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VIZING'S CONJECTURE AND THE ONE–HALF ARGUMENT

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Abstract

The domination number of a graph G is the smallest order, $\gamma(G)$, of a dominating set for G. A conjecture of V. G. Vizing [5] states that for every pair of graphs G and H, $\gamma(G \,\Box\, H) \geq \gamma(G)\gamma(H)$, where $G \,\Box\, H$ denotes the Cartesian product of G and H. We show that if the vertex set of G can be partitioned in a certain way then the above inequality holds for every graph H. The class of graphs G which have this type of partitioning includes those whose 2-packing number is no smaller than $\gamma(G) - 1$ as well as the collection of graphs considered by Barcalkin and German in [1]. A crucial part of the proof depends on the well-known fact that the domination number of any connected graph of order at least two is no more than half its order.

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1. Introduction and Terminology

We consider only finite, simple, undirected graphs. The vertex set of a graph G will be denoted by $V(G)$ and its edge set by $E(G)$. For a subset D of $V(G)$, $\langle D\rangle$ is the induced subgraph on the vertices of D. The *neighborhood* of $v \in V(G)$ is $N(v) = \{x|xv \in E(G)\}$ and $N(D) = \cup_{x\in D}N(x)$; $N[v] = N(v)\cup\{v\}$ is the *closed* neighborhood of v and $N[D] = \cup_{v\in D}N[v]$. The set D is called a *dominating set* for G if $N[D] = V(G)$. The cardinality of a smallest dominating set for G is the *domination number* of G and is denoted by $\gamma(G)$. We will refer to any dominating set of G having cardinality $\gamma(G)$ as a *γ-set* of G. $C \subseteq V(G)$ will be called a *clique* if $\langle C\rangle$ is a (not necessarily maximal) complete graph.

A set $I \subseteq V(G)$ is a *2-packing* of G if $N[x]\cap N[y] = \emptyset$ for every pair $x, y \in I$, $x \neq y$. $P_2(G)$, the *2-packing number* of G, is the cardinality of the largest 2-packing of G. Note that since a dominating set in G must contain at least one vertex from every closed neighborhood of G, it is immediate that $\gamma(G) \geq P_2(G)$.

The *Cartesian product*, $G \Box H$, of graphs G and H is the graph with vertex set $V(G)\times V(H)$ and where two vertices are adjacent when they are equal in one coordinate and adjacent in the other. That is, (u, v) and (x, y) are adjacent in $G \Box H$ when either $u = x$ and $vy \in E(H)$, or $ux \in E(G)$ and $v = y$.

Let h be any vertex of H. We let G_h represent the induced subgraph $\langle\{(x, h)|x \in V(G)\}\rangle$. Note that G_h is isomorphic to G. We refer to this subgraph as *level* h of $G \Box H$. We shall say that level t is a *neighboring level* of level h if and only if t and h are neighbors in H. Similarly, H_g for $g \in V(G)$ will denote the subgraph of $G \Box H$ induced by the set of vertices $\{(g, y)|y \in V(H)\}$. If $S \subset V(G)$, then $H_S = \langle S \times V(H)\rangle$. This subgraph of $G \Box H$ is isomorphic to $\langle S\rangle \Box H$.

We are interested in the conjecture first suggested by Vizing [5]: *for all graphs G and H, $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$.* We will say that *Vizing's conjecture is true for a graph G* if the above inequality is true for every graph H.

Suppose that D is any dominating set for $G \Box H$ and (g, h) is any vertex not in D. Since D must intersect the neighborhood of (g, h), D must contain either a vertex (g, t) where level t is a neighboring level of level h or a vertex (s, h) where vertices s and g are adjacent in G.

Our approach will be to partition the vertices of G into $\gamma(G)$ sets, say $S_1, S_2, \ldots, S_{\gamma(G)}$, and then to show that each H_{S_i} contains at least $\gamma(H)$ vertices of any dominating set D of $G \Box H$, or if some of these induced subgraphs have fewer than $\gamma(H)$ members of D then there must be sufficient extras in those which do to compensate for the shortage.

The simplest case occurs when one is guaranteed that each H_{S_i} has at least $\gamma(H)$ elements of D. For instance, it has been noted by a number of authors (see [1] and [4]) that if $P_2(G) = \gamma(G)$, then $\gamma(G \,\square\, H) \geq \gamma(G)\gamma(H)$. This follows by observing that for each vertex v in a maximum 2-packing of G there must be sufficient vertices in D to dominate H_v, and these vertices can only be from $H_{N[v]}$.

For example, if G is the 6-cycle having consecutive vertices a, b, c, d, e, f, then to dominate H_b only vertices from $H_a \cup H_b \cup H_c$ can be used. Similarly, H_e is dominated only from $H_d \cup H_e \cup H_f$. Hence at least $2\gamma(H)$ will be required in any dominating set of $C_6 \,\square\, H$.

The following observation proves useful in working with Vizing's conjecture. It can be used to conclude that Vizing's conjecture is true for a given class of graphs after proving it for those graphs in the class which are edge-maximal with respect to the domination number.

Lemma 1.1. *If G is a spanning subgraph of G' such that $\gamma(G) = \gamma(G')$ and if Vizing's conjecture is true for G', then it is also true for G.*

Proof. Let G and G' be as in the statement of the lemma and let H be any graph. Since the domination number of a spanning subgraph is always at least as large as that of the original graph, it follows that $\gamma(G \,\square\, H) \geq \gamma(G' \,\square\, H) \geq \gamma(G')\gamma(H) = \gamma(G)\gamma(H)$. ∎

Another very useful result is that of Barcalkin and German [1] which is a more general partitioning condition than the 2-packing one mentioned above.

Theorem 1.2. *Suppose G is a spanning subgraph of a graph G' such that $\gamma(G) = \gamma(G')$ and such that $V(G')$ can be partitioned into $\gamma(G')$ subsets each of which induces a clique in G'. Then Vizing's conjecture is true for G.*

To see that the previous case, where G has a 2-packing $\{v_1, v_2, \ldots, v_{\gamma(G)}\}$, is actually a special case of this theorem it suffices to observe that sufficient edges can be added to make each $N[v_i]$ a clique. Any vertices of G not included in $\cup_{i=1}^{\gamma(G)} N[v_i]$ can be assigned to any of the cliques. The resulting graph G' has domination number $\gamma(G)$. Thus in the C_6 example above, edges ac and df can be added to C_6 to form two triangles. However, this result also handles such diverse cases as $K_n \,\square\, K_n$ (as it partitions into n cliques) and C_7 with vertices (in order) a, b, c, d, e, f, g. Here one can produce three cliques (one K_3 and two K_2's) by adding the two edges from a to vertices d and e. In the first example note that the 2-packing number is much smaller than the domination number. Since in C_7 one cannot, for example, join vertices a and c without lowering the domination number, this example illustrates the care that must be taken to check if the Barcalkin and German result applies.

Our intent is to consider graphs G which cannot be partitioned into $\gamma(G)$ cliques (nor are they spanning subgraphs of such) and extend the partition concept to include some of them. For example, the four graphs in Figure 1 are potential candidates.

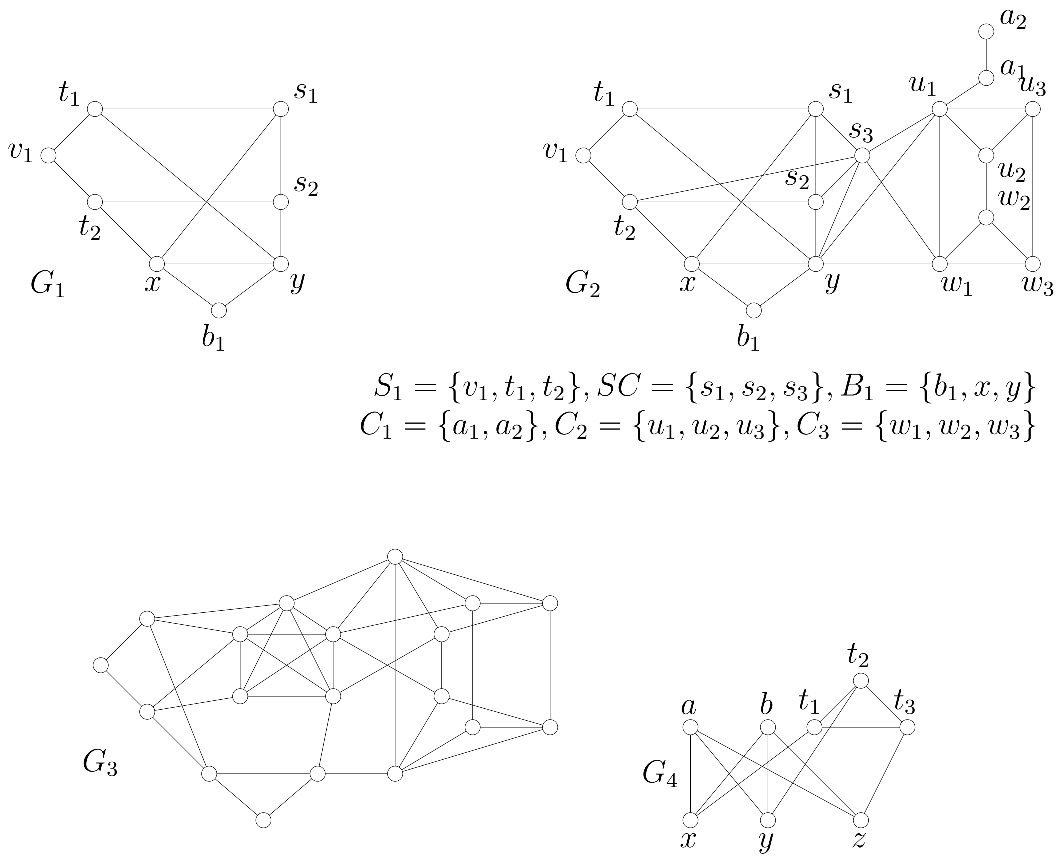

Figure 1

2. Illustration of Proof Technique

Before we consider graphs of a more general nature, we will illustrate the main new argument that we employ on a very small graph. In particular, consider $G = C_4$ with vertices (in order) a, b, c, d. Although this graph can be partitioned into $\gamma(G)$ cliques (for instance, $\{a, b\}$ and $\{c, d\}$), and so Theorem 1.2 states that Vizing's conjecture is true for G, we will show another argument that can be used.

For an arbitrary graph H let D be any dominating set of $G \,\square\, H$. If H_a is dominated by $D \cap H_a$, then $|D \cap H_a| \geq \gamma(H)$. If there are some vertices of H_a not dominated in this way, then we make a list of these "missing levels" for H_a. That is, vertex x of H is on the missing level list for H_a if $D \cap H_a$ contains no vertex of the form (a, y) for $y \in N[x]$. Since D dominates (a, x) it follows that D must contain a vertex from level x in either H_b or H_d. Similarly, any vertex of H_c not dominated by $D \cap H_c$ appears on the missing level list for H_c.

We will now show there are sufficient elements, say m, of D in $H_b \cup H_d$ so that some set of m vertices could be carefully chosen to dominate the missing levels in H_a and those in H_c. Consider the members of $D \cap H_d$ and project them onto H_b. That is, if $(d, i) \in D$, then this is projected to (b, i). Consider the set A of vertices in H_b consisting of the original members of $D \cap H_b$ as well as the projected elements from $D \cap H_d$. In general the subgraph F of H_b induced by A will consist of components of order two or more as well as singleton vertices. The effect they had as far as H_a and H_c were concerned was to dominate (although a given one may not actually be required) the vertices in the corresponding levels. But consider any component C of F of order two or more. All of the vertices of C can be dominated by a γ-set of C, and a γ-set of C will contain no more than $\frac{1}{2}|C|$ vertices. Hence for any such component we could count half the elements of D in that component towards H_a and half towards H_c. They would be sufficient to dominate all missing levels (in H_a and H_c) that correspond to the vertices in C. Of course, there may be fewer missing levels than there are vertices in C, but the worst case is that they are all missing.

Now consider a component of F of order one consisting of the single vertex (b, i). This means that no neighbor of (b, i) is in $H_b \cap D$ and no neighbor of (d, i) is in $H_d \cap D$. If i appears on the missing level list of only one of H_a or H_c, then we can count (b, i) there. On the other hand, if i appears on the missing level list at both H_a and H_c, then since D

contains no vertices from neighboring levels of level i in $H_b \cup H_d$, it follows that D contains both (b, i) and (d, i) since both of these vertices must be dominated by D. Hence we have a "duplicate" and can count one of these for H_a and the other for H_c. Note that since these arose from an isolated vertex component there is no danger of counting one of these vertices from D in two different ways.

We have shown that there are sufficient extra vertices of D in $H_b \cup H_d$ to be "earmarked" for H_a and H_c to dominate any missing levels. Hence D must contain at least $2\gamma(H)$ vertices and so Vizing's conjecture is true for G. It should be noted that one cannot consider the missing levels one at a time and simply project the corresponding vertices from $D \cap H_b$ or $D \cap H_d$ to either H_a or H_c where needed, because in so doing it is possible to leave portions of the graphs H_b and H_d not yet considered no longer dominated.

As another illustration of this technique consider $G = G_4$ as shown in Figure 1. Let H be any graph and consider any dominating set D of $G \Box H$. If $D \cap H_a$ does not dominate H_a, then make a list of the missing levels for H_a. Similarly, make a missing level list for any vertices of H_b that are not dominated by $D \cap H_b$. For any fixed vertex, say t_1, of the triangle project all the elements of $D \cap (H_{t_2} \cup H_{t_3})$ onto H_{t_1}. Let W represent the set of vertices in H_{t_1} which were already in D together with those which are images of the projections. If there are any vertices in H_{t_1} which are not dominated by W, they form a missing level list for the triangle.

First note that if vertex h is on the missing level list of the triangle, then it must be the case that (x, h), (y, h) and (z, h) are all in D in order to dominate the "triangle part" of level h. Hence we will be able to count one member of D for missing level h in the triangle and still have two members of D for H_a and H_b if required.

Now project the vertices in $D \cap (H_y \cup H_z)$ onto H_x. Let F denote the subgraph of H_x induced by the resulting set A of vertices from $H_x \cap D$ together with the projection images. As in the previous example, the effect on H_a and H_b of some vertex in $D \cap (H_x \cup H_y \cup H_z)$ is to dominate the corresponding vertices in the same level and not in any neighboring levels. Any component of F of order at least two can be dominated by a subset of cardinality at most half the order of the component. Thus we can compensate for any missing level in either H_a or H_b which corresponded to a vertex in such a component. If the triangle were also missing that level, then as noted above we have duplicates at that level.

Suppose $\{(x, h)\}$ is a component of F. If only one of H_a or H_b is missing level h, then we can count the vertex (x,h) where it is missing. In the event that h is on the missing level list for both H_a and H_b, then either there are at least two of (x,h), (y,h) and (z,h) in D (and hence we can count one for each of H_a and H_b), or only one of these is in D. In this case there must be at least two of (t_1,h), (t_2,h) and (t_3,h) in D to dominate $\{(x,h),(y,h),(z,h)\}$. Hence again we have a duplicate in the triangle at that level along with the single member from (x,h), (y,h) and (z,h) for H_a and H_b. This counting gives $|D| \geq 3\gamma(H) = \gamma(G_4)\gamma(H)$. Thus Vizing's conjecture is true for G_4.

3. The Main Theorem

Consider a graph G with $\gamma(G) = n = k+t+m+1$ and such that $V(G)$ can be partitioned into $S \cup SC \cup BC \cup C$, where $S = S_1 \cup S_2 \cup \ldots \cup S_k$, $BC = B_1 \cup B_2 \cup \ldots \cup B_t$, and $C = C_1 \cup C_2 \cup \ldots \cup C_m$. Each of $SC, B_1, \ldots, B_t, C_1, \ldots, C_m$ induces a clique. Every vertex of SC (special clique) has at least one neighbor outside SC whereas each of $B_1, \ldots, B_t$ (the buffer cliques), say B_i, has at least one vertex, say b_i, which has no neighbors outside B_i. Each $S_i \in \{S_1, S_2, \ldots, S_k\}$ is star-like in that it contains a star centered at a vertex v_i which is adjacent to each vertex in $T_i = S_i - \{v_i\}$. The vertex v_i has no neighbors besides those in T_i. Although other pairs of vertices in T_i may be adjacent (and hence S_i does not necessarily induce a star), S_i does *not* induce a clique nor can more edges be added in $\langle S_i \rangle$ without lowering the domination number of G. Furthermore, there are no edges between vertices in S and vertices in C. For ease of reference we will say such a graph G is of *Type* $\mathcal{X}$.

It should be noted that a graph of Type $\mathcal{X}$ need not have a clique having the properties of SC, and any of t, m or k is allowed to be 0. However, if such an SC is not in G, then $\gamma(G) = n = k+t+m$. Also, if SC is not present and BC is empty, but S as well as C are not empty, then the graph is disconnected. SC can not be the only one of these which is nonempty since by definition its vertices must have neighbors outside SC. As illustrations of graphs of Type $\mathcal{X}$ see G_1, G_2 and G_3 in Figure 1.

Theorem 3.1. *Let G be a graph of Type $\mathcal{X}$. Vizing's conjecture is true for G.*

Proof. Let H be an arbitrary graph and suppose D is any dominating set for $G \square H$. For each set, an S_i or B_i, in the partition of $S \cup BC$, there are sufficient elements of D to entirely dominate a copy of H. In particular, consider $S_j \in S$ and the copy of H represented by H_{v_j}. Any vertex (v_j, h) of H_{v_j} not dominated by $D \cap H_{v_j}$ must be dominated by a neighboring vertex in level h and hence by some vertex from $D \cap H_{T_j}$. In either case, all of H_{v_j} is dominated by $D \cap H_{S_j}$. Similarly, for each buffer clique B_j, there must be sufficient vertices in $D \cap H_{B_j}$ to entirely dominate H_{b_j} since b_j has no neighbors in $V(G) - B_j$.

However, H_{SC} and H_{C_j} may not be entirely dominated by vertices of $D \cap H_{SC}$ and $D \cap H_{C_j}$, respectively. We must show that there are sufficient elements of D to yield at least $\gamma(H)$ for each of the sets in $\{SC, C_1, \ldots, C_m\}$ as well as $\{S_1, \ldots, S_k, B_1, \ldots, B_t\}$.

To obtain a measure of the shortfall at each clique, say K, in $\{SC, C_1, \ldots, C_m\}$, select one vertex $w \in K$ and project all the elements of $D \cap H_K$ onto H_w. That is, a vertex $(u, h) \in (D \cap H_K) - H_w$ is projected onto the vertex (w, h). If all of H_w is dominated by the elements of D originally in H_w or those projected onto H_w, then there is no shortfall as we must already have at least $\gamma(H)$ elements of D in H_K. On the other hand, if certain vertices (levels) of H_w are not so dominated, form a missing level list for H_w consisting of these. Observe that if vertex h is in the missing level list at H_w this means that no vertex of H_K in level h is in D nor was any vertex from a neighboring level to level h in $D \cap H_K$. This implies that all vertices from level h in H_K must be dominated by neighbors in level h in D corresponding to other sets in the partition of $V(G)$.

For example, let $G = G_1$ in Figure 1 and $H = P_5$ with vertices labeled 1,2,3,4,5 in order. Suppose the only vertex in $H_{s_1} \cap D$ is $(s_1, 1)$, and $(s_2, 1)$ and $(s_2, 5)$ are the only members of $H_{s_2} \cap D$. When projected onto H_{s_1} vertex 3 would be a missing level. Hence either $(t_1, 3)$ or $(x, 3)$ (or both) must belong to D to dominate $(s_1, 3)$. Similarly, $(t_2, 3)$ or $(y, 3)$ must belong to D to dominate $(s_2, 3) \in H_{s_2}$.

Note that there may well be duplicates at certain levels when the projecting occurs as is the case with level 1 in H_{s_1}. We note the extra occurrences in a particular clique of a level h vertex in D (i.e., all but one) may be required for counting towards any shortages at that level in other cliques. For instance in Figure 1, if both $(x, 3)$ and $(y, 3)$ are in D, one of these is sufficient for H_{b_1}, and the other could be counted towards the missing level 3 in SC.

We must modify this argument for the sets $T_1, T_2, \ldots, T_k$. First observe that because of the structure of G, $(u, h) \in D \cap H_{T_j}$ is only needed to dominate vertex (v_j, h) in H_{v_j} (not any vertex in a neighboring level in H_{v_j}). The vertices of $D \cap H_{T_j}$ may help dominate a missing level of H_{SC} as well. Although they may dominate vertices in some H_{T_r} or H_{B_p}, both H_{S_r} and H_{B_p} have sufficient members of D to dominate a copy of H and do not require external assistance in the count. Now for each d, $1 \le d \le k$, choose a vertex $e \in T_d$ and project all members of $D \cap T_d$ onto H_e. The projected vertices will induce a subgraph of H_e consisting of isolated vertices or of components of order two or more.

Let F be one of these components of order at least two. For $(e, f) \in F$ the corresponding vertex $(v_d, f) \in H_{v_d}$ may need to be dominated. Thus it is possible that $\gamma(F)$ vertices from F must be available to dominate H_{v_d}. But $\gamma(F) \le \frac{1}{2}|F|$ and hence we could count $\gamma(F)$ for H_{v_d} and $\gamma(F)$ for H_{SC}, even though these vertices may not be needed at H_{SC}. All components of order two or more could be treated similarly. The singleton components will be treated separately.

Let us now proceed to consider $G \,\square\, H$. For each clique Q in $\{SC, C_1, C_2, \ldots, C_m\}$ fix a vertex $a \in Q$ and project $D \cap H_Q$ onto H_a. Create a missing level list for each such Q. As discussed before, in the case of each member of $\{S_1, \ldots, S_k, B_1, \ldots, B_t\}$ there are sufficient members of D to dominate a copy of H, and hence there are no missing levels.

Now consider each vertex of H which is a missing level in at least one clique. Suppose a total of r cliques other than SC are missing level h. Then every vertex in level h corresponding to a vertex from one of these r cliques must be dominated by a member of D (in level h) corresponding to a vertex from another clique, possibly in $SC \cup BC$. Suppose there are a total of s such cliques. That is, there are s cliques which have at least one element of D at level h which is adjacent to at least one vertex in the set of r cliques missing level h. Call the set of vertices at level h in D from these s cliques D_r. The claim is that $|D_r| \ge r + s$, since in G_h the set D_r dominates the s cliques it belongs to as well as the set of r cliques missing level h. For if there were fewer than $r + s$ vertices in D_r, then we can extend D_r to a dominating set of G_h by including $(v_1, h), (v_2, h), \ldots, (v_k, h)$ and one per clique at level h for the cliques not in the $r + s$ already considered. This resulting dominating set has cardinality less than n which is a contradiction.

In case SC is missing level h but level h of SC is entirely dominated from neighboring cliques (as would be possible if $G = G_1$ in Figure 1), the same argument applies. If SC is missing level h but level h of SC is not entirely dominated by vertices of D in neighboring cliques, then we must consider elements of D in $T_1 \cup T_2 \cup \ldots \cup T_k$. (For example, this would necessarily be the case if $G = G_3$ in Figure 1 since two of the vertices of SC are not adjacent to vertices in any of the cliques.)

For each j, $1 \le j \le k$, choose $w_j \in T_j$ and project $D \cap H_{T_j}$ onto H_{w_j}. The projected vertices will, for each T_j, induce a subgraph F_j of H_{w_j} consisting of components of order two or more and singleton vertices. There are two possibilities:

Case 1. SC is missing level h and for every i, $1 \le i \le k$, T_i has either no elements of D or one element of D in level h. Furthermore, for each T_i with exactly one element of D at level h, no vertex from a neighboring level to level h in H_{T_i} belongs to D (hence, when projected, singletons would result). In addition, for each such T_i, in the corresponding H_{v_i}, neither (v_i, h) nor any vertex from a neighboring level to level h belongs to D. Let S' denote the collection of all S_i corresponding to such a T_i.

In this case suppose a total of r_1 cliques besides SC are missing level h. These r_1 cliques, as well as SC and the set of, say r_2, members of S' must be dominated at level h by the r_2 members of D in S' as well as vertices of D which are neighbors in level h but in other cliques. Suppose there are a total of s such cliques. That is, there are s cliques which have at least one element of D in level h which is adjacent to at least one vertex in SC, in a member of S', or in one of the r_1 cliques missing level h. But if there were fewer than $r_1 + r_2 + s + 1$ such elements of D, then all of G_h could be dominated by enlarging this set to include v_j for each $S_j \notin S'$ and one vertex from each of the cliques not counted above. This is a contradiction since the resulting dominating set for G_h would have fewer than n vertices.

Case 2. SC is missing level h and there is at least one j, $1 \le j \le k$, such that T_j has either

(2a) two or more elements of D in level h (and hence there will be duplicates when projected), or

(2b) a level h member of D as well as a neighboring level member of D (and hence, when projected, a component of order two or more will result), or

(2c) a level h vertex in D, and $D \cap H_{v_j}$ has either a vertex in level h or a vertex from a neighboring level of level h.

First, if any F_j contains a level h vertex in a component L of order two or more, then L can be dominated in $\gamma(L) \le \frac{1}{2}|L|$ elements, and so $\gamma(L)$ of the vertices in L could be counted towards dominating H_{v_j} and $\gamma(L)$

towards dominating those missing levels of SC. If no F_i contains the level h vertex in a component of order at least two, then either (2a) holds which means there is a duplicate level h vertex in D available for SC, or (2c) holds, in which case H_{v_j} does not require the level h vertex in $D \cap T_j$ and so it can be counted for dominating SC.

Hence any missing level h of SC can be handled by either Case 1 or Case 2. We have shown that $|D| \geq n\gamma(H)$ and so Vizing's conjecture is true for G. ∎

The following more general result is an immediate corollary of Theorem 3.1 and Lemma 1.1.

Corollary 3.2. *If G is a graph of Type $\mathcal{X}$ and F is a spanning subgraph of G such that $\gamma(F) = \gamma(G)$, then Vizing's conjecture is true for F.*

We also note that the result of Barcalkin and German, Theorem 1.2, is a special case of this theorem.

Corollary 3.3. *If G is a graph as in Theorem 3.1 except that the set S is empty, then Vizing's conjecture is true for G.*

As mentioned in Section 1, if $\gamma(G) = P_2(G)$ then Vizing's conjecture is true for G. The following corollary of Theorem 3.1 shows that this can now be extended.

Corollary 3.4. *If G is a graph and $\gamma(G) = P_2(G) + 1$, then Vizing's conjecture is true for G.*

Proof. Let $P_2(G) = k$ and $\gamma(G) = k + 1$. Suppose $\{v_1, v_2, \ldots, v_k\}$ is a maximum 2-packing of G. For each i, $1 \leq i \leq k$, let S_i be the subgraph of G induced by $N[v_i]$ and let $W = V(G) - \cup_{1 \leq j \leq k} N[v_i]$. Add edges if necessary to make W into a clique SC, and if possible, add edges in each $N[v_i]$ as long as the domination number of the resulting graph is not smaller than $\gamma(G)$. It is clear that the resulting graph G' has domination number $k + 1$ and satisfies the hypothesis of Theorem 3.1. The fact that Vizing's conjecture is true for G now follows from Corollary 3.2. ∎

Note that each of G_2 and G_3 in Figure 1 satisfies the hypothesis of Theorem 3.1 but is not covered by Theorem 1.2 or Corollary 3.4.

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