lfloor \dfrac{q}{m-4+3\lfloor \frac{m-4}{3} \rfloor} \rfloor \lfloor \dfrac{m-4}{3} \rfloor \ge t$ when $m \ge 7$
or $\lfloor \dfrac{q}{m-1} \rfloor \ge t$ when $5 \le m \le 6$,
then $G[P \cup \{u_1, u_2\}]$ has a C_m or a (u_1, u_2)-path of length either $m-2$ or $m-3$.

Proof. Suppose that $G[P \cup \{u_1, u_2\}]$ has neither C_m nor a (u_1, u_2)-path of length $m-2$ or $m-3$. If $v_i \in N(u_1)$ (resp. $N(u_2)$), for some $i \le q-m+4$, then $v_{i+m-4}, v_{i+m-5} \notin N(u_2)$ (resp. $N(u_1)$) and $v_{i+m-2} \notin N(u_1)$ (resp. $N(u_2)$) for $i \le q-m+2$. So if $v_i, v_{i+2} \in N(u_1)$ (resp. $N(u_2)$) for some $i \le q-m+2$, then $v_{i+m-2} \notin N(u_1) \cup N(u_2)$. If $v_i \in N(u_1)$ (resp.$N(u_2)$) and $v_{i+1} \in N(u_2)$ (resp. $N(u_1)$) for some $i \le q-m+1$, then $v_{i+m-4} \notin N(u_1) \cup N(u_2)$.

We claim that among $\{v_i, v_{i+1}, v_{i+2}, v_{i+m-4}, v_{i+m-3}, v_{i+m-2}\}$, there is at least one common nonadjacency of u_1 and u_2. To show the claim, assume, without loss of generality, that $u_1v_i \in E(G)$. Then by using the statements above, we deduce that if v_{i+m-2} and v_{i+m-4} are not common nonadjacency, $v_{i+2}u_1 \notin E(G)$, $v_{i+1}u_2 \notin E(G)$. Hence either one of the two edges $v_{i+2}u_2$, $v_{i+1}u_1$ is in $E(G)$ or $v_{i+2}u_2, v_{i+1}u_1 \notin E(G)$ that imply that v_{i+m-3} is a common nonadjacency. The claim is proved.

It follows that when $q \ge m+2$ and $m \ge 10$, there is at least one common nonadjacency in $\{v_1, v_2, v_3, v_{m-3}, v_{m-2}, v_{m-1}\}$ and one in $\{v_4, v_5, v_6, v_m, v_{m+1}, v_{m+2}\}$. So $P[v_1, v_{m+2}]$ has at least two common nonadjacencies of u_1 and u_2. When $m \le 9$ and $q \ge 16$, there is one common nonadjacency in $P[v_1, v_8]$ and one in $P[v_9, v_{16}]$. These give a contradiction of the degree sum condition if $t = 2$. So we assume $t \ge 3$. Then if $m \ge 7$, by considering the sets $\{v_{f+3i}, v_{f+3i+1}, v_{f+3i+2}, v_{f+3i+m-4}, v_{f+3i+m-3}, v_{f+3i+m-2}\}$ with $0 \le i \le \lfloor \frac{m-4}{3} \rfloor - 1$, we know that there is at least one common nonadjacency in every set and so there are at least $\lfloor \frac{m-4}{3} \rfloor$ common nonadjacencies in

any subpath $P[f, f+m-4+3\lfloor\frac{m-4}{3}\rfloor-1]$ of $m-4+3\lfloor\frac{m-4}{3}\rfloor$ vertices. Thus u_1 and u_2 have at least $\lfloor\frac{q}{m-4+3\lfloor\frac{m-4}{3}\rfloor}\rfloor\lfloor\frac{m-4}{3}\rfloor \geq t$ common nonadjacencies in P. This implies that $d_P(u_1)+d_P(u_2) \leq q-t$, a contradiction to the hypothesis. If $5 \leq m \leq 6$, there is at least one common nonadjacency in every set $\{v_{1+i(m-1)}, v_{2+i(m-1)}, v_{3+i(m-1)}, v_{m-3+i(m-1)}, v_{m-2+i(m-1)}, v_{m-1+i(m-1)}\}$ for $0 \leq i \leq \lfloor\frac{q}{m-1}\rfloor-1$. Then we have at least $\lfloor\frac{q}{m-1}\rfloor \geq t$ common nonadjacencies and again $d_P(u_1)+d_P(u_2) \leq q-t$, a contradiction. ■

Theorem 4. *If G is a hamiltonian graph of order $n \geq 20$ such that there exists a pair of nonadjacent vertices u and v satisfying $d(u)+d(v) \geq n+z$ where $z=0$ if n is odd and $z=1$ if n is even, then G contains cycles C_m for all $3 \leq m \leq max(d_C(u,v)+1, \frac{n+19}{13})$, $d_C(u,v)$ being the distance of u and v on a hamiltonian cycle of G.*

Consider the graph drawn in Figure 3 and obtained from a hamiltonian cycle $C = c_1c_2...c_nc_1$ by adding all edges c_ic_j, for $i=1$ or $\lfloor\frac{n+1}{2}\rfloor$ and for any j, $p+2 \leq j \leq \lfloor\frac{n+1}{2}\rfloor-p-1$ and $\lfloor\frac{n+1}{2}\rfloor+p+1 \leq j \leq n-p$ where p is an integer between 2 and $\frac{n-5}{4}$. The degree sum of c_1 and $c_{\lfloor\frac{n+1}{2}\rfloor}$ is $2(n-4p+2)$ and the graph does not contain cycles of length l, $n-p+1 \leq l \leq n-1$.

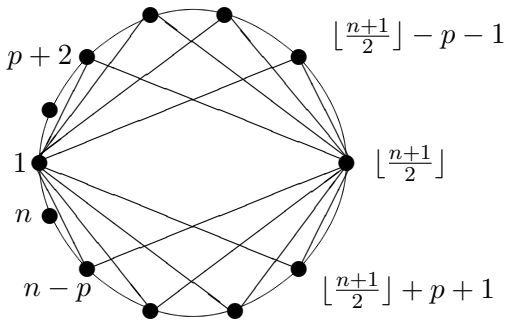

Figure 3

This example shows that even if there exist in a hamiltonian graph two nonadjacent vertices whose degree sum is very large, this graph is not necessarily pancyclic.

Proof of Theorem 4. Let $C = c_1c_2...c_n$ be a hamiltonian cycle of G, and without loss of generality, let $u = c_1, v = c_p$ with $\frac{n+2}{2} \leq p \leq n-2$. Thus the condition $m \leq \frac{n+19}{13}$ implies $m \leq \frac{2p+17}{13}$ and $d_C(u,v) = n+1-p$. It is easy to see that there exists some i such that both c_i and c_{i+1} are adjacent

to one of u and v, and there exist some j' and j'', $j' \neq j''$, such that $c_{j'}$ and $c_{j''}$ are adjacent to both of u and v. Therefore $C_3, C_4 \subset G$.

Henceforth we assume $m \geq 5$ and suppose that G does not contain any C_m, for some $m \leq \max(d_C(u,v)+1, \frac{n+19}{13})$. Put $A = G[c_1, c_2, ..., c_p]$ and $B = G[c_p, c_{p+1}, ..., c_n, c_1]$. If $d_A(u)+d_A(v) \geq p$, by Theorem 2, $C_m \subset A \subset G$. So there exists an integer $t \geq 2$ such that $d_A(u)+d_A(v) = p-t+1$, and hence $d_B(u)+d_B(v) \geq n+z-p+t-1 = |B|+z+t-3$. Since $d_B(u)+d_B(v) \leq 2|B|-4$, we also have $|B| \geq t+z+1$ and thus $t \leq n-p+1$. If $m \leq |B|$ and $t+z \geq 3$, then $C_m \subset B \subset G$ by Theorem 2. So we assume that $t = 2$ or $t \geq 3$ and $m \geq |B|+1 = n-p+3$.

By Lemma 2(1) applied to the path $B' = B - \{u, v\}$ of $n-p$ vertices which satisfies $d_{B'}(u) + d_{B'}(v) \geq |B'| + z + t - 1$, there is some $c_s \in N(u) \cap N(v) \cap B$. When $t = 2$, by the same lemma applied to $A' = A - \{u, v\}$, there is also some $c'_s \in N(u) \cap N(v) \cap A$.

When $t \geq 3$, by Lemma 2(2) applied to B', there exists some c_r in B' such that u is adjacent to one of c_r and c_{r+1} and v is adjacent to the other one.

When $t = 2$, if there exists some c_r in B' such that u is adjacent to one of c_r and c_{r+1} and v is adjacent to the other one, we apply Theorem 3 to the graph $G[A \cup \{c_s, c_r, c_{r+1}\}]$ and have all cycles $C_m,\ 4 \leq m \leq |A|+2$. So we assume there is no such c_r in B'. It follows from Lemma 2(2) that $z = 0$, $n-p$ is odd and $N_{B'}(u) = N_{B'}(v) = \{c_{p+1}, c_{p+3}, ..., c_n\}$ In this case, by Lemma 2(2) applied to A', there exists some c_r in A' such that u is adjacent to one of c_r and c_{r+1} and v is adjacent to the other one, and by symmetry we may assume $r \geq \frac{p}{2}$. Moreover, using $c'_s,\ c_r,\ c_{r+1}$ in A and the neighborhoods $N_{B'}(u) = N_{B'}(v) = \{c_{p+1}, c_{p+3}, ..., c_n\}$, we have all cycles $C_m,\ m \leq |B|+2$. So only the following two possibilities remain to be studied.

Case 1. $t = 2$, $\frac{p}{2} \leq r \leq p-1$ and $m \geq |B|+3 = n-p+5$.

Case 2. $t \geq 3$ and $m \geq |B|+1 = n-p+3$.

Note that $t+2 \leq m \leq \frac{2p+17}{13}$ and, since $n-p+3 \leq \frac{n+19}{13}$, $n \leq \frac{13p-20}{12}$ and by $n \geq 13m-19$, $p \geq 12m-16$. In the remainder of the proof, the justification of all the omitted details is based on these inequalities.

We begin with the proof of the following two claims and note that G contains no (u,v)-path of length $m-2$ or $m-3$ avoiding c_s or c_r and c_{r+1} for otherwise G would contain a cycle C_m.

Claim 1. *In Case* 1*, there is no common adjacency of* u *and* v *in* $C[c_m, c_{\frac{p}{2}-m+3}]$.

In Case 2, there is no common adjacency of u and v in $C[c_{m+t-3}, c_{p-m-t+4}]$.

***Proof of Claim* 1.** When $t = 2$ and $c_r \in C[\frac{p}{2}, p-1]$, to avoid a (u,v)-path of length $m-2$ or $m-3$ that does not contain c_s, c_r nor c_{r+1}, if $c_j \in N(u) \cap N(v) \cap C[c_m, c_{\frac{p}{2}-m+3}]$ then $c_{j+m-5}, c_{j+m-4} \notin N(u) \cup N(v)$ (note that $(\frac{p}{2} - m + 3) + m - 4 < \frac{p}{2} \leq r$). Let $P' = c_1 c_2 ... c_{j+m-6}$ and $P'' = c_{j+m-3} c_{j+m-2} ... c_p$. Since $2m-6 \leq j+m-6$ and $p-(j+m-4) \geq 2m-6$, and since $d_{P' \cup P''}(u) + d_{P' \cup P''}(v) = p - 1 = |P' \cup P''| + 1$, $C_m \subset G$ by Lemma 3, a contradiction.

When $t \geq 3$, there exists some (u,v)-path of length l in B for any l, $2 \leq l \leq t$ by Lemma 4. To avoid a C_m, there does not exist any (u,v)-path of length d in A for any d, $m - t \leq d \leq m-2$. If $c_j \in N(u) \cap N(v) \cap C[c_{m+t-3}, c_{p-4m+10}]$, then $[N(u) \cup N(v)] \cap C[c_{j+m-t-2}, c_{j+m-4}] = \emptyset$ and clearly $c_{j+m-2} \notin N(u) \cup N(v)$.

If $e(\{c_{j+m-3}\}, \{u,v\}) \leq 1$, we let $P' = c_1 c_2 ... c_{j+m-t-3}$ and $P'' = c_{j+m-1} c_{j+m} ... c_p$. Since $j+m-t-3 \geq 2m-6$ and $p-j-m+2 \geq 2m-6$, and since

$$d_{P' \cup P''}(u) + d_{P' \cup P''}(v) \geq p - t = |P' \cup P''| + 1,$$

$C_m \subset G[P' \cup P''] \subset G$ by Lemma 3, a contradiction.

Therefore $c_{j+m-3} \in N(u) \cap N(v)$ and by the previous argument, replacing c_j by c_{j+m-3}, we obtain $c_{j+2m-6} \in N(u) \cap N(v)$ since we still have $p-j-2m+5 \geq 2m-6$. We deduce that $[N(u) \cup N(v)] \cap C[c_{j+m-2}, c_{j+m-4+t}] = \emptyset$. Since $j+m \leq (j+m-4+t)+1$, the paths $P' = c_1 c_2 ... c_{j+m-t-3}$ and $P'' = c_{j+m} c_{j+m+1} ... c_p$ satisfy $|P'| = j+m-t-3 \geq 2m-6$, $|P''| = p-j-m+1 \geq 2m-6$ and $d_{P' \cup P''}(u) + d_{P' \cup P''}(v) = p-t-1 = |P' \cup P''| + 1$. Thus by Lemma 3, we also have $C_m \subset G[P' \cup P''] \subset G$, a contradiction.

So $(N(u) \cap N(v)) \cap C[c_{m+t-3}, c_{p-4m+10}] = \emptyset$. As $p - 4m + 10 \geq \frac{p+1}{2}$, the result follows by symmetry.

Claim 2. *In Case 1, there exist some q_1 and q_2 such that $2m-5 \leq q_1 < q_2 \leq \frac{p}{2} - m + 3$ and c_{q_1}, $c_{q_2} \notin N(u) \cup N(v)$.*
In Case 2, there exist some $q_1, q_2, ..., q_t$ such that $2m-5 \leq q_1 < q_2 < ... < q_t \leq p - 2m + 6$ and $\{c_{q_1}, c_{q_2}, ..., c_{q_t}\} \cap (N(u) \cup N(v)) = \emptyset$.

***Proof of Claim* 2.** Let $P = c_{2m-5} c_{2m-4} ... c_{\frac{p}{2}-m+3}$ in Case 1 and $P = c_{2m-5} c_{2m-4} ... c_{p-2m+6}$ in Case 2. In each case, by Claim 1 and since $m \geq 5$ and $m \geq t+2$, u and v have no common adjacency on P.

When $t = 2$ and $c_r \in C[\frac{p}{2}, p-1]$ we have $|P| = \frac{p}{2} - 3m + 9 \geq max\{16, m+2\}$ since $p \geq 12m - 16$ and $m \geq 5$. By Claim 1 and Lemma 5, to avoid a C_m or a (u,v)-path of length $m-2$ or $m-3$ in $G[P \cup \{u,v\}]$ (note that $c_r \notin P$), $d_P(u) + d_P(v) \leq |P| - 2$ and thus q_1 and q_2 exist.

When $t \geq 3$ we have $|P| \geq 8m-4$ since $p \geq 12m-16$. Some arithmetical verifications allow us to check that $\lfloor \frac{8m-4}{m-4+3\lfloor \frac{m-4}{3} \rfloor} \rfloor \lfloor \frac{m-4}{3} \rfloor \geq m-2$ if $m \geq 7$ and that $\lfloor \frac{8m-4}{m-1} \rfloor \geq m-2$ if $5 \leq m \leq 6$. So by Claim 1 and Lemma 5, to avoid a C_m or a (u,v)-path of length $m-2$ or $m-3$ in $G[P \cup \{u,v\}]$, $d_P(u) + d_P(v) \leq |P| - t$ and thus $q_1,\ q_2, ..., q_t$ exist. Claim 2 is proved.

To complete the proof, note that by Claims 1 and 2, the path $Q = c_{q_1} c_{q_1+1} ... c_{q_t}$ satisfies $d_Q(u) + d_Q(v) \leq |Q| - t$. The two paths $Q' = c_1 c_2 ... c_{q_1-1}$ and $Q'' = c_{q_t+1} c_{q_t+2} ... c_p$ have at least $2m-6$ vertices. Moreover $d_{Q' \cup Q''}(u) + d_{Q' \cup Q''}(v) \geq (p-t+1) - (|Q| - t) = |Q'| + |Q''| + 1$. Therefore, by Lemma 3, $C_m \subset G[Q' \cup Q''] \subset G$, the final contradiction. The proof of Theorem 4 is complete. ■

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