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KP-DIGRAPHS AND *CKI*-DIGRAPHS SATISFYING THE *k*-MEYNIEL'S CONDITION

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Abstract

A digraph D is said to satisfy the k-Meyniel's condition if each odd directed cycle of D has at least k diagonals.

The study of the k-Meyniel's condition has been a source of many interesting problems, questions and results in the development of Kernel Theory.

In this paper we present a method to construct a large variety of kernel-perfect (resp. critical kernel-imperfect) digraphs which satisfy the k-Meyniel's condition.

Primary keywords: digraph, kernel, independent set of vertices, absorbing set of vertices, kernel-perfect digraph, critical-kernel-imperfect digraph, τ-system, τ_1-system.

Secondary keywords: indepedent kernel modulo Q, co-rooted tree, τ-construction, τ_1-construction.

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1. Introduction

For general concepts we refer the reader to [1]. If D is a digraph, then $V(D)$ and FD or $F(D)$ will denote the sets of vertices and arcs of D respectively. We write $D_0 \subseteq D$ (resp: $D_0 \subseteq^* D$) whenever D_0 is a subdigraph (resp: induced subdigraph) of D. For $S_1, S_2 \subset V(D)$, the arc $u_1 u_2$ of D will be called an $S_1 S_2$-*arc* provided that $u_1 \in S_1$ and $u_2 \in S_2$; $D[S_1]$ will denote the subdigraph of D induced by S_1 and $D[S_1, S_2]$ the subdigraph of D with vertex set $S_1 \cup S_2$ whose arcs are the $S_1 S_2$-arcs of D. The *asymmetrical part* of D (resp: *symmetrical part* of D), which is denoted by $Asym\ D$ (resp:

Sym D) is the spanning subdigraph of D whose arcs are the asymmetrical (resp: symmetrical) arcs of D.

The set $I \subset V(D)$ is *independent* if $FD[I] = \emptyset$. A *kernel* N of D is an independent set of vertices such that for every $z \in (V(D) - N)$ there exists a $z\,N$-arc in D. A *semikernel* S of D is an independent set of vertices such that for every $z \in (V(D) - S)$ for which there exists an $S\,z$-arc, there also exists a $z\,S$-arc.

A digraph D is called

(i) *quasi KP-digraph* if every proper induced subdigraph of D has a kernel,

(ii) *kernel-perfect digraph* or *KP-digraph* if every induced subdigraph of D has a kernel,

(iii) *critical kernel-imperfect* or *CKI-digraph* if D is a quasi KP-digraph and has no kernel.

It was proved by Neumann-Lara in [9] that D is a KP-digraph iff every induced subdigraph of D has a non empty semikernel. We will say that a digraph A is a *co-rooted tree* if A is an asymmetrical digraph whose underlying graph is a tree and there exists one and only one vertex $v \in F(A)$ (the *co-root* of A) such that there is no arc in A whose initial endvertex is v.

Let $C = (1, 2, \ldots, m, 1)$ be a directed cycle of D, we denote by $\ell(C)$ its length, for $i \neq j \quad i, j \in V(C)$ we denote by (i, C, j) the ij-directed path contained in C and we denote by $\ell(i, C, j)$ its length; an arc $f = ij \in (FD - FC)$ is a *diagonal* of C iff $i \neq j,\ i, j \in V(C)$ and $\ell(i, C, j) < \ell(C) - 1$ and f is a *pseudodiagonal* when $\ell(i, C, j) \leq \ell(C) - 1$.

A digraph D is said to satisfy the k-Meyniel's condition if each odd directed cycle of D has at least k diagonals.

The study of the k-Meyniel's condition has been a source of many interesting problems, questions and results in the development of Kernel Theory (see by example [2], [3], [4], [5], [6]).

In this paper we present a method to construct a large variety of kernel-perfect (resp. critical kernel-imperfect) digraphs which satisfy the k-Meyniel's condition. This method is also the basis in the study of extensions of kernel-perfect digraphs to critical kernel-imperfect digraphs (see [8]).

Theorem 1.1 [8]. *Let D_1, D_2 and D be digraphs such that $V(D_1) \cap V(D_2) = \{v\}$ and $D = D_1 \cup D_2$. Then D is a KP-digraph iff D_1 and D_2 are KP-digraphs.*

Theorem 1.2. *Let G be a connected graph without cycles and for each $e = w_1 w_2 \in E(G)$ let γ_e be a digraph such that $\{w_1, w_2\} \subseteq V(\gamma_e)$, $V(\gamma_e) \cap V(G) = \{w_1, w_2\}$. Suppose that the digraphs $(\gamma_e - V(G))_{e \in E(G)}$ are mutually disjoint. The digraph $D = \bigcup_{e \in E(G)} \gamma_e$ is a KP-digraph iff γ_e is a KP-digraph for each $e \in E(G)$.*

Proof. Theorem 1.2 follows directly from Theorem 1.1 proceeding by induction on $|V(G)|$. ■

Theorem 1.3 [6]. *Supppose that $V(D)$ has a partition $\{V_1, V_2\}$ such that every $V_1 V_2$-arc in D is symmetric and $D[V_1]$ and $D[V_2]$ are KP-digraphs. Then D is a KP-digraph.*

Theorem 1.4 [6]. *If D is a CKI-digraph, there is no a partition $\{V_1, V_2\}$ of $V(D)$ such that $D[V_1, V_2] \subseteq Sym\ D$; in other words $Asym\ D$ is strongly connected.*

2. τ_1-System and τ_1-Construction

Definition 2.1. Let D be a multidigraph and $u \in V(D)$; a partition $\pi_u = \{u_-^0, u_-^1, \ldots, u_-^{m(u)-1}, u_+\}$ of $F_u(D) = F_u^+(D) \cup F_u^-(D)$ will be called a *τ-partition* in u if it satisfies the following two properties:

(1) $u_-^i \subseteq F_u^-(D)$ for each $i \in \{0, 1, \ldots, m(u) - 1\}$.

(2) $u_+ = F_u^+(D)$.

$F_u^+(D)$ (resp: $F_u^-(D)$) denotes the set of arcs of D whose initial (resp: terminal) endvertex is u.

When π_u is a τ-partition in u we denote by $\overline{\pi}_u$ the set

$$\overline{\pi}_u = \left\{ \ (u, u_+), (u, u_-^i) \mid i \in \{ \ 0, 1, \ldots, m(u) - 1 \ \} \ \right\}.$$

Definition 2.2. A triple $t_0 = (D_0, U, \mathrm{A})$ will be called a *τ_0-system* if it satisfies the following two properties:

(1) D_0 is a multidigraph, $U \subseteq V(D_0)$.

(2) $\mathrm{A} = (A_u)_{u \in U}$ is a family of co-rooted trees with $V(A_u) = \overline{\pi}_u$ where π_u is a τ-partition in u, (u, u_+) is the co-root of A_u and $|\pi_u| \geq 2$.

For each $u \in U$ and $f \in F_u(D)$ we denote by $\pi_u(f)$ the element of π_u containing f.

If $t_0 = (D_0, U, \mathrm{A})$ is a τ_0-system, then $\tau_0(t_0)$ denotes the digraph defined as follows:

$$\begin{aligned} V(\tau_0(t_0)) &= V(D_0 - U) \cup \bigcup_{u \in U} V(A_u)\,, \\ F(\tau_0(t_0)) &= \{\, f^* \mid f \in F\,D_0 \,\} \end{aligned}$$

for each $f = w\,z \in FD_0$, f^* is defined by

$$f^* = \begin{cases} f, & \text{when } \{\, w, z \,\} \subseteq (V(D_0) - U), \\ w(z, \pi_z(f)), & \text{when } w \in (V(D_0) - U) \text{ and } z \in U, \\ (w, w_+)z, & \text{when } w \in U \text{ and } z \in (V(D_0) - U), \\ (w, w_+)(z, \pi_z(f)), & \text{when } \{\, w, z \,\} \subseteq U\,. \end{cases}$$

Definition 2.3. A pair $t_1 = (t_0, \gamma)$ will be called a τ_1-*system* if $t_0 = (D_0, U, \mathrm{A})$ is a τ_0- system and $\gamma = (\gamma_u)_{u \in U}$ is a family, where $\gamma_u = (\gamma_u^f)_{f \in F(A_u)}$ is a family of internally disjoint directed paths. Moreover, if $f = w_1\,w_2$, then γ_u^f is a $w_1\,w_2$-directed path of positive even length and $V(\gamma_u^f) \cap V(A_u) = \{\, w_1, w_2 \,\}$. Also we denote $t_1 = (D_0, U, \mathrm{A}, \gamma)$.

Note that $V(\gamma_{u_1}^{f_1}) \cap V(\gamma_{u_2}^{f_2}) = \emptyset$ for any $f_1 \in F\,A_{u_1}$, $f_2 \in F\,A_{u_2}$ and $u_1 \neq u_2$.

If $t_1 = (t_0, \gamma)$ is a τ_1-system, then we denote $\tau_1(t_1) = \tau_0(t_0) \cup \bigcup_{u \in U} \bigcup_{f \in F\,A_u} \gamma_u^f$.

Definition 2.4 [5]. If D is a digraph and $N, Q \subset V(D)$, $N^c = V(D) - N$, $Q^c = V(D) - Q$, N is said to be an independent kernel *modulo* Q (i.k. mod Q) of D iff

(i) N is independent,

(ii) For every $w \in N^c \cap Q^c$ there exists a $w\,N$-arc.

Observation 2.1. If D is a directed path of positive even length say $D = (u_0, u_1, \ldots, u_{2n}), n \geq 1$, then D satisfies the following properties:

(i) If N is an i.k. mod $\{u_{2n}\}$ of D, then $u_0 \in N$ iff $u_{2n} \in N$.

(ii) $\{u_i \mid i = 2k,\ 0 \leq k \leq n\,\}$ is an i.k. mod $\{u_{2n}\}$, in fact it is a kernel of D which contains $\{\, u_0, u_{2n} \,\}$.

(iii) $N = \{u_{2i+1} \mid 0 \leq i \leq n-1\}$ is an i.k. mod $\{u_{2n}\}$ of D such that $\{u_0, u_{2m}\} \subseteq N^c$.

Definition 2.5. Let D be a multidigraph, $R, T \subseteq V(D)$; T will be called *R-homogeneous* whenever $T \subseteq R$ or $T \subseteq (V(D) - R)$.

Lemma 2.1. *Let A be a co-rooted tree with co-root a_0, $|V(A)| \geq 2$ and $(\gamma^f)_{f=(u_f,v_f)\in F A}$ a family of internally disjoint directed paths of positive even length such that γ^f is a $u_f\, v_f$-directed path and $V(\gamma^f) \cap V(A) = \{u_f, v_f\}$. If N is an i.k.* mod $\{a_0\}$ *of $\bigcup_{f\in FA} \gamma^f$, then $V(A)$ is N-homogeneous. Moreover, when $V(A) \subseteq N^c$ there is no $a_0\, N$-arc in $\bigcup_{f\in FA} \gamma^f$.*

Proof. The proof is by induction on $|V(A)|$. If $|V(A)| = 2$ the result is a directed consequence of Observation 2.1 (i). Suppose that $|V(A)| > 2$ and let $g = u\,w \in F(A)$ be an arc such that $\delta^-_A(u) = 0$, N be an i.k. mod $\{a_0\}$ of $\bigcup_{f\in FA} \gamma^f$ and $A_0 = A - \{u\}$. Clearly we have:

(1) $N \cap \bigcup_{f\in FA_0} V(\gamma^f)$ is an i.k. mod $\{a_0\}$ of $\bigcup_{f\in FA_0} \gamma^f$ (because $\delta^-_A(u) = 0$) thus by the inductive hypothesis $V(A_0)$ is N-homogeneous.

(2) $N \cap V(\gamma^g)$ is an i.k. mod $\{w\}$ of γ^g and Observation 2.1 (i) implies $\{u, w\}$ is N-homogeneous.

It follows from (1) and (2) that $V(A)$ is N-homogeneous. When $V(A) \subseteq N^c$ it follows from the choice of a_0 that there is no $a_0 \bigcup_{f\in FA} V(\gamma^f)$-arc, so there is no $a_0\, N$-arc in $\bigcup_{f\in FA} \gamma^f$). ■

Theorem 2.1. *Let $t_1 = (D_0, U, \mathrm{A}, \gamma)$ be a τ_1-system. If D_0 has a kernel, then $D = \tau_1(t_1)$ has a kernel.*

Proof. Let N_0 be a kernel of D_0, Observation 2.1 implies that for each $u \in U$ and $f = w_1\, w_2 \in F(A_u)$ there exist $N^i_{u,f}$, $i \in \{0, 1\}$ independent kernels mod $\{w_2\}$ of γ^f_u such that $\{ w_1, w_2 \} \subseteq N^0_{u,f}$ and $\{ w_1, w_2 \} \subseteq \left(N^1_{u,f} \right)^c$. It is easy to see by using Lemma 2.1 that

$$\begin{aligned} N &= [N_0 \cap (V(D_0) - U)] \cup \left(\bigcup_{u\in N_0 \cap U} \bigcup_{f\in FA_u} N^0_{u,f} \right) \\ &\quad \cup \left(\bigcup_{u\in N_0 \cap U^c} \bigcup_{f\in F(A_u)} N^1_{u,f} \right) \end{aligned}$$

is a kernel of $\tau_1(t_1)$. ■

Theorem 2.2. *Let* $t_1 = (D_0, U, \mathrm{A}, \gamma)$ *be a* τ_1*-system. If* $D = \tau_1(t_1)$ *has a kernel, then* D_0 *has a kernel.*

Proof. Let N be a kernel of D; it is easy to see that for each $u \in U$, $N \cap (\bigcup_{f \in FA_u} V(\gamma_u^f))$ is an i.k. mod $\{ (u, u_+) \}$ of $\bigcup_{f \in FA_u} \gamma_u^f$ and Lemma 2.1 implies $V(A_u)$ is $N \cap (\bigcup_{f \in FA_u} V(\gamma_u^f))$-homogeneous and hence $V(A_u)$ is N-homogeneous and when $V(A_u) \subseteq N^c$ there is no

$$(u, u_+) \left[N \cap \left(\bigcup_{f \in FA_u} V(\gamma_u^f) \right) \right]\text{-arc}$$

in D and it follows that

$$N_0 = \left[N \cap (V(D) - \bigcup_{u \in U} \bigcup_{f \in FA_u} V(\gamma_u^f)) \right] \cup \{ u \in U \mid V(A_u) \subseteq N \}$$

is a kernel of D_0. ■

Definition 2.6. Let A be a co-rooted tree, a subset S of $V(A)$ will be called an *initial section* of A if for each $w \in A$ such that there exists a wS-directed path in A, we have $w \in S$.

Clearly the empty set is an initial section of any co-rooted tree.

Theorem 2.3. *Let* $t_1 = (D_0, U, \mathrm{A}, \gamma)$ *be a* τ_1*-system. Suppose that for each non trivial family* $\mathrm{S} = (S_u)_{u \in U}$*, where* S_u *is an initial section of* A_u*, the digraph* $D_0 - \bigcup_{u \in U} \{f \in FD_0 \mid f^* \text{ incides in } S_u \}$ *is a* KP*-digraph (for each* $f \in FD_0, f^*$ *denotes the arc of* $\tau_0(t_0)$ *defined as in Definition* 2.2*). If every proper induced subdigraph of* D_0 *has a kernel, then every proper induced subdigraph of* $D = \tau_1(t_1)$ *has a kernel.*

Proof. First we recall that if G and H are digraphs then $G \cap H$ denote the digraph whose vertex set is $V(G) \cap V(H)$ and $A(G \cap H) = A(G) \cap A(H)$. Now, if Theorem 2.3 were false, D would contain a proper induced CKI-subdigraph. Let H be a proper induced CKI-subdigraph of D. First we will prove that for each $u \in U$,

$$H \cap D \left[\bigcup_{f \in FA_u} V(\gamma_u^f) \right] = D \left[\bigcup_{f \in F(A_u - S_u)} V(\gamma_u^f) \right],$$

where S $= (S_u = V(A_u) - V(H))_{u \in U}$ is a family such that S_u is an initial section of A_u.

Let $u \in U$, when $H \cap D\left[\bigcup_{f \in FA_u} V(\gamma_u^f)\right] = D\left[\bigcup_{f \in FA_u} V(\gamma_u^f)\right]$, then $S_u = \emptyset$ satisfies the required properties. If

$$H \cap D\left[\bigcup_{f \in FA_u} V(\gamma_u^f)\right] \subsetneq D\left[\bigcup_{f \in FA_u} V(\gamma_u^f)\right]$$

since

$$H \cap D\left[\bigcup_{f \in FA_u} V(\gamma_u^f)\right] = \bigcup_{f \in FA_u} H \cap D\left[V(\gamma_u^f)\right]$$

then there exists $f = w_1 w_2 \in FA_u$ such that $H \cap D\left[V(\gamma_u^f)\right] \subsetneq D\left[V(\gamma_u^f)\right] = \gamma_u^f$. Since $Asym\ H$ is strongly connected (see Theorem 1.3) Definition 2.3 implies

$$H \cap D\left[V(\gamma_u^f)\right] \subseteq D\left[\{w_1, w_2\}\right] ;$$

now we consider $A_u^{w_1} = A_u\left[\{z \in V(A_u) |\right.$ there exists a zw_1-directed path contained in $A_u\}]$ and

$$H_{w_1} = H \cap D\left[\bigcup_{f \in FA_u^{w_1}} V(\gamma_u^f)\right].$$

So, we have that $H_{w_1} = \emptyset$ since, if $H_{w_1} \neq \emptyset$ then $H_2 = H\left[V(H) - V(H_{w_1})\right]$ is a KP-digraph such that $H_2 \cap D\left[V(\gamma_u^f)\right] \subseteq D\left[\{w_2\}\right]$ (since $H \cap D\left[V(\gamma_u^f)\right] \subseteq D\left[\{w_1, w_2\}\right]$). Furthermore, since for each $f \in FA_u, \gamma_u^f$ is a KP-digraph and, A_u is a co-rooted tree, Theorem 1.2 implies that $\bigcup_{f \in FA_u} \gamma_u^f$ is a KP-digraph and clearly $H_{w_1} \subseteq^* \left(\bigcup_{f \in FA_u} \gamma_u^f\right)$ so H_{w_1} is a KP-digraph; Definition 2.3 and $H_2 \cap D\left[V(\gamma_u^f)\right] \subseteq \{w_2\}$ imply there is no $H_{w_1}H_2$-arcs in H and using Theorem 1.3 we conclude that H is a KP-digraph which is impossible. So, we have proved that $H_{w_1} = \emptyset$ and then

$$H \cap D\left[\bigcup_{f \in FA_u} V(\gamma_u^f)\right] \subseteq^* D\left[\bigcup_{f \in F(A_u - A_u^{w_1})} V(\gamma_u^f)\right]$$

for each $f \in FA_u$ such that

$$H \cap D\left[\ V(\gamma_u^f)\right] \subsetneq D\left[V(\gamma_u^f)\ \right],$$

and this implies

$$H \cap D\left[\bigcup_{f\in FA_u} V(\gamma_u^f)\right] = D\left[\bigcup_{f\in F(A_u-S_u)} V(\gamma_u^f)\right],$$

where $S_u = S_u^1 \cup S_u^2$, $S_u^1 = \{w \in V(A_u) \mid$ there exists $f = wz \in FA_u$ such that

$$H \cap D\left[V(\gamma_u^f)\right] \subsetneq \gamma_u^f\right\}$$

and $S_u^2 = \bigcup_{w\in S_u^1} V(A_u^w)$. Clearly, S_u is an initial section of A_u.

Let H_0 be a subdigraph (not necessarily induced) of D_0 obtained from H by identifying $\bigcup_{f\in FA_u} V(\gamma_u^f)$ with u, for each $u \in U$ such that $H \cap D\left[\bigcup_{f\in FA_u} V(\gamma_u^f)\right] \neq \emptyset$. So, we have that $H \cong \tau_1(H_0, U_0, (A_u - S_u)_{u\in U_0}, \gamma_0)$ where, $U_0 = U \cap V(H_0)$ and γ_0 is the restriction of γ to $\bigcup_{u\in U} F(A_u - S_u)$.

Now we will prove that H_0 has a kernel.

If $S_u = \emptyset$ for each $u \in U$, then there exists

$$z \in \left(V(D) - \bigcup_{u\in U}\bigcup_{f\in FA_u} V(\gamma_u^f) \cap (V(D) - V(H))\right)$$

and hence H_0 is a proper induced subdigraph of D_0 and the hypothesis implies H_0 has a kernel.

If $S_u \neq \emptyset$ for some $u \in U$, then H_0 is an induced subdigraph of $D_0 - \bigcup_{u\in U'} \{\ f \in FD_0 \mid f^*$ incides in $\ S_u\ \}$, where $U' = \{\ u \in U \mid S_u \neq \emptyset\ \}$ (f^* is defined as in Definition 2.2) and the hypothesis implies that H_0 has a kernel.

Since H_0 has a kernel and $H = \tau_1(H_0, U_0, (A_u - S_u)_{u\in U_0}, \gamma_0)$, it follows from Theorem 3.1 that H has a kernel contradicting that H is a *CKI*-digraph. ∎

Theorem 2.4. *Let $t_1 = (D_0, U, \mathrm{A}, \gamma)$ be a τ_1-system. If every proper induced subdigraph of $D = \tau_1(t_1)$ has a kernel, then every proper induced subdigraph of D_0 has a kernel.*

Proof. Let D'_0 be a proper induced subdigraph of D_0 and

$$D' = \tau_1(D'_0, U', (A_u)_{u \in U'}, \gamma'),$$

where $U' = (U \cap V(D'_0)$ and γ' is the restriction of γ to $\bigcup_{u \in U'} FA_u$; since D' is a proper induced subdigraph of D, we have that D' has a kernel and Theorem 2.2 implies that D'_0 has a kernel. ■

Theorem 2.5. *Let $t_1 = (D_0, U, \mathrm{A}, \gamma)$ be a τ_1-system such that for every non trivial family $\mathrm{S} = (S_u)_{u \in U}$, where S_u is an initial section of A_u, the digraph $D_0 - \bigcup_{u \in U} \{f \in FD_0 | f^* incides\ in S_u\}$ is a KP-digraph. Then $\tau_1(t_1)$ is a KP-digraph (resp: CKI-digraph) if and only if D_0 is a KP-digraph (resp: CKI-digraph).*

3. τ_1-Constructions

In these section we present a method to realize in a simple way some τ_1-constructions and we obtain a large variety of KP digraphs and CKI-digraphs satisfying the k-Meyniel's condition.

Let D_0 be a multidigraph, $U \subseteq V(D_0), <^p$ be a total order in $\{v(f) = \{u_1, u_2\} \mid f$ is an $u_1 u_2$-arc$\}$, and $<^{u_1 u_2}$ be a total order in $\{ f \in FD_0 \mid f$ is an $u_1 u_2$-arc$\}$. We will denote by $<$ the total order defined in

$$\bigcup_{u \in U} \{(v(f), f) \mid f \in F(Sym\ D_0) \cap F_u^-(D_0)\}$$

as follows: $(v(f), f) < (v(g), g)$ if and only if $v(f) <^p v(g)$ or $v(f) = v(g) = \{u_1, u_2\}$ and $f <^{u_1 u_2} g$. And for each $u \in U$ we will denote by $u_-(f) = \{f\}$ when $f \in F(Sym\ D_0) \cap F_u^-(D_0)$; $\overset{0}{u}_- = F(Asym\ D_0) \cap F_u^-(D_0)$, $u_+ = F_u^+(D_0)$,

$$\Pi_u = \left\{u_+, \overset{0}{u}_-, u_-(f) \mid f \in F(Sym\ D_0) \cap F_u^-(D_0)\right\}.$$

(clearly Π_u is a τ-partition in u), $A_u^<$ the $\overset{0}{u}_- u_+$-directed path defined as follows

$$A_u^< = \left(\overset{0}{u}_-, u_-(f_1), u_-(f_2), \ldots, u_-(f_r), u_+\right),$$

where

$$(v(f_1), f_1) < (v(f_2), f_2) < \ldots < (v(f_r), f_r)$$

and

$$\{f_1, \ldots, f_r\} = F\, Sym\ D_0 \cap F_u^- D_0 \,.$$

Finally we denote by $A^< = (A_u^<)_{u \in U}$.

Theorem 3.1. *Let D_0 be a multidigraph which is a quasi KP-digraph and $t_0 = (D_0, U, A^<)$ any τ_0-system defined as at the begining of this section. For any non trivial family $S = (S_u)_{u \in U}$, where S_u is an initial section of $A_u^<$, $(D_0 - \bigcup_{u \in U} \{f \in FD_0 \mid f^* \text{ incides in } S_u\}) = D_0(S)$ is a KP-digraph.*

Proof. Suppose that there exists a non trivial family $S = (S_u)_{u \in U}$, where S_u is an initial section of $A_u^<$, such that $D_0(S)$ is not a KP-digraph and let D_1 be a CKI-digraph which is an induced subdigraph of $D_0(S)$; since $D_0(S)$ is a proper subdigraph of D_0, we have that D_1 is not an induced subdigraph of D_0 and there exists an uv-arc in $FD_0\,[V(D_1)] - F(D_1)$ and S_v is not empty, so $\overset{0}{v_-} \cap FD_1 = \emptyset$. Since D_1 is a CKI-digraph, Theorem 1.4 implies that there exists some wv-arc in $Asym\ D_1$ and $\emptyset = \overset{0}{v_-} \cap FD_1 = F\, Asym\ D_0 \cap F_v^-(D_0) \cap FD_1$ implies $wv \in F(Sym\ D_0)$, so there exists $w \in V(D_1)$ such that $\delta^+_{D_1'}(w) \neq 0$, where $D_1' = Asym\ D_1 \cap\ Sym\ D_0$. Futhermore, if $zv \in F_z^+(D_1')$, then $S_z \neq \emptyset$, $\overset{0}{z_-} \cap F(D_1) = \emptyset$ and since D_1 is a CKI-digraph, Theorem 1.4 implies $F(Asym\ D_1) \cap F_z^-(D_1') \neq \emptyset$ and then there exists $wz \in F(D_1')$; hence $\delta^+_{D_1'}(w) \neq 0$. We have proved:

(a) there exists $w \in V(D_1')$ such that $\delta^+_{D_1'}(w) \neq 0$.

(b) if $\delta^+_{D_1'}(z) \neq 0$, then $\delta^-_{D_1'}(z) \neq 0$.

It follows that D_1' contains a directed cycle $\mathcal{C} = (w_0, f_0, w_1, f_1, \ldots, w_n, f_n, w_0)$ where $\{w_0, \ldots, w_n\} \subseteq V(D_1')$, $\{f_0, \ldots, f_n\} \subseteq FD_1'$. Since $<^p$ is a total order in $\{v(f) \mid f \in Sym\, D_0\}$ and $\mathcal{C} \subseteq D_1'$, it follows that for some $i \in \{0, 1, \ldots, n\}$, $\{w_{i-1}, w_i\} <^p \{w_i, w_{i+1}\}$ (the indices are taken mod $n+1$). It follows from the definition of $t_1 = (D_0, U, A^<)$ that

$$A^<_{w_i}\,[\{w_{i\_}(g) \mid g \text{ is a } w_{i+1}w_i \text{ -arc}\}]$$

is a subpath of the subpath of $A^<_{w_i}$ between the vertices $w_{i\_}(f_{i-1})$ and w_i^+ and since $f_{i-1} \in F(\mathcal{C}) \subseteq FD_1'$ it follows $w_{i\_}(f_{i-1}) \notin S_{w_i}$ and then $\{w_{i\_}(g) \mid g \text{ is a } w_{i+1}w_i\text{-arc}\} \cap S_{w_i} = \emptyset$. Since $f_i \in \mathcal{C} \subseteq D_1'$ and

$$\{w_{i\_}(g) \mid g \text{ is a } w_{i+1}w_i\text{-arc}\} \cap S_{w_i} = \emptyset\,,$$

there exists some $w_{i+1}w_i$-arc in D_0 which is also in $D_0(S)$ and since D_1 is an induced subdigraph of D_0, it follows that $f_i \in F(Sym\ D_1)$ contradicting $f_i \in F(D'_1)$.

A digraph D is said to satisfy the *k-Meyniel's condition* if each odd directed cycle of D has at least k diagonals and we write D satisfies $M(k)$.

Let D_0 be a digraph, we will denote by $D_0^{(k)}$ the multidigraph obtained from D_0 by adding to each symmetrical arc the multiplicity k. ■

Lemma 3.1. *If D_0 is a digraph such that every odd directed cycle has a symmetrical arc and $t_1 = (D_0^{(k)}, V(D_0), \mathrm{A}^{<}, \gamma)$, then $\tau_1(t_1)$ satisfies $M(k)$.*

Proof. Let $\mathcal{C}$ be an odd directed cycle contained in $\tau_1(t_1)$; since $\bigcup_{f \in FA_u^{<}} \gamma_u^f$ is a directed path of even lenght, we have that $\mathcal{C}'$ is the digraph obtained from $\mathcal{C}$ by identifying $\bigcup_{f \in FA_u^{<}} \gamma_u^f$ with u for each $u \in V(D_0)$; $\mathcal{C}'$ is an odd directed cycle in D_0^k and clearly $\mathcal{C} \cong t_1(\mathcal{C}', V(\mathcal{C}'), \mathrm{A}^{<}/V(\mathcal{C}'), \gamma')$, where γ' is the restriction of γ to $\bigcup_{u \in V(\mathcal{C}')} \bigcup_{f \in FA_u^{<}} \gamma_u^f$ and Definition 2.3 implies that each pseudodiagonal of $\mathcal{C}'$ is a diagonal of $\mathcal{C}$. ■

As a direct consequence of Theorems 3.1, 2.5 and Lemma 3.1 we obtain.

Theorem 3.2. *If D_0 is a KP-digraph (resp. CKI-digraph) such that every odd directed cycle has a symmetrical arc and $t_1 = (D_0^{(k)}, V(D_0), \mathrm{A}^{<}, \gamma)$, then $\tau_1(t_1)$ is a KP-digraph (resp. CKI-digraph) which satisfies $M(k)$.*

Corollary 3.1. *For each natural number k, there exists some KP-digraph (resp. CKI-digraph) D_k which satisfies the k-Meyniel's condition.*

Proof. Define the digraph $C = \overrightarrow{C}_n\ (j_1, \ldots, j_k)$ by $V(C) = \{\ 0, 1, \ldots, n-1\ \}$, $F(C) = \{\ uv \mid v - u \equiv j_s (\mathrm{mod}\ n)$ for $s = 1, \ldots, k\ \}$ and denote $D_0 = \overrightarrow{C}_n$ $(1, \pm 2, \ldots, \pm r)$ for an even natural number $n \not\equiv 0 \ (\mathrm{mod}\ r+1)$. In [6] it was proved that D_0 is a CKI-digraph; so it follows from Theorems 3.1, 2.3 and Lemma 3.1 that $\tau_1(t_1)$ is a CKI-digraph which satisfies $M(k)$. ■

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