

GENERALIZED LIST COLOURINGS OF GRAPHS

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Abstract

We prove: (1) that $\text{ch}_{\mathcal{P}}(G) - \chi_{\mathcal{P}}(G)$ can be arbitrarily large, where $\text{ch}_{\mathcal{P}}(G)$ and $\chi_{\mathcal{P}}(G)$ are \mathcal{P} -choice and \mathcal{P} -chromatic numbers, respectively, (2) the (\mathcal{P}, L) -colouring version of Brooks' and Gallai's theorems.

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1. INTRODUCTION AND NOTATION

We consider finite undirected graphs without loops and multiple edges. The vertex set of a graph G is denoted by $V(G)$ and the edge set by $E(G)$. The notation $H \subseteq G$ means that H is a subgraph of G . The vertex induced (we will say briefly: induced) subgraph H of G is denoted by $H \leq G$. We say that G contains H whenever G contains a subgraph isomorphic to H . In general, we follow the notation and terminology of [5].

Let \mathcal{I} denote the set of all mutually nonisomorphic graphs. If \mathcal{P} is a nonempty subset of \mathcal{I} , then \mathcal{P} will also denote the property that a graph is

a member of the set \mathcal{P} . We shall use the terms *set of graphs* and *property of graphs* interchangeably.

A property \mathcal{P} of graphs is said to be *induced hereditary* (shortly: *hereditary*) if whenever $G \in \mathcal{P}$ and H is a vertex induced subgraph of G , then also $H \in \mathcal{P}$. For convenience, the empty set \emptyset will be regarded as the set inducing the subgraph with any property \mathcal{P} .

A property \mathcal{P} is *additive*, if for each graph G all of whose components have the property \mathcal{P} it follows that $G \in \mathcal{P}$. Let us denote by \mathbf{M} and \mathbf{M}^a the set of all hereditary and additive hereditary properties, respectively. Any hereditary property \mathcal{P} of graphs is uniquely determined by the set $\mathcal{C}(\mathcal{P})$ of all *forbidden subgraphs* defined by

$$\mathcal{C}(\mathcal{P}) = \{H \in \mathcal{I} : H \notin \mathcal{P} \text{ but } (H - v) \in \mathcal{P} \text{ for any } v \in V(H)\}.$$

It is easy to prove that a property $\mathcal{P} \in \mathbf{M}$ is additive if and only if all forbidden subgraphs $H \in \mathcal{C}(\mathcal{P})$ are connected.

A hereditary property $\mathcal{P} \in \mathbf{M}$ is said to be nontrivial if $\mathcal{P} \neq \mathcal{I}$. For a nontrivial property $\mathcal{P} \in \mathbf{M}$ there exists a number $c(\mathcal{P})$ called the *completeness* of \mathcal{P} defined as $\sup\{k : K_{k+1} \in \mathcal{P}\}$, and $c(\mathcal{P}) = \infty$ if for every k , $K_{k+2} \in \mathcal{P}$. Let $\delta(\mathcal{P}) = \min\{\delta(H) : H \in \mathcal{C}(\mathcal{P})\}$.

Let us denote by $\mathcal{O} = \{G : G \in \mathcal{I}, E(G) = \emptyset\}$. For this property we have $\mathcal{C}(\mathcal{O}) = \{K_2\}$ and $\delta(\mathcal{O}) = 1$.

A \mathcal{P} -*partition* (*colouring*) of a graph G is a partition (V_1, \dots, V_n) of $V(G)$ such that the subgraph $\langle V_i \rangle$ induced by the set V_i has property \mathcal{P} for each $i = 1, \dots, n$. If (V_1, \dots, V_n) is a \mathcal{P} -partition of a graph G , then the corresponding vertex colouring f is defined by $f(v) = i$ whenever $v \in V_i$, for $i = 1, \dots, n$. The smallest integer n for which G has \mathcal{P} -partition is called the \mathcal{P} -*chromatic* (or \mathcal{P} -*vertex-partition*) *number* of G and is denoted by $\chi_{\mathcal{P}}(G)$. The \mathcal{O} -chromatic number is the ordinary chromatic number. See [1] for a survey and more details.

Let G be a graph and let $L(v)$ be a list of colours (as above, positive integers) prescribed for the vertex v , and $\mathcal{P} \in \mathbf{M}$. A (\mathcal{P}, L) -*colouring* is a graph \mathcal{P} -colouring $f(v)$ with the additional requirement that for all $v \in V(G)$, $f(v) \in L(v)$. If G admits a (\mathcal{P}, L) -colouring, then G is said to be (\mathcal{P}, L) -*colourable*. The graph G is (k, \mathcal{P}) -*choosable* if it is (\mathcal{P}, L) -colourable for every list L of G satisfying $|L(v)| = k$ for every $v \in V(G)$. The \mathcal{P} -*choice number* $\text{ch}_{\mathcal{P}}(G)$ is the smallest natural number k such that G is (k, \mathcal{P}) -choosable.

Vizing [6] and Erdős, Rubin and Taylor [3] independently introduce the idea of considering (\mathcal{O}, L) -colouring and (k, \mathcal{O}) -choosability.

The aim of this paper is to prove some extensions of two basic theorems in the colouring theory of graphs, namely the Brooks [2] and Gallai [4] theorems. If $L(v)$ is the same for all vertices of G , this results generalize also the corresponding results of [1]. Moreover, we prove that $\text{ch}_{\mathcal{P}}(G) - \chi_{\mathcal{P}}(G)$ can be arbitrarily large.

2. BEHAVIOUR OF THE \mathcal{P} -CHOICE NUMBER

To prove the main theorem of this section, use well-known Pigeonhole Principle in the following form.

Pigeonhole Principle. *Suppose that q_1, \dots, q_t are positive integers. If $X = X_1 \cup \dots \cup X_t$ is a partition of the set X and $|X| \geq (\sum_{i=1}^t q_i) - t + 1$, then $|X_i| \geq q_i$ for some $i \in \{1, \dots, t\}$.*

Theorem 1. *Let $\mathcal{P} \in \mathbf{M}^a$ and $1 \leq c(\mathcal{P}) < \infty$. Then for any nonnegative integer s there exists a graph G_s such that $\text{ch}_{\mathcal{P}}(G_s) - \chi_{\mathcal{P}}(G_s) > s$.*

Proof. For a given $c(\mathcal{P})$ let the function $g(l)$ be defined by

$$g(l) = \left\lceil \frac{l(l-1) - 2 + c(\mathcal{P})(c(\mathcal{P}) + 3)}{2c(\mathcal{P})(c(\mathcal{P}) + 3)} \right\rceil - 1 - l$$

The function $g(l)$ tends to infinity with $l \rightarrow \infty$. Hence, for given s and $c(\mathcal{P})$ there is an integer $l_0 \geq 3$ such that $g(l_0) \geq s$. For this integer l_0 , let

$$(*) \quad b = \left\lceil \frac{l_0(l_0 - 1) - 2 + c(\mathcal{P})(c(\mathcal{P}) + 3)}{2c(\mathcal{P})(c(\mathcal{P}) + 3)} \right\rceil - 1$$

Note that $b \geq s + l_0 \geq 3$.

Let the graph G_s be given by the join $H_1 + \dots + H_{l_0}$ of totally disconnected graphs H_i , $i = 1, \dots, l_0$, all of order $t = \binom{2b-1}{b}$, i.e., G_s is a complete l_0 -partite graph with t elements in each part.

Since $\mathcal{P} \in \mathbf{M}^a$ and $K_1 \in \mathcal{P}$, then $H_i \in \mathcal{P}$ for $i = 1, \dots, l_0$. Hence, $\chi_{\mathcal{P}}(G_s) \leq l_0$. Note that it is sufficient to prove that G_s is not (b, \mathcal{P}) -choosable, i.e., $\text{ch}_{\mathcal{P}}(G_s) > b$. By this and $(*)$ we have $\text{ch}_{\mathcal{P}}(G_s) - \chi_{\mathcal{P}}(G_s) > b - l_0 \geq s$.

To prove this, let $C = \{1, \dots, 2b-1\}$ be a set of colours and let $[C]^b = \{A : A \subseteq C, |A| = b\}$. Suppose that this set is indexed as follows $\{A_j : j = 1, \dots, t\}$. Besides, let $V(H_i) = \{v_{ij} : j = 1, \dots, t\}, i = 1, \dots, l_0$. Define a list L for the graph G_s by $L(v_{ij}) = A_j$ for all i and j . Let f be a (\mathcal{P}, L) -colouring of G_s . Note that

(a) For any colouring of the graph H_i from its lists we need at least b different colours for any $i = 1, \dots, l_0$.

(b) For $i_1, i_2 \in \{1, \dots, l_0\}, i_1 \neq i_2$, there is $j_1, j_2 \in \{1, \dots, t\}$ and a colour $c \in C$ such that $f(v_{i_1 j_1}) = f(v_{i_2 j_2}) = c$. This follows by (a) and $|C| = 2b-1$. The colour c is said to be $\{i_1, i_2\}$ -colour. Let $X = \{\{i_1, i_2\} : \{i_1, i_2\}\text{-colour}\}$ and $X_c = \{\{i_1, i_2\} : c \text{ is } \{i_1, i_2\}\text{-colour}\}, c \in C$.

Let be given a sequence of integers $q_1 = \dots = q_{2b-1} = \binom{c(\mathcal{P})+2}{2}$.

Note that $X = X_1 \cup \dots \cup X_{2b-1}$. By above and (*) we have

$$|X| = \binom{l_0}{2} \geq \sum_{i=1}^{2b-1} q_i - (2b-1) + 1.$$

Hence, by Pigeonhole Principle it follows that there is $c_0 \in C$, such that $|X_{c_0}| \geq q_{c_0} = \binom{c(\mathcal{P})+2}{2}$. It implies that there are at least $c(\mathcal{P})+2$ pairwise different integers $i_1, \dots, i_{c(\mathcal{P})+2} \in \{1, \dots, l_0\}$ and for any i_r there is an integer $j, 1 \leq j \leq t$, such that $f(v_{i_r j}) = c_0$. By above and the definition of G_s , the subgraph of G_s induced by vertices with the assigned colour c_0 contains a complete graph of order $c(\mathcal{P})+2$, i.e., it is not (b, \mathcal{P}) -choosable. Hence, $\text{ch}_{\mathcal{P}}(G_s) > b$. ■

3. (\mathcal{P}, L) -CRITICAL GRAPHS

For a nontrivial property $\mathcal{P} \in \mathbf{M}$ a graph G is said to be (\mathcal{P}, L) -critical if G has no (\mathcal{P}, L) -colouring but $G - v$ is (\mathcal{P}, L) -colourable for all $v \in V(G)$.

Lemma 1. *If $\mathcal{P} \in \mathbf{M}$ and G is (\mathcal{P}, L) -critical, then $d_G(v) \geq \delta(\mathcal{P}) \mid L(v) \mid$ for any vertex v of G .*

Proof. Suppose that $d_G(u) < \delta(\mathcal{P}) \mid L(u) \mid$ for a vertex $u \in V(G)$. Then there is $i \in L(u)$ which is used in colouring of the vertices of $N_G(u)$ less than $\delta(\mathcal{P})$ times. Therefore, the vertex u can be coloured by i , $\langle V_i \rangle_G \in \mathcal{P}$ and G is not (\mathcal{P}, L) -critical, a contradiction. ■

Let $\mathcal{P} \in \mathbf{M}$, G be (\mathcal{P}, L) -critical and $x \in V(G)$. Define a new list assignment

$$L^x(v) = \begin{cases} \{l\}, & v = x, \\ L(v), & \text{otherwise,} \end{cases}$$

where $l \notin L(v)$ for $v \in V(G)$.

Since, by the definition, $G - w$ is (\mathcal{P}, L) -colourable, then G is (\mathcal{P}, L^w) -colourable. Thus, for a vertex w of a (\mathcal{P}, L) -critical graph G we shall say that f is a (\mathcal{P}, L^w) -colouring of G , whenever $f(w) = l$ and $f(v) \in L(v)$ for $v \neq w$.

Note that the list L^x is always created from the list L by assignment of the colour l to the vertex x and preserving the remaining assignments.

Let us denote by $S(G) = \{v : v \in V(G), \ d_G(v) = \delta(\mathcal{P}) \mid L(v)\}$.

From Lemma 1 we have immediately the following lemma.

Lemma 2. *Let G be a (\mathcal{P}, L) -critical, $w \in S(G)$. Then for any (\mathcal{P}, L^w) -colouring f of G , $|N_G(w) \cap V_i| = \delta(\mathcal{P})$ for any $i \in L(w)$, where $V_i = \{v : v \in V(G), f(v) = i\}$. ■*

Lemma 3. *Let G be a (\mathcal{P}, L) -critical, $u, v \in S(G)$, $uv \in E(G)$, and let f be a (\mathcal{P}, L^v) -colouring of G . Then there is a (\mathcal{P}, L^u) -colouring f' of G such that $f'(v) = f(u)$ and $f'(w) = f(w)$ for all $w \in V(G) - \{u, v\}$.*

Proof. Since $u, v \in S(G)$, by Lemma 2 we have $f(u) \in L(v)$. From this, the definition of the (\mathcal{P}, L^x) -colouring and again Lemma 2, the required f' colouring follows. ■

Lemma 4. *Let G be a (\mathcal{P}, L) -critical and $Q : v_0v_1\dots v_m$ be a walk in G such that $V(Q) = \{v_0, v_1, \dots, v_m\} \subseteq S(G)$. Let f be a (\mathcal{P}, L^{v_0}) -colouring of G . Then there is a (\mathcal{P}, L^{v_m}) -colouring f' of G such that $f'(v_i) = f(v_{i+1})$ for $i = 1, \dots, m-1$, $f'(v_m) = f(v_0) = l$ and $f'(w) = f(w)$ for all $w \in V(G) - V(Q)$.*

Proof. By applying Lemma 3 to the consecutive adjacent vertices of Q we obtain the required f' colouring. ■

The procedure described by Lemma 4 will be called (\mathcal{P}, L^x) -recolouring of Q .

Lemma 5. *Let G be a (\mathcal{P}, L) -critical and $C : v_0v_1\dots v_mv_0$ be a cycle in G with $V(C) = \{v_0, v_1, \dots, v_m\} \subseteq S(G)$ and let f be a (\mathcal{P}, L^{v_0}) -colouring of G . Then there is a (\mathcal{P}, L^{v_0}) -colouring f' of G such that $f'(v_i) = f(v_{i+1})$ for $i = 0, 1, \dots, m-1$, $f'(v_m) = f(v_1)$, $f'(w) = f(w)$ for all $w \in V(G) - V(C)$.*

Proof. By (\mathcal{P}, L^x) recolouring of $Q : v_0v_1 \dots v_mv_0$ it follows. \blacksquare

Lemma 6. *Let $C : v_0v_1 \dots v_mv_0$ be an even cycle in a (\mathcal{P}, L) -critical graph G with $V(C) \subseteq S(G)$. If there exists a vertex v_j which is not incident to any diagonal of C , then in any (\mathcal{P}, L^{v_j}) -colouring f of G all vertices of C but v_j have the same colour.*

Proof. By the assumption, $N_C(v_j) = \{v_{j-1}, v_{j+1}\}$. Let f be an arbitrary (\mathcal{P}, L^{v_j}) -colouring of G . By Lemma 2 the vertex v_j has $\delta(\mathcal{P})$ neighbours in each colour from its list. Lemma 4 implies that this property is preserved after (\mathcal{P}, L^x) -recolouring of a walk $W : v_{j+1} \dots v_mv_0 \dots v_{j-1}$. Since C is an even cycle we have that all vertices of C but v_j have the same colour. \blacksquare

Lemma 7. (Dirac, see [4], p.170). *If each even cycle in the block B of a graph G has at least two diagonals in G , then the block B is a complete subgraph of G .* \blacksquare

Theorem 2. *Let $\mathcal{P} \in \mathbf{M}$, G be a (\mathcal{P}, L) -critical graph. Then any block of $\langle S(G) \rangle_G$ is one of the following types:*

- (i) B is a complete graph,
- (ii) B is a $\delta(\mathcal{P})$ -regular graph belonging to $\mathcal{C}(\mathcal{P})$,
- (iii) $B \in \mathcal{P}$ and $\Delta(B) \leq \delta(\mathcal{P})$,
- (iv) B is an odd cycle.

Proof. We have considered three cases.

Case 1. The block B of $\langle S(G) \rangle_G$ contains no even cycles.

Then either $B = K_2$ or B contains an odd cycle C_{2p+1} . In the case when $B \neq C_{2p+1}$, either C_{2p+1} has a diagonal in $\langle S(G) \rangle_G$ or there exists a vertex u of B not belonging to C_{2p+1} . In both cases there is an even cycle in B , a contradiction.

Case 2. There is an even cycle $C_{2p} : v_0v_1 \dots v_{2p-1}v_0$ in B which contains a vertex v_j not incident to any diagonal of C_{2p} .

Let f be a (\mathcal{P}, L^{v_j}) -colouring of G . By Lemma 6 all vertices of C_{2p} but v_j have the same colour. Let us suppose that $f(v_i) = a, i \neq j$. We are going to prove that all vertices of B but v_j are coloured in f by a . Suppose that there exists a vertex $z \neq v_j$ such that $f(z) \neq a$. Then since B is 2-connected and $z \notin V(C_{2p})$ there exists a cycle $C' : v_jv_{j+1} \dots z \dots v_j$. By applying Lemma 5 to the cycle C' we can obtain (\mathcal{P}, L^{v_j}) -colouring of G such that the vertices $v_{j+1}, \dots, v_{2p-1}, v_0, \dots, v_j$ of C_{2p} are not coloured the same, a contradiction. Thus, by Lemma 2, we have $\Delta(B) \leq \delta(\mathcal{P})$. If B is $\delta(\mathcal{P})$ -regular, then

$B \in \mathcal{C}(\mathcal{P})$; otherwise v_j could be recoloured by a , which contradicts that G is (\mathcal{P}, L) -critical. If there is a vertex $u \in V(B)$ with $d_B(u) < \delta(\mathcal{P})$, then according to Lemma 4 by (\mathcal{P}, L^x) -recolouring of a walk $Q : v_j \dots u$, we obtain (\mathcal{P}, L^u) -colouring of G with all vertices of B but u coloured by a . Since $d_B(u) < \delta(\mathcal{P})$, we have $B \in \mathcal{P}$.

Case 3. Each vertex of any even cycle C in B is incident with at least one diagonal of C .

In this case, by Lemma 7, B is a complete subgraph of G . ■

4. (k, \mathcal{P}) -CHOICE CRITICAL GRAPHS. GENERALIZATIONS OF GALLAI'S AND BROOKS' THEOREMS

For a nontrivial property $\mathcal{P} \in \mathbf{M}$, a graph G is said to be (*vertex*) (k, \mathcal{P}) -choice critical if $\text{ch}_{\mathcal{P}}(G) = k \geq 2$ but $\text{ch}_{\mathcal{P}}(G - v) < k$ for all vertices v of G . According to the previous definitions, it follows immediately that if G is $(k + 1, \mathcal{P})$ -choice critical, then G is (\mathcal{P}, L) -critical with some list $|L(v)| = k$ for all $v \in V(G)$.

Hence, by Theorem 2 we have the following generalization of Gallai's Theorem.

Theorem 3. *Let $\mathcal{P} \in \mathbf{M}$ and G be a (k, \mathcal{P}) -choice critical graph. Then any block of $< S(G) >_G$ is one of the following types:*

- (i) B is a complete graph,
- (ii) B is a $\delta(\mathcal{P})$ -regular graph belonging to $\mathcal{C}(\mathcal{P})$,
- (iii) $B \in \mathcal{P}$ and $\Delta(B) \leq \delta(\mathcal{P})$,
- (iv) B is an odd cycle.

Note that in Theorem 3, $S(G) = \{v : v \in V(G), d_G(v) = \delta(\mathcal{P})(k - 1)\}$.

Lemma 8. *Let $\mathcal{P} \in \mathbf{M}^a$ and let G be a connected graph with $|L(v)| \geq \frac{d_G(v)}{\delta(\mathcal{P})}$ for all $v \in V(G)$, and let there exist a vertex $v_0 \in V(G)$ such that $|L(v_0)| \geq \frac{d_G(v_0)+1}{\delta(\mathcal{P})}$. Then G is (\mathcal{P}, L) -colourable.*

Proof. The proof is by induction on the order of G . Let $|V(G)| = 1$. Then $G = K_1$, $|L(v)| \geq 1$ for $v \in V(G)$ and Lemma is true.

Assume the Lemma holds for all graphs of order $\leq n$. Let $|V(G)| = n + 1$ and let L be a list satisfying the assumptions of the Lemma. Consider the graph $G - v_0$ and its list $\tilde{L}(v) = L(v)$ for all $v \in V(G) - \{v_0\}$. Since G is connected, then in each component G' of $G - v_0$ there is a vertex u such that $uv_0 \in E(G)$. Thus,

$$|\tilde{L}(u)| \geq \frac{d_G(u)}{\delta(\mathcal{P})} = \frac{d_{G'}(u)+1}{\delta(\mathcal{P})}.$$

For the remaining vertices of each component

$$|\tilde{L}(v)| \geq \frac{d_G(v)}{\delta(\mathcal{P})} \geq \frac{d_{G'}(v)}{\delta(\mathcal{P})} \text{ holds.}$$

Then, by the induction hypothesis, each component G' of $G - v_0$ is (\mathcal{P}, \tilde{L}) -colourable. Since $\mathcal{P} \in \mathbf{M}^a$, we have (\mathcal{P}, \tilde{L}) -colourability f' of $G - v_0$. We will prove that f' can be extended to (\mathcal{P}, L) -colourability f of G . Suppose, contrary to our claim. It implies that the vertex v_0 has at least $\delta(\mathcal{P})$ neighbours in each colour class V_i for $i \in L(v_0)$. Thus,

$$d_G(v_0) \geq \delta(\mathcal{P}) |L(v_0)| \geq \delta(\mathcal{P}) \frac{d_G(v_0)+1}{\delta(\mathcal{P})} = d_G(v_0) + 1, \text{ a contradiction.} \quad \blacksquare$$

Theorem 4. *Let $\mathcal{P} \in \mathbf{M}^a$ and G be a connected graph other than*

- (i) *a complete graph of order $n\delta(\mathcal{P}) + 1, n \geq 0$,*
- (ii) *a $\delta(\mathcal{P})$ -regular graph belonging to $\mathcal{C}(\mathcal{P})$,*
- (iii) *an odd cycle if $\mathcal{P} = \mathcal{O}$.*

Then

$$\text{ch}_{\mathcal{P}}(G) \leq \left\lceil \frac{\Delta(G)}{\delta(\mathcal{P})} \right\rceil.$$

Proof. By Lemma 8, the Theorem is true for all not regular graphs and for any graph G with $\Delta(G) \not\equiv 0 \pmod{\delta(\mathcal{P})}$. So, let G be a regular graph with $\Delta(G) = (k-1)\delta(\mathcal{P})$. Since $G \neq K_1$ we have $k \geq 2$. Now, suppose the assertion of the Theorem is false for a list L , with $|L(v)| = k-1$ for all $v \in V(G)$. By Lemma 8 it follows that for each vertex $v \in V(G)$ and each component G' of $G - v$ there exists (\mathcal{P}, L) -colouring. Since, $\mathcal{P} \in \mathbf{M}^a$ we have $\text{ch}_{\mathcal{P}}(G - v) \leq k-1$ for all $v \in V(G)$. Thus, $\text{ch}_{\mathcal{P}}(G) = k$, i.e., G is (k, \mathcal{P}) -choice critical. Since $S(G) = V(G)$, then by Theorem 3, for $k=2$ it follows that G is a $\delta(\mathcal{P})$ -regular graph with $\text{ch}_{\mathcal{P}}(G) > 1$. Hence, $G \in \mathcal{C}(\mathcal{P})$, a contradiction.

If $k \geq 3$, also by Theorem 3, we have that G is a complete graph of order $(k-1)\delta(\mathcal{P}) + 1$ or G is an odd cycle (only in the case when $\mathcal{P} = \mathcal{O}$), a contradiction. \blacksquare

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