Discussiones Mathematicae
Graph Theory 16 (1996) 17–26

RADII AND CENTERS IN ITERATED LINE DIGRAPHS

Martin Knor

Slovak Technical University, Faculty of Civil Engineering
Department of Mathematics, Radlinského 11, 813 68 Bratislava, Slovakia

and

L'udovít Niepel

Comenius University, Faculty of Mathematics and Physics
Mlynská dolina, 842 15 Bratislava, Slovakia

Abstract

We show that the out-radius and the radius grow linearly, or "almost" linearly, in iterated line digraphs. Further, iterated line digraphs with a prescribed out-center, or a center, are constructed. It is shown that not every line digraph is admissible as an out-center of line digraph.

Keywords: center, digraph, line digraph, radius.

1991 Mathematics Subject Classification: 05C20, 05C12.

1. Introduction

Line digraphs are useful and important in various problems. In this paper we concentrate on iterated line digraphs, and on their radii (in [4] the diameter and average distance of iterated line digraphs are studied). In [1] Aigner proved the following theorem.

Theorem A. *If D is a strongly connected digraph, then $r^+(D) \leq r^+(L(D)) \leq r^+(D) + 1$.*

Here, r^+ denotes the out-radius of a digraph. We extend this result to a larger class of digraphs (see Lemma 3.2). This enables us to examine the behavior of radii of iterated line digraphs (Theorems 3.3, 3.4, and 3.6). In particular, it is shown that if D is strongly connected then the out-radius of $L^i(D)$ increases linearly with i while the radius of $L^i(D)$ differs from i by at most a constant (depending only on D). This should be contrasted with iterated line graphs, the radius of which satisfies inequalities [6]:

$$i - \sqrt{2\log_2 i} + c_G < r(L^i(G)) < i - \sqrt{2\log_2 i} + c'_G,$$

and hence its growth is slower than linear. (Here, c_G and c'_G are constants depending only on G, and G is any graph different from a path, a cycle, and a claw $K_{3,1}$.)

Further, we consider centers in iterated line digraphs. It is easily seen that for every digraph D there is a digraph H having D as its center (for the graph version of this result, see [3, p.41]). We generalize this observation by investigating conditions under which $L^i(D)$ is a center of some $L^i(H)$ for all i (or for all i up to some fixed number). Finally, we show that there exist line digraphs which are not out-centers of any line digraphs. This may again be compared with the case of graphs, where every line graph serves as a center of some line graph [5].

The outline of this paper is as follows. In section 2 we build our basic tool for counting the distances in iterated line digraphs. In section 3 we examine the functions $r^+(L^i(D))$ and $r(L^i(D))$. Finally, section 4 is devoted to out-centers and centers in iterated line digraphs.

2. Preliminaries

Let D be a digraph. As usual, by $V(D)$ we denote the node set of D and by $E(D)$ the arc set of D; $id_D(u)$ denotes the input degree and $od_D(u)$ the output degree of a node u in D. If u and v are nodes in D, then $d_D(u,v)$ denotes the length of the shortest path from u to v in D. If there is no path from u to v, we set $d_D(u,v) = \infty$. Throughout the paper, by a path (a cycle) we always mean a directed path (a directed cycle).

The line digraph $L(D)$ of a digraph D is a digraph whose nodes are the arcs of D, with two nodes uv and xy joined by an arc in $L(D)$ if and only if $v = x$. If D has no arcs, then $L(D)$ is an empty digraph. By $L^0(D)$ we denote the digraph D. The *i-iterated line digraph* of D, $L^i(D)$, is $L(L^{i-1}(D))$ where $i \geq 1$.

Let u be a node in D. Then:

out-eccentricity of u is $e^+_D(u) = \max\{d_D(u,v) : v \in V(D)\}$;
in-eccentricity of u is $e^-_D(u) = \max\{d_D(v,u) : v \in V(D)\}$;
eccentricity of u is $e_D(u) = \max\{e^+_D(u), e^-_D(u)\}$.

Using various eccentricities we obtain various radii and various centers. The *out-radius* $r^+(D)$ (*in-radius* $r^-(D)$, *radius* $r(D)$) is the minimum value of

$e^+_D(u)$ ($e^-_D(u)$, $e_D(u)$) over all nodes u of D; and the *out-center* $C^+(D)$ (*in-center* $C^-(D)$, *center* $C(D)$) is the subgraph of D induced by nodes with the minimum out-eccentricity (in-eccentricity, eccentricity).

Let D' arise from D by reversing the orientation of all arcs. Then $e^-_{D'}(u) = e^+_D(u)$ for every node u in D, and hence, $r^+(D) = r^-(D')$ and $C^+(D) = C^-(D')$. This observation allows us to restrict our considerations to radii r^+ and r and to centers C^+ and C only.

Definitions not included here can be found in [2] or [3].

Let D be a digraph, and let u be a node in $L^i(D)$. Then the 0-*history* of u, $B^0(u)$, is simply (u); and for $j \geq 1$ the j-*history* of u, $B^j(u)$, is a sequence of nodes $(x_0, x_1, \ldots, x_j)$ of $L^{i-j}(D)$ such that $(x_0x_1, x_1x_2, \ldots, x_{j-1}x_j)$ is the $j-1$-history of u.

Clearly, the sequence $x_0, x_1, \ldots, x_j$ determines a trail in $L^{i-j}(D)$, and there is one-to-one correspondence between the j-histories (i.e., the trails of length j in $L^{i-j}(D)$) and the nodes in $L^i(D)$. The j-history $B^j(u)$ will be abbreviated to a history $B(u)$ if $i = j$. Note that the history allows us to represent a node of $L^i(D)$ in D. The following lemma enables us to count distances in $L^i(D)$ using the distances in D.

Lemma 2.1. *Let $x_0, x_1, \ldots, x_n$ be the shortest trail in D (if such exists) such that $(x_0, x_1, \ldots, x_i) = B(u)$ and $(x_{n-i}, x_{n-i+1}, \ldots, x_n) = B(v)$. Then $d_{L^i(D)}(u, v) = n - i$. Moreover, $d_{L^i(D)}(u, v) = \infty$ if there is no required trail in D.*

Proof. Let a and b be two adjacent nodes in $L^i(D)$, $B(a) = (a_0, \ldots, a_i)$ and $B(b) = (b_0, \ldots, b_i)$. Moreover, let $B^1(a) = (x, y)$ and $B^1(b) = (w, z)$. Then a and b are adjacent in $L^i(D)$ if and only if $y = w$. Thus $a_j = b_{j-1}$, $1 \leq j \leq i$, and $a_0, a_1, \ldots, a_i, b_i$ determines a trail in D. This implies $d_{L^i(D)}(u, v) = n-i$ as $x_0, x_1, \ldots, x_n$ is the shortest trail in D satisfying the assumptions of the lemma.

Clearly, if there is no required trail in $L^i(D)$, then $d_{L^i(D)}(u, v) = \infty$. ■

We remark that in Lemma 2.1 the subtrail $x_i, x_{i+1}, \ldots, x_{n-i}$ is a path and $d_D(x_i, x_{n-i}) = n - 2i$.

3. Radii in Iterated Line Digraphs

First we introduce results concerning the out-radius of iterated line digraphs.

Lemma 3.1. *Let D be a digraph with out-radius $t < \infty$ and let u be a node in the out-center of D such that $id_D(u) = 0$. If $L^i(D)$ is not empty, then either $t-i \leq r^+(L^i(D)) \leq t$ or $r^+(L^i(D)) = \infty$.*

Proof. Suppose that $r^+(L^i(D)) < \infty$. Since $L^i(D)$ is not empty, there is a node x in $L^i(D)$ such that $B(x) = (u, x_1, \ldots, x_i)$. Now $id_D(u) = 0$ implies that x is the unique node in the out-center of $L^i(D)$.

Let z be a node in $L^i(D)$, $B(z) = (z_0, \ldots, z_i)$, and let $u, y_1, \ldots, y_n$ be the shortest trail in D such that $y_{n-i+j} = z_j$, $0 \le j \le i$. Then $y_j = x_j$, $1 \le j \le i$, as $r^+(L^i(D)) = e^+_{L^i(D)}(x) < \infty$. By Lemma 2.1, $d_{L^i(D)}(x, z) = d_{L^i(D)}(u, z_0)$, and hence $r^+(L^i(D)) \le t$.

Let v be a node in D with $d_D(u, v) = t$. Suppose that $t \ge i$. Then there is a node z in $L^i(D)$ with $B(z) = (z_0, \ldots, z_{i-1}, v)$. Now $d_{L^i(D)}(x, z) = d_D(u, z_0) \ge t-i$, and hence $r^+(L^i(D)) \ge t-i$. ■

By Lemma 3.1 if a digraph D with out-radius $t < \infty$ contains no cycle and $L^i(D)$ is not empty, then either $t-i \le r^+(L^i(D)) \le t$ or $r^+(L^i(D)) = \infty$.

Lemma 3.2. *Let D be a digraph with out-radius $t < \infty$ and let u be a node in the out-center of D such that $id_D(u) \ge 1$. Then $t \le r^+(L^i(D)) \le t+i$.*

Proof. As $id_D(u) \ge 1$, there is a node v in D with $vu \in E(D)$. Since $r^+(D) < \infty$, we have $d_D(u, v) < \infty$, and thus u lies in a cycle in D. Hence there is a node x in $L^i(D)$ with $B(x) = (x_0, \ldots, x_{i-1}, u)$. Since $e^+_D(u) = t$, we have $r^+(L^i(D)) \le e^+_{L^i(D)}(x) \le e^+_D(u) + i = t + i$, by Lemma 2.1.

Let z be a node in the out-center of $L^i(D)$, $B(z) = (z_0, \ldots, z_i)$, and let w be a node in D such that $d_D(z_i, w) = e^+_D(z_i)$. Clearly, there is a node, say y, in $L^i(D)$ such that $B(y) = (y_0, \ldots, y_{i-1}, w)$ for some $y_0, \ldots, y_{i-1}$. Now $r^+(L^i(D)) \ge d_{L^i(D)}(z, y) \ge d_D(z_i, w) = t$. ■

Theorem 3.3. *Let D be a digraph no two of whose cycles are joined by a path, and assume that D contains at least one cycle. Then there are numbers k and t such that for every $i \ge k$ either $r^+(L^i(D)) = t$ or $r^+(L^i(D)) = \infty$.*

Proof. Since D contains cycles, no two of which are joined by a path, there are numbers j_D and i_D such that $L^i(D)$ is isomorphic to $L^{i+j_D}(D)$ for every $i \ge i_D$ (see e.g. [2, Theorem 10.9.1]). Suppose that there is $k \ge i_D$ such that $r^+(L^k(D)) = t < \infty$. Distinguish two cases:

(i) There is a node u in $L^k(D)$ with $id_{L^k(D)}(u) = 0$. Then u is in the out-center of $L^k(D)$. By Lemma 3.1 either $r^+(L^{k+i}(D)) = \infty$ or $t = r^+(L^{k+j_D}(D)) \le r^+(L^{k+i}(D)) \le r^+(L^k(D)) = t$ for all i, $0 \le i \le j_D$.

(ii) $id_{L^k(D)}(u) \ge 1$ for every node u in $L^k(D)$. By Lemma 3.2 we have $t = r^+(L^k(D)) \le r^+(L^{k+i}(D)) \le r^+(L^{k+j_D}(D)) = t$ for all i, $0 \le i \le j_D$. Thus, $r^+(L^i(D)) = t$ or $r^+(L^i(D)) = \infty$ for every $i \ge k$, as required. ■

Now we prove the main result concerning the out-radius.

Theorem 3.4. *Let D be a digraph containing two cycles joined by a path (possibly of length 0). Then either there are k and t such that $r^+(L^i(D)) = i+t$ for every $i \geq k$ or there is k such that $r^+(L^i(D)) = \infty$ for every $i \geq k$.*

Proof. Let $C_1 = (a_1, a_2, \ldots, a_{l_1}, a_1)$ and $C_2 = (b_1, b_2, \ldots, b_{l_2}, b_1)$ be two cycles in D joined by a path. Suppose that there is $j \geq 0$ such that $r^+(L^j(D)) < \infty$. Distinguish two cases:

(i) There is a node u in $L^j(D)$ with $id_{L^j(D)}(u) = 0$. Then $B(u) = (u_0, u_1, \ldots, u_j)$ and $id_D(u_0) = 0$. Since $r^+(L^j(D)) = e^+_{L^j(D)}(u) < \infty$, there are paths P_1 and P_2 from u_j to C_1 and C_2, respectively. For $k = 1, 2$ denote by B_k the trail starting at u_0, traversing $B(u)$, P_k, and then continuing once around the cycle C_k. Clearly, both B_1 and B_2 can be completed to i-histories, say $B^i(x^i)$ and $B^i(y^i)$, for every sufficiently large i. As $id_{L^i(D)}(x^i) = id_{L^i(D)}(y^i) = 0$ and $x^i \neq y^i$, we have $r^+(L^i(D)) = \infty$.

(ii) $id_{L^j(D)}(u) \geq 1$ for every node u in $L^j(D)$. Then $r^+(L^j(D)) \leq r^+(L^i(D)) \leq r^+(L^j(D)) + i - j$ for every $i \geq j$, by Lemma 3.2. Let x and y be nodes in $L^i(D)$, $i \geq j$, such that $B(x)=(a_1, a_2, \ldots, a_{l_1}, a_1, a_2, \ldots)$ and $B(y)=(b_1, b_2, \ldots, b_{l_2}, b_1, b_2, \ldots)$. The cycles C_1 and C_2 may have some common paths. Let l be the maximum length of a path common to C_1 and C_2. Let z be a node in the out-center of $L^i(D)$ and $B(z) = (z_0, \ldots, z_i)$. Suppose that $d_{L^i(D)}(z, x) < i - l$. By Lemma 2.1 $z_{i-l-1}, z_{i-l}, \ldots, z_i$ lies on C_1, and hence $d_{L^i(D)}(z, y) \geq i - l$. Thus, $i - l \leq r^+(L^i(D)) \leq r^+(L^j(D)) + i - j$ for every $i \geq j$.

By Lemma 3.2 $r^+(L^{i+1}(D)) \leq r^+(L^i(D)) + 1$ for every $i \geq j$, so that $r^+(L^{i+1}(D)) = r^+(L^i(D)) + 1$ with finitely many (at most $r^+(L^j(D)) - j + l$) exceptions. Hence, there are numbers k and t such that $r^+(L^i(D)) = i + t$ for every $i \geq k$, as required. ∎

From now on we consider the radius (as opposed to the out-radius) of iterated line digraphs. Since $r(D) < \infty$ if and only if D is strongly connected, we consider only nontrivial strongly connected digraphs.

Lemma 3.5. *For each nontrivial strongly connected digraph D we have $r(L(D)) \geq r(D)$.*

Proof. Let x be a node in $L(D)$ with $B(x) = (u, v)$. We show that $e_{L(D)}(x) \geq e_D(u)$.

Let z be a node in D for which $e^-_D(u) = d_D(z,u)$, and let y be a node in $L(D)$ such that $B(y) = (z,w)$ for some w. By Lemma 2.1 $d_{L(D)}(y,x) \geq d_D(z,u)$, and hence $e^-_{L(D)}(x) \geq e^-_D(u)$.

Similarly, if z' satisfies $e^+_D(u) = d_D(u,z')$, choose y' from $V(L(D))$ such that $B(y') = (z',w')$. Then $e^+_{L(D)}(x) \geq d_{L(D)}(x,y') \geq d_D(u,z') = e^+_D(u)$.

Thus, $e_{L(D)}(x) \geq e_D(u)$ for each node x in $L(D)$, $B(x) = (u,v)$, and hence $r(L(D)) \geq r(D)$. ■

Theorem 3.6. *Let D be a nontrivial strongly connected digraph different from a cycle. Then there are t and t', such that $i+t \leq r(L^i(D)) \leq i+t'$ for every $i \geq 0$.*

Proof. Clearly, D contains two cycles joined by a path (possibly of length 0). Moreover, the digraph $L^i(D)$ is not empty and $r(L^i(D)) < \infty$ for every $i \geq 0$. By Theorem 3.4. there are k' and t' such that $r^+(L^i(D)) = i+t'$ for every $i \geq k'$, and hence there is $t \leq t'$ such that $i + t \leq r(L^i(D))$ for every $i \geq 0$.

Let u be a central node in D, and let C be the shortest cycle in D containing u, $C = (u, a_2, \ldots, a_l, u)$. Then there is a node x in $L^l(D)$ such that $B(x) = (u, a_2, \ldots, a_l, u)$. We have $r(L^l(D)) \leq e_{L^l(D)}(x) \leq e_D(u) + l = r(D) + l$, by Lemma 2.1. Analogously, if $j > 1$, then $r(L^{jl}(D)) \leq jl + r(D)$ (take a trail going j times around C for $B(x)$). By Lemma 3.5 we have $r(L^i(D)) \leq r(L^{(j+1)l}(D)) \leq (j+1)l + r(D)$ for all i, $jl < i \leq (j+1)l$, and hence $r(L^i(D)) \leq i+r(D)+l$ for every $i \geq 0$. ■

4. Centers in Iterated Line Digraphs

First we introduce results concerning the out-centers. In what follows, by $H \supseteq D$ we mean that D is a subgraph of H, and if D and H are isomorphic, we write $D \cong H$.

Theorem 4.1. *Let D be a nontrivial strongly connected digraph. Then there is a digraph H, $H \supseteq D$, such that $C^+(L^i(H)) = L^i(D)$ for every $i \geq 0$.*

Proof. Let $d = \max\{2, d(D)\}$, where $d(D)$ denotes the diameter of D. As D is strongly connected, we have $d < \infty$. Let $V(H) = V(D) \cup \{a_1, b_1, \ldots, a_d, b_d\}$ and $E(H) = E(D) \cup \{ua_1, ub_1 : u \in V(D)\} \cup \{a_j a_{j+1}, b_j b_{j+1} : 1 \leq j \leq d-1\} \cup \{a_d a_{d-1}, b_d b_{d-1}\}$ (see Figure 1). Let $i \geq 0$. We show that $C^+(L^i(H)) = L^i(D)$.

Let u be a node in $L^i(D)$. For every node x in D, it holds that $e^+_H(x) = d_H(x, a_d) = d$. Since there is a node v in $L^i(H)$ with $B(v) = (a_d, a_{d-1}, a_d, \ldots)$, we have $e^+_{L^i(H)}(u) = d + i$, by Lemma 2.1.

Let u be a node from $V(L^i(H)) - V(L^i(D))$ with $B(u) = (x_0, \ldots, x_i)$. Then either $d_H(x_i, a_d) = \infty$ or $d_H(x_i, b_d) = \infty$ as x_i is not in D. ■

A digraph D is *antisymmetric* if and only if $uv \notin E(D)$ for every $vu \in E(D)$. We remark that if D is an antisymmetric strongly connected digraph, then there is an antisymmetric digraph H, $H \supseteq D$, such that $C^+(L^i(H)) = L^i(D)$ for every $i \geq 0$. (Just replace the two-cycles at a_{d-1} and b_{d-1} by three-cycles.) In contrast with Theorem 4.1, if D is not strongly connected we cannot guarantee the existence of H for which $C^+(L^i(H)) \cong L^i(D)$, even if $i = 1$.

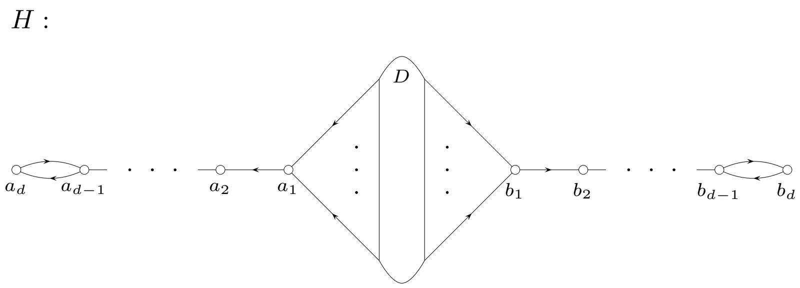

Figure 1

Theorem 4.2. *Let D be a digraph that is not strongly connected with $C^+(D) \neq D$ and $id_D(u) \geq 2$ for every node u in D. Then there is no digraph H for which $C^+(L(H)) \cong D$.*

Proof. Suppose that H satisfies $C^+(L(H)) \cong D$. Let $F = C^+(L(H))$, and let $t = r^+(L(H))$. Clearly, $e^+_{L(H)}(u) = t < \infty$ for every node u in F. As D is not strongly connected, there are nodes x and y in F such that $d_F(x, y) = \infty$ while $d_{L(H)}(x, y) \leq t$. Denote by P a path from x to y in $L(H)$. Let w be the last node on P that is not in F and let v be a successor of w in P. Since $id_F(v) \geq 2$, there are nodes v_1 and v_2 in F such that $v_1v, v_2v \in E(F)$. Then $e^+_{L(H)}(v_i) = t$, $i = 1, 2$, and $e^+_{L(H)}(w) > t$.

It is well-known that if x_1y_1, x_1y_2, and x_2y_1 are arcs in a line digraph, then so is x_2y_2 (see e.g. [2, Theorem 10.8.4]). Thus, wu is an arc in $L(H)$ if and only if so are v_iu, $i = 1, 2$. Now if u is from $V(L(H)) - \{v_1, w\}$ we have $d_{L(H)}(w, u) = d_{L(H)}(v_1, u)$, while $d_{L(H)}(w, v_1) = d_{L(H)}(v_2, v_1) \leq t$. Hence $e^+_{L(H)}(w) \leq t$, a contradiction. ∎

Let D_1 be a digraph on $n \geq 5$ nodes, say $u_1, u_2, \ldots, u_n$, and let $E(D_1) = \{u_iu_{i+1},\ u_iu_{i+2} : 1 \leq i \leq n\}$ (the addition is modulo n). Let D_1' be isomorphic to and distinct from D_1, and let D consist of D_1, D_1', and all arcs joining the nodes of D_1 to nodes in D_1'. Then $L(D)$ satisfies the assumptions of Theorem 4.2, and hence there is an infinite number of antisymmetric digraphs D such that $L(D) \not\cong C^+(L(H))$ for every digraph H.

From now on we consider the centers (instead of the out-centers) in iterated line digraphs. Here, the situation is different since each i- iterated line digraph is admissible as a center of i-iterated line digraph, $i \geq 0$.

Theorem 4.3. *Let D be a digraph and let $j \geq 0$. If $L^j(D)$ is not empty then there is a digraph H, $H \supseteq D$, such that $C(L^i(H)) = L^i(D)$ for every $i \leq j$.*

Proof. Let $k = \left\lceil \frac{j+1}{2} \right\rceil$. Add to D $2(k+1)$ new nodes $a_1, a_2, \ldots, a_{k+1}, b_1, b_2, \ldots, b_{k+1}$, and $4k$ arcs $a_ia_{i+1}, a_{i+1}a_i, b_ib_{i+1}, b_{i+1}b_i$, $1 \leq i \leq k$. Moreover, subdivide each of the added arcs not incident with a_{k+1} or b_{k+1} (see Figure 2 for the case $k = 2$). Finally, join every node u of D to a_1 and b_1 by paths of length k, and join a_1 and b_1 to every node u in D by another pair of paths of length k. Do this so that all $4|V(D)|$ paths are pairwise internally node disjoint (that is, we have appended $4(k-1)|V(D)|$ new vertices).

Denote by H the resulting digraph. Then $e^+_H(x) = e^-_H(x) = 3k-1$ for every node x of D (although D can be disconnected), and both $e^+_H(x)$ and $e^-_H(x)$ are greater than $3k-1$ for every node x from $V(H) - V(D)$. Let $i \leq j$. We show $C(L^i(H)) = L^i(D)$.

Let u be a node in $L^i(D)$. As $e_H(x) = 3k-1$ for every node x of D, we have $e_{L^i(H)}(u) \leq 3k-1+i$, by Lemma 2.1. Let v be a node in $L^i(H)$, for which $B(v) = (\ldots, a_{k+1}, a_k, a_{k+1})$. Then $d_{L^i(H)}(v, u) = 3k-1+i$, and hence $e_{L^i(H)}(u) = 3k-1+i$.

Let u be a node from $V(L^i(H)) - V(L^i(D))$, $B(u) = (x_0, x_1, \ldots, x_i)$. Since $i < 2k$ either x_0 or x_i is not in D. Suppose that $x_0 \notin V(D)$. Let v and z be nodes in $L^i(H)$ such that $B(v) = (\ldots, a_k, a_{k+1})$ and $B(z) = (\ldots, b_k, b_{k+1})$. Then, by Lemma 2.1, $d_{L^i(H)}(v, u) \geq 3k+i$ or $d_{L^i(H)}(z, u) \geq 3k+i$, and hence $e^-_{L^i(H)}(u) \geq 3k+i$. Similarly, $e^+_{L^i(H)}(u) \geq 3k+i$ if $x_i \notin V(D)$, and hence $e_{L^i(H)}(u) \geq 3k+i$. ∎

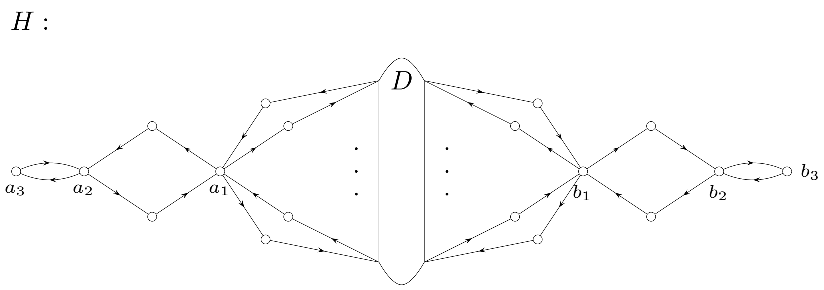

Figure 2

We remark that if D is an antisymmetric digraph and $L^j(D)$ is not empty, then there is an antisymmetric digraph H, $H \supseteq D$, such that $C(L^i(H)) = L^i(D)$ for every $i \leq j$. (Just choose $k = \max\{2, \lceil \frac{j+1}{2} \rceil\}$ and replace the two-cycles at a_{k+1} and b_{k+1} by three-cycles.) Theorem 4.3 is best possible in a sense, as shown by Corollary 4.5 and Corollary 4.6 for digraphs and antisymmetric digraphs, respectively.

Theorem 4.4. *Let D be a digraph with a nonempty arc set, each arc of which is contained in a cycle of length at most l. Then there is no H, $H \supset D$, such that $r(L(H)) \geq l$ and $C(L^i(D)) = L^i(D)$ for every $i \geq 0$.*

Proof. Suppose that there is $H \supset D$ satisfying the assumptions of theorem. Since $H \supset D$, we have $E(H) \supset E(D)$ and $r(L(H)) = t < \infty$.

Write $H' = L(H)$ and $D' = L(D)$. Then $V(H') - V(D') \neq \emptyset$ and H' is strongly connected. Thus, there is a cycle, say C_1, in H' passing through a node, say x, of D' such that $C_1 \not\subseteq D'$. Let C_2 be a cycle of length $l_2 \leq l$ in D' passing through the node x (i.e; the arc x of D). Denote by l_1 the length of C_1. There are $a \in V(L^{l_1 l_2}(H')) - V(L^{l_1 l_2}(D'))$ and $b \in V(L^{l_1 l_2}(D'))$ such that $V(B(a)) = V(C_1)$, $V(B(b)) = V(C_2)$, and x is the initial and terminal node of both $B(a)$ and $B(b)$. (The $B(a)$ circles l_2 times around C_1 and $B(b)$ circles l_1 times around C_2.) In what follows we show that $e_{L^{l_1 l_2}(H')}(a) \leq e_{L^{l_1 l_2}(H')}(b)$.

We have $e_{H'}(x) = r(H') = t$. Suppose that $e_{H'}(x) = e^+_{H'}(x)$ (the case $e_{H'}(x) = e^-_{H'}(x)$ can be proved similarly). Then there is a node y in H' such that $d_{H'}(x, y) = t$. Since H' is strongly connected, there is a node u in $L^{l_1 l_2}(H')$ with $B(u) = (y, \ldots)$. As $t \geq l \geq l_2$, y is not in C_2, and hence $e_{L^{l_1 l_2}(H')}(b) \geq d_{L^{l_1 l_2}(H')}(b, u) = t + l_1 l_2$, by Lemma 2.1.

Now let v be a node in $L^{l_1 l_2}(H')$ with $B(v) = (v_0, \ldots, v_{l_1 l_2})$. Then $d_{L^{l_1 l_2}(H')}(a, v) \leq d_{H'}(x, v_0) + l_1 l_2 \leq t + l_1 l_2$ and $d_{L^{l_1 l_2}(H')}(v, a) \leq d_{H'}(v_{l_1 l_2}, x) + l_1 l_2 \leq t + l_1 l_2$, and hence $e_{L^{l_1 l_2}(H')}(a) \leq r + l_1 l_2 \leq e_{L^{l_1 l_2}(H')}(b)$, a contradiction. ∎

Corollary 4.5. *Let D be a digraph, each arc of which lies in a two-cycle and assume that $C(D) \neq D$. Then there is no H, $H \supseteq D$, such that $C(L^i(H)) = L^i(D)$ for every $i \geq 0$.*

Proof. Suppose that H satisfies $C(L^i(H)) = L^i(D)$ for every $i \geq 0$. Let $e = u_1 u_2$ be an arc in D. Then also $f = u_2 u_1 \in E(D)$, and since $C(D) \neq D$ there is another arc, say $g = v_1 v_2$, in D. Now $v_1 \neq u_2$ or $v_1 \neq u_1$, and hence $d_{L(H)}(e, g) \geq 2$ or $d_{L(H)}(f, g) \geq 2$, respectively. Thus $r(L(H)) \geq 2$, contradicting Theorem 4.4. ∎

Corollary 4.6. *Let D be an antisymmetric digraph, each arc of which lies in a triangle and assume that $C(D) \neq D$. Then there is no antisymmetric digraph H, $H \supseteq D$, such that $C(L^i(H)) = L^i(D)$ for every $i \geq 0$.*

Proof. Suppose that H satisfies $C(L^i(H)) = L^i(D)$ for every $i \geq 0$. Let $e = uv$ be an arc in D and let T be a triangle in D containing e. Since $C(T) = T$, there is an arc f in D such that $f \notin E(T)$. As D is strongly connected, we may assume that $f = uz$, $z \neq v$. Since H is antisymmetric, $d_H(v, u) \geq 2$, and hence $d_{L(H)}(e, f) \geq 3$. Thus $r(L(H)) \geq 3$, which contradicts Theorem 4.4. ∎

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Received 4 October 1994
Revised 18 April 1995